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Kyoto University
AUTOMORPHIC GREEN FUNCTIONS ON ARITHMETIC QUOTIENTS OF TYPE IV SYMMETRIC DOMAIN

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1. Introduction

This article is a short summary of the forthcoming paper:

‘Automorphic Green functions associated with the secondary spherical functions’
(Takayuki Oda and Masao Tsuzuki)

Let $G := O_0(n, 2)$ be the identity component of the orthogonal group with signature $(n+, 2-)$ and $K := G \cap \text{diag}(O(n), O(2))$ a maximal compact subgroup of $G$. The Lie algebra $\mathfrak{g} := \text{Lie}(G)$ is identified with the space of matrices $X \in \text{Mat}_{n+2}(\mathbb{R})$ satisfying $^tXI_{n,2}+I_{n,2}X = O$ with the bracket product $[X, Y] = XY - YX$. Let $E_{ij}$ $(1 \leq i, j \leq n+2)$ be the usual matrix unit of $\text{Mat}_{n+2}(\mathbb{R})$.

The homogeneous manifold $G/K$ is a symmetric space of type IV, which is a Hermitian symmetric domain with the $G$-invariant complex structure coming from the adjoint action $J := \text{ad}(Z_0)|p$ with $Z_0 := E_{n+1,n+2} - E_{n+2,n+1} \in \mathfrak{g}$, the orthogonal complement of $\mathfrak{f}$ in $\mathfrak{g}$ with respect to the Killing form $B$ of $\mathfrak{g}$. The $K$-invariant alternating form $\tilde{\omega}(X, Y) := (8n)^{-1}B(X, J(Y))$ on $p$ is uniquely extended to a $G$-invariant $C^\infty$ differential form $\omega$ of $(1, 1)$ type on $G/K$, by which $G/K$ is a Kähler manifold.

Any arithmetic subgroup $\Gamma$ of $G$ acts discontinuously on $G/K$ through bi-holomorphic automorphisms of $G/K$. When $\Gamma$ is neat, taking the quotient by $\Gamma$ we have a Kähler manifold $\Gamma\backslash G/K$ with Kähler form $\omega_{\Gamma\backslash G/K}$ such that the quotient map $\pi : G/K \rightarrow \Gamma\backslash G/K$ is holomorphic and $\pi^*\omega_{\Gamma\backslash G/K} = \omega$.

Consider the symmetric subgroup $H = O_0(n-1, 2)$ consisting of fixed points of the involution $\sigma$ of $G$ defined by $\sigma(g) = SgS$ with $S := \text{diag}(E_{n-1}, -1, E_2)$. We assume that $H$ is 'T-rational' in a proper sense. In particular the invartiant volume of $\Gamma_H\backslash H/K_H$ is finite, where $\Gamma_H := \Gamma \cap H$ and $K_H := H \cap K$. Let $D$ be the image of the natural holomorphic map $\Gamma_H\backslash H/K_H \rightarrow \Gamma\backslash G/K$. Then $D$ is a closed complex analytic subset of $\Gamma\backslash G/K$ with complex codimension 1, which defines a closed current $\delta_D$ by integration

$$\langle \delta_D, \alpha \rangle = \int_{D_{ns}} j^*\alpha, \quad \alpha \in A_c(\Gamma\backslash G/K).$$

Here $D_{ns}$ denotes the smooth locus of $D$ and $A_c(M)$ denotes the space of compactly supported smooth forms on a complex manifold $M$.

Then our aim here is to explain an explicit construction of the Green current for $D$ following [1]. Though a similar construction for the 'unitary case' (i.e., for the modular divisors in an arithmetic quotient of a complex-hyperball) is proved to be possible, we focus only on the 'orthogonal case' setting aside the 'unitary case' for simplicity of presentation.
2. SECONDARY SPHERICAL FUNCTIONS

Let $a$ be the maximal abelian subspace $\mathbb{R}Y_0$ of $\mathfrak{p} \cap \mathfrak{q}$ with the basis $Y_0 = E_{n,n+1} + E_{n+1,n}$. Here $\mathfrak{q}$ is the orthogonal complement of $\mathfrak{h} := \text{Lie}(H)$. Then the group $G$ is a union of the double cosets $Ha_tK (t \geq 0)$ with $a_t := \exp(tY_0)$. We introduce two functions $\phi_s^{(2)}$ and $\psi_s$ with singularities on $G$.

2.0.1. The function $\phi_s^{(2)}$. There exists a unique family of functions $\phi_s^{(2)} (\text{Re}(s) > n/2)$ such that

- $\phi_s^{(2)}$ is a $C^\infty$-function on $G - HK$ and $(H, K)$-invariant, i.e.,
  \[ \phi_s^{(2)}(hgk) = \phi_s^{(2)}(g) \quad \forall h \in H, \forall g \in G - HK, \forall k \in K. \]
- $\phi_s^{(2)}$ satisfies the differential equation
  \[ \Omega \phi_s^{(2)}(g) = (s^2 - (n/2)^2) \phi_s^{(2)}(g), \quad g \in G - HK. \]
- There exists a positive $\delta$ such that $\phi_s^{(2)}(\exp(tY_0)) - \log(t)$ is bounded on the interval $(0, \delta)$.
- $\phi_s^{(2)}(a_t)$ decays exponentially as $t$ getting large:
  \[ \phi_s^{(2)}(a_t) = O(e^{-(\text{Re}(s)+n/2)t}) \quad (t \to +\infty). \]

([1, Proposition 2.4.2]).

We have the explicit formula:

\[
\phi_s^{(2)}(a_t) = -\frac{1}{2} \frac{\Gamma((s+n/2)/2) \Gamma((s-n/2)/2+1)}{\Gamma(s+1)} \times (\cosh t)^{-((s+n/2)/2)} F_1 \left( \frac{s+n/2}{2}, \frac{s-n/2}{2}+1; s+1; \frac{1}{\cosh^2} \right), \quad (t > 0).
\]

([1, 2.5]).

2.0.2. The function $\psi_s$. Let $\mathfrak{p}_\pm$ be the $\pm \sqrt{-1}$-eigen space of the complex linear extension of $J$ to $\mathfrak{p}_\mathbb{C}$. Then $\mathfrak{p}_+ = \sum_{i=0}^{n-1} \mathbb{C} X_i$ and $\mathfrak{p}_- = \sum_{i=0}^{n-1} \mathbb{C} \overline{X}_i$ with

\[
X_0 = E_{n,n+1} + E_{n+1,n} + \sqrt{-1}(E_{n,n+2} + E_{n+2,n}),
\]
\[
X_i = E_{i,n+1} + E_{n+1,i} + \sqrt{-1}(E_{i,n+2} + E_{n+2,i}), \quad 1 \leq i \leq n-1.
\]

Let $\{\omega_i\}$ and $\{\overline{\omega}_i\}$ be the dual basis of $\{X_i\}$ and $\{\overline{X}_i\}$ respectively. Put

\[
\nu_{11} := \frac{1}{4} \left( \sum_{i=1}^{n-1} \omega_i \wedge \overline{\omega}_i - (n-1)\omega_0 \wedge \overline