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ON THE GAUSS HYPERGEOMETRIC SERIES WITH ROOTS OUTSIDE THE UNIT DISK

髙野勝男 [Takano Katsuo] 茨城大学理学部
岡崎宏光 [Okazaki Hiromitsu] 熊本大学教育学部

1. INTRODUCTION

It is known in [9] that the normed conjugate product of gamma functions such as

$$\frac{2}{\pi}\Gamma(1-ix)\Gamma(1+ix)=\frac{2}{\pi}\frac{1}{\Pi_{n=1}^{\infty}(1+x^2/n^2)}, \tag{1}$$

is an infinitely divisible density. In the process in showing the infinite divisibility of the probability distribution with density (1) a family of polynomials with roots outside the unit disk appeared. From the infinite divisibility of the above probability distribution and from numerical analysis of roots of the terminating hypergeometric series we conjectured that the following density function consisting of normed conjugate product of gamma functions is an infinitely divisible density.

$$c|\frac{\Gamma(m+ix)}{\Gamma(m)}|^2=\frac{c}{\Pi_{n=0}^{\infty}(1+x^2/(m+n)^2)} \qquad (m\in N) \tag{2}$$

(cf. [1. 6.1.25]) In this case the Gauss hypergeometric series appears in general form and it is much more complicated than the case $m=1$. We are necessary to study the location of roots of the Gauss hypergeometric series in showing the infinite divisibility of the probability distribution with density (2). In this paper we will show that many Gauss hypergeometric series have roots outside the unit disk.

2. ON THE GAUSS HYPERGEOMETRIC SERIES

In what follows, suppose taht $a_1 = m$, $a_2 = m+1, \ldots, a_n = m+n-1$ and consider the following density function instead of (2),

$$f(x) = \frac{c}{\Pi_{j=1}^{n}(x^2 + a_j^2)} \tag{3}$$

where c is a normalized constant to be satisfied by the following

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

The probability density function $f(x)$ is an approximation of the above right hand side of (2) in the sense of weak limit. Let us consider a characteristic function of the density fucntion (3). It holds that

$$\begin{aligned} \phi(t) &= \int_{-\infty}^{\infty} e^{itx} \frac{c}{\Pi_{j=1}^{n}(x^2 + a_j^2)} dx \\ &= \pi c \sum_{j=1}^{n} \frac{\exp(-a_j|t|)}{a_j \Pi_{l=1, l\neq j}^{n}(-a_j^2 + a_l^2)}, \quad -\infty < t < \infty. \end{aligned} \tag{4}$$

If we set $x = \exp(-|t|)$ then we obtain a polynomial such as the following form,

$$\phi(t) = \pi c \sum_{j=1}^{n} \frac{x^{a_j}}{a_j \Pi_{l=1, l\neq j}^{n}(-a_j^2 + a_l^2)}, \quad 0 \leq x \leq 1,$$

and we have a complex polynomial,

$$P_{n-1}(z) = (-1)^{n-1} a_n \Pi_{l=1}^{n-1}(-a_n^2 + a_l^2) \sum_{j=1}^{n} \frac{z^{a_j - m}}{a_j \Pi_{l=1, l\neq j}^{n}(-a_j^2 + a_l^2)}. \tag{5}$$

We will use the symbol $g_n(z)$ in place of $P_n(z)$. They are concretely as follows.

$$g_0(z) = 1 \tag{6}$$

$$g_1(z) = 1 + \frac{(-1)(2m)}{2m+2} z \tag{7}$$

$$g_2(z) = 1 + \frac{(-2)(2m)}{2m+3} z + \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)} \frac{z^2}{2!} \tag{8}$$

$$\begin{aligned} g_3(z) &= 1 + \frac{(-3)(2m)}{2m+4} z + \frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)} \frac{z^2}{2!} \\ &+ \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+4)(2m+5)(2m+6)} \frac{z^3}{3!} \end{aligned} \tag{9}$$

....

$$
\begin{aligned}
g_n(z) &= 1 + \frac{(-n)(2m)}{2m+n+1} z + \frac{(-n)(-n+1)(2m)(2m+1)}{(2m+n+1)(2m+n+2)} \frac{z^2}{2!} \\
&+ \frac{(-n)(-n+1)(-n+2)(2m)(2m+1)(2m+2)}{(2m+n+1)(2m+n+2)(2m+n+3)} \frac{z^3}{3!} + \cdots \\
&+ \frac{(-n)(-n+1)\cdots(-n+k-1)(2m)(2m+1)(2m+2)\cdots(2m+k-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+n+k)} \\
&\cdot \frac{z^k}{k!} + \cdots \\
&+ \frac{(-n)(-n+1)\cdots(-2)(-1)(2m)(2m+1)(2m+2)\cdots(2m+n-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+2n)} \frac{z^n}{n!} \\
&= {}_2F_1(2m, -n; 2m+n+1; z) \qquad (10)
\end{aligned}
$$

Two trigonometirc sums are coming from the polynomials $g_n(z)$. Consider the unit circle $C: \ z = e^{i\theta} \ (0 \le \theta \le 2\pi)$ and

$$g_n(e^{i\theta}) \ = \ {}_2F_1(2m, -n; 2m+n+1; e^{i\theta}). \qquad (11)$$

It is often convenient for us to treat the polynomial $z^m g_n(z)$ in place of $g_n(z)$. Let us set

$$u(m, n; \theta) = Re\ e^{im\theta} g_n(e^{i\theta}), \qquad (12)$$

$$v(m, n; \theta) = Im\ e^{im\theta} g_n(e^{i\theta}). \qquad (13)$$

We have

$$
\begin{aligned}
u(m, n; \theta) &= \cos m\theta + \frac{(-n)(2m)}{2m+n+1} \cos(m+1)\theta \\
&+ \frac{(-n)(-n+1)(2m)(2m+1)}{(2m+n+1)(2m+n+2)} \frac{\cos(m+2)\theta}{2!} \\
&+ \frac{(-n)(-n+1)(-n+2)(2m)(2m+1)(2m+2)}{(2m+n+1)(2m+n+2)(2m+n+3)} \frac{\cos(m+3)\theta}{3!} + \cdots \\
&+ \frac{(-n)(-n+1)\cdots(-n+k-1)(2m)(2m+1)(2m+2)\cdots(2m+k-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+n+k)} \\
&\cdot \frac{\cos(m+k)\theta}{k!} \\
&+ \cdots \\
&+ \frac{(-n)(-n+1)\cdots(-2)(-1)(2m)(2m+1)(2m+2)\cdots(2m+n-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+2n)} \\
&\cdot \frac{\cos(m+n)\theta}{n!} \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
v(m,n;\theta) &= \sin m\theta + \frac{(-n)(2m)}{2m+n+1}\sin(m+1)\theta \\
&+ \frac{(-n)(-n+1)(2m)(2m+1)}{(2m+n+1)(2m+n+2)}\frac{\sin(m+2)\theta}{2!} \\
&+ \frac{(-n)(-n+1)(-n+2)(2m)(2m+1)(2m+2)}{(2m+n+1)(2m+n+2)(2m+n+3)}\frac{\sin(m+3)\theta}{3!} + \cdots \\
&+ \frac{(-n)(-n+1)\cdots(-n+k-1)(2m)(2m+1)(2m+2)\cdots(2m+k-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+n+k)} \\
&\cdot \frac{\sin(m+k)\theta}{k!} \\
&+ \cdots \\
&+ \frac{(-n)(-n+1)\cdots(-2)(-1)(2m)(2m+1)(2m+2)\cdots(2m+n-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+2n)} \\
&\cdot \frac{\sin(m+n)\theta}{n!}. \qquad (15)
\end{aligned}
$$

It can be shown that $u(m,n;\theta)$ and $v(m,n;\theta)$ do not always make a Jordan curve when θ runs through the interval $[-\pi/2, \pi/2]$. See the figures after a conjecture in the last section.

3. THE HYPERGEOMETRIC SERIES HAS NOT ROOTS ON THE UNIT CIRCLE

It is known in [1] that the Gauss hypergeometric series is a solution of a differential equation. That is, $g_n(z)$ satisfies the hypergeometric equation.

$$z(1-z)\frac{d^2}{dz^2}g_n(z) + (c-(a+b+1)z)\frac{d}{dz}g_n(z) - abg_n(z) = 0. \qquad (16)$$

In the above equation we assume $a = 2m$, $b = -n$ and $c = 2m+n+1$. We are possibly able to make use of a property of two independent solutios of the second order differential equations and obtain the following

Theorem 1. *If $2 \le m$ and $2 \le n \le 10$ the Gauss hypergeometric series $g_n(z)$ has not roots on the unit circle.*

Proof. If $z^m g_n(z)$ has not roots on the unit circle then $g_n(z)$ has not roots on the unit circle. In order to show that $z^m g_n(z)$ has not roots on the unit circle we will show that the following relation

$$r(\theta) = u(m,n;\theta)v'(m,n;\theta) - u'(m,n;\theta)v(m,n;\theta) = c(m,n)(1-\cos\theta)^{n-1} \tag{17}$$

holds, where $c(m,n)$ is positive constant not depending on the variable θ. If and only if $\theta_0 = 0,\ 2\pi$ then $r(\theta_0) = 0$. But we have $\cos km\theta_0 = 1 = x$ and $u(m,n;\theta_0) = const \cdot \phi(0) > 0$ and so $v'(m,n;\theta_0) = 0$ and we obtain an identity

$$\begin{aligned}
& m + \frac{(-n)(2m)(m+1)}{2m+n+1} + \frac{(-n)(-n+1)(2m)(2m+1)}{(2m+n+1)(2m+n+2)}\frac{(m+2)}{2!} \\
+ & \frac{(-n)(-n+1)(-n+2)(2m)(2m+1)(2m+2)}{(2m+n+1)(2m+n+2)(2m+n+3)}\frac{(m+3)}{3!} + \cdots \\
+ & \frac{(-n)(-n+1)\cdots(-n+k-1)(2m)(2m+1)(2m+2)\cdots(2m+k-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+n+k)} \\
\cdot & \frac{(m+k)}{k!} + \cdots \\
+ & \frac{(-n)(-n+1)\cdots(-2)(-1)(2m)(2m+1)(2m+2)\cdots(2m+n-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+2n)} \\
\cdot & \frac{(m+n)}{n!} = 0
\end{aligned} \tag{18}$$

for $n = 2, ..., 10$.

The case of $n = 2$: We have

$$\begin{aligned}
u(m,2;\theta) &= \cos m\theta + \frac{(-2)(2m)}{2m+3}\cos(m+1)\theta \\
&+ \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)}\frac{\cos(m+2)\theta}{2!}
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
v(m,2;\theta) &= \sin m\theta + \frac{(-2)(2m)}{2m+3}\sin(m+1)\theta \\
&+ \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)}\frac{\sin(m+2)\theta}{2!}.
\end{aligned} \tag{20}$$

We see that

$$\begin{aligned}
r(\theta) &= Re\Big\{e^{-im\theta}g_2(e^{-i\theta})\frac{1}{i}\frac{d}{d\theta}\{e^{im\theta}g_2(e^{i\theta})\}\Big\} \qquad (21) \\
&= \Big\{1 + \frac{(-2)(2m)}{2m+3}\cos\theta + \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)}\frac{\cos 2\theta}{2!}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \quad \Big\{m + \frac{(-2)(2m)}{2m+3}(m+1)\cos\theta \\
&+ \quad \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)}(m+2)\frac{\cos 2\theta}{2!}\Big\} \\
&+ \quad \Big\{\frac{(-2)(2m)}{2m+3}(m+1)\sin\theta + \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)}(m+2)\frac{\sin 2\theta}{2!}\Big\} \\
&\cdot \quad \Big\{\frac{(-2)(2m)}{2m+3}\sin\theta + \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)}\frac{\sin 2\theta}{2!}\Big\}. \qquad (22)
\end{aligned}
$$

For simplicity, set $y = \cos\theta$ and substitute the following idenstities, $\cos 2\theta = 2y^2 - 1$ and $\sin 2\theta = \sin\theta \cdot (2y)$, in the last memeber of (22). Then we see that

$$
\begin{aligned}
r(\theta) \quad &= \quad \Big\{1 - \frac{2(2m)}{2m+3}y + \frac{2\cdot 1(2m)(2m+1)(2y^2-1)}{(2m+3)(2m+4)2!}\Big\} \\
&\cdot \quad \Big\{m - \frac{2(2m)(m+1)}{2m+3}y + \frac{2\cdot 1(2m)(2m+1)(m+2)(2y^2-1)}{(2m+3)(2m+4)2!}\Big\} \\
&+ \quad (1-y^2)\Big\{-\frac{2(2m)}{2m+3} + \frac{2\cdot 1(2m)(2m+1)(2y)}{(2m+3)(2m+4)2!}\Big\} \\
&\cdot \quad \Big\{-\frac{2(2m)(m+1)}{2m+3} + \frac{2\cdot 1(2m)(2m+1)(m+2)(2y)}{(2m+3)(2m+4)2!}\Big\} \\
&= \quad \frac{2(2m)(2m+1)(2m+2)}{(2m+3)(2m+4)}(1-y)^2 \qquad (23)
\end{aligned}
$$

and we obtain (17) for the case $n = 2$.

The case of $n = 3$: We have

$$
\begin{aligned}
&u(m,3;\theta) = \cos m\theta + \frac{(-3)(2m)}{2m+4}\cos(m+1)\theta \\
&+ \quad \frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)}\frac{\cos(m+2)\theta}{2!} \\
&+ \quad \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+5)(2m+6)(2m+7)}\frac{\cos(m+3)\theta}{3!} \qquad (24)
\end{aligned}
$$

and

$$
\begin{aligned}
&v(m,3;\theta) = \sin m\theta + \frac{(-3)(2m)}{2m+4}\sin(m+1)\theta \\
&+ \quad \frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)}\frac{\sin(m+2)\theta}{2!} \\
&+ \quad \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+4)(2m+5)(2m+6)}\frac{\sin(m+3)\theta}{3!}. \qquad (25)
\end{aligned}
$$

We see that

$$
\begin{aligned}
r(\theta) &= Re\Big\{e^{-im\theta}g_3(e^{-i\theta})\frac{1}{i}\frac{d}{d\theta}\{e^{im\theta}g_3(e^{i\theta})\}\Big\} && (26)\\
&= \Big\{1+\frac{(-3)(2m)}{2m+4}\cos\theta+\frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)}\frac{\cos 2\theta}{2!}\\
&+ \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+5)(2m+6)(2m+7)}\frac{\cos 3\theta}{3!}\Big\}\\
&\cdot \Big\{m+\frac{(-3)(2m)}{2m+4}(m+1)\cos\theta\\
&+ \frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)}(m+2)\frac{\cos 2\theta}{2!}\\
&+ \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+5)(2m+6)(2m+7)}(m+3)\frac{\cos 3\theta}{3!}\Big\}\\
&+ \Big\{\frac{(-3)(2m)}{2m+4}(m+1)\sin\theta\\
&+ \frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)}(m+2)\frac{\sin 2\theta}{2!}\\
&+ \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+5)(2m+6)(2m+7)}(m+3)\frac{\sin 3\theta}{3!}\Big\}\\
&\cdot \Big\{\frac{(-3)(2m)}{2m+4}\sin\theta+\frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)}\frac{\sin 2\theta}{2!}\\
&+ \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+5)(2m+6)(2m+7)}\frac{\sin 3\theta}{3!}\Big\}. && (27)
\end{aligned}
$$

Substituting the following idenstities, $\cos 2\theta = 2y^2-1$, $\cos 3\theta = 4y^3-3y$ and $\sin 2\theta = \sin\theta\cdot(2y)$, $\sin 3\theta = \sin\theta\cdot(4y^2-1)$ in the last memeber of (27), then we see that

$$
\begin{aligned}
r(\theta) &= \Big\{1-\frac{3(2m)}{2m+4}y+\frac{3\cdot 2(2m)(2m+1)(2y^2-1)}{(2m+4)(2m+5)2!}\\
&- \frac{3!(2m)(2m+1)(2m+2)(4y^3-3y)}{(2m+4)(2m+5)(2m+6)3!}\Big\}\\
&\cdot \Big\{m-\frac{3(2m)(m+1)}{2m+4}y+\frac{3\cdot 2(2m)(2m+1)(m+2)(2y^2-1)}{(2m+4)(2m+5)2!}\\
&- \frac{3!(2m)(2m+1)(2m+2)(m+3)(m+3)(4y^3-3y)}{(2m+4)(2m+5)(2m+6)3!}\Big\}\\
&+ (1-y^2)\Big\{-\frac{3(2m)}{2m+4}+\frac{3\cdot 2(2m)(2m+1)(2y)}{(2m+4)(2m+5)2!}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3!(2m)(2m+1)(2m+2)(4y^2-1)}{(2m+4)(2m+5)(2m+6)3!}\Big\} \\
&\cdot \Big\{-\frac{3(2m)(m+1)}{2m+4} + \frac{3\cdot 2(2m)(2m+1)(m+2)(2y)}{(2m+4)(2m+5)2!} \\
&- \frac{3!(2m)(2m+1)(2m+2)(m+3)(m+3)\{4y^2-1\}}{(2m+4)(2m+5)(2m+6)3!}\Big\} \\
&= \frac{2^2(2m)(2m+1)(2m+2)(2m+3)}{(2m+4)(2m+5)(2m+6)}(1-y)^3 \qquad (28)
\end{aligned}
$$

and we obtain (17) for the case $n = 3$. Repeating this method for the cases $n = 4, 5, 6, ..., 10$ we obtain the assertion of theorem. q.e.d.

4. THE HYPERGEOMETRIC SERIES HAS ROOTS OUTSIDE THE UNIT DISK

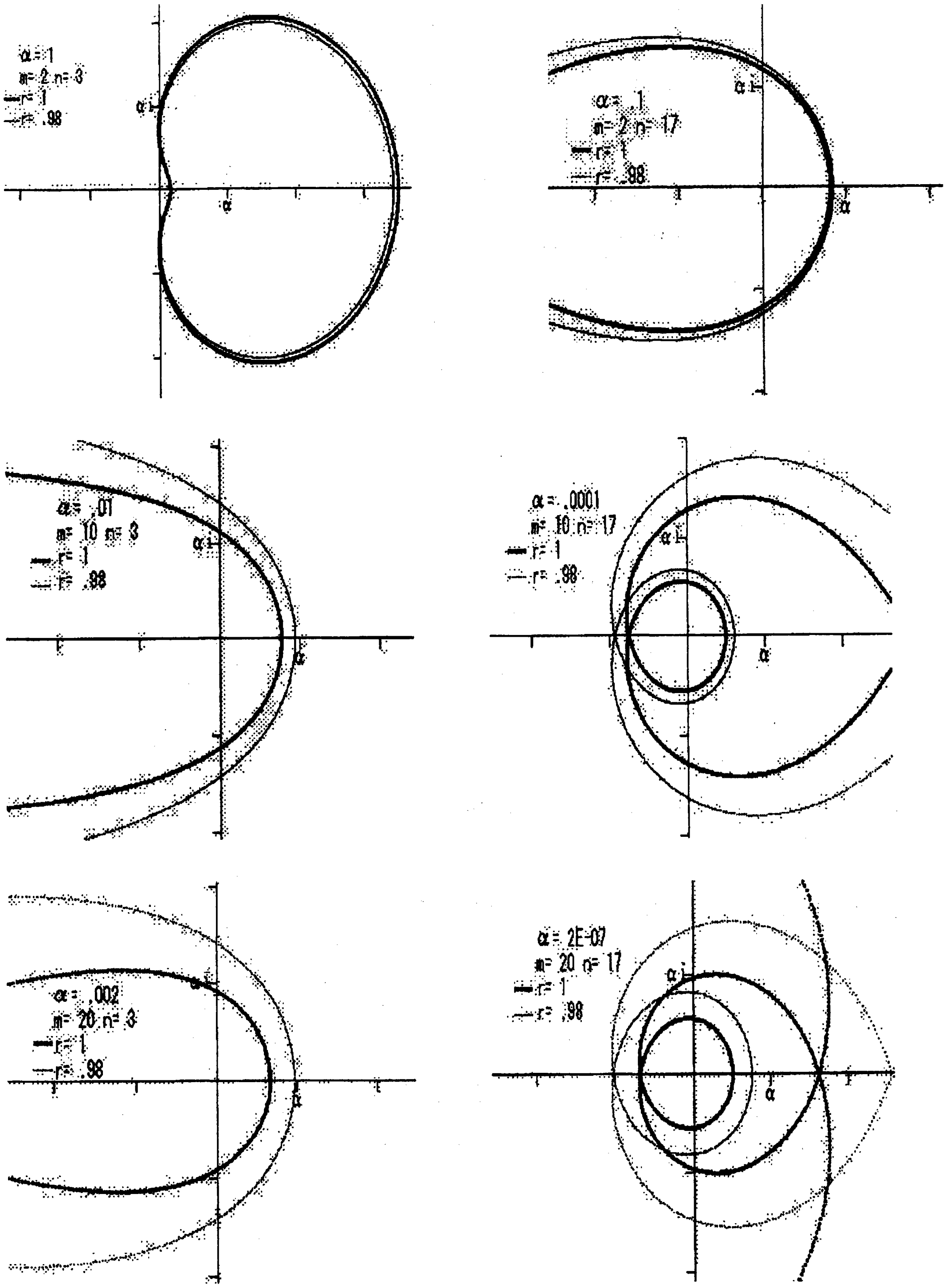
If $m = 1$ it is known in [8] that the roots of $g_n(z)$ appears outside the closed unit disk. If $n = 1$ the root of $g_1(z)$ is $z_1 = m + 1/m$ and if $n = 2$ the two roots of $g_2(z)$ are

$$z_1 = \frac{m+1}{2m+1}\Big(\frac{m+2}{m+1} + i\sqrt{\frac{3(m+2)}{m}}\Big), \quad z_2 = \frac{m+1}{2m+1}\Big(\frac{m+2}{m+1} - i\sqrt{\frac{3(m+2)}{m}}\Big)$$

for all $m \in N$. These roots are outside the unit disk. We obtain the following computational result.

Conjecture: *If $2 \le m \le 20$ and $3 \le n \le 17$ the Gauss hypergeometric series $g_n(z)$ has roots outside the closed unit disk.*

By the following graphs which were drawn with the computer, we are able to conclude that this conjecture is true. If the value of $g_n(z)$ is most near 0, then z is a point on the unit circle since it can be seen from the curve that the origin is outside the range of the hypergeometric series with the domain of the unit disk and $g_n(z)$ is not equal to 0.


α= 1
m= 2 n= 3
r= 1
r= .98
α= .1
m= 2 n= 17
r= 1
r= .98
α= .01
m= 10 n= 3
r= 1
r= .98
α= .0001
m= 10 n= 17
r= 1
r= .98
α= .002
m= 20 n= 3
r= 1
r= .98
α= 2E-07
m= 20 n= 17
r= 1
r= .98

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, New York, Dover, 1970.

[2] L. Bondesson, On the infinite divisibility of the half-Cauchy and other decreasing densities and probability functions on the nonnegative line, *Scand. Acturial J.*, (1985), 225-247.

[3] M. J. Goovaerts, L. D'Hooge, and N. De Pril, On the infinite divisibility of the product of two Γ-distributed stochastical variables, *Applied mathematics and computation*, 3 (1977), 127-135.

[4] D. H. Kelker, Infinite divisibility and variance mixtures of the normal distribution, *Ann. Math. Statist.*, 42 (1971), 802–808.

[5] G. Sansone & J. Gerretsen, Lectures on the theory of functions of a complex variable, 1. Holomorphic functions, P. Noordhoff-Groningen, 1960

[6] K. Sato, Class L of multivariate distributions and its subclasses, *J.Multivaria Anal.* 10 (1980), 207-232.

[7] F. W. Steutel, Preservation of infinite divisibility under mixing and related topics, *Math. Centre Tracts, Math. Centre, Amsterdam,* 33. 1970

[8] K. Takano, On a family of polynomials with zeros outside the unit disk, *International J. comput. Numer. Anal. Appl.*, 1. No.1 (2002),369-382.

[9] K. Takano, On infinite divisibility of normed product of Cauchy densities, *J. Comput. Applied Math.*, 150(2003),253-263.

[10] O. Thorin, On the infinite divisibility of the Pareto distribution, *Scand. Acturial. J.*, (1977), 31–40.

Katsuo TAKANO
Department of Mathematics
Faculty of Science, Ibaraki University
Mito-city, Ibaraki 310, JAPAN
e-mail:ktaka@mx.ibaraki.ac.jp

Hiromitsu OKAZAKI
Department of Mathematics
Faculty of Education, Kumomoto University
Kumomoto-city, Kumomoto 860, JAPAN
e-mail:okazaki@educ.kumomoto-u.ac.jp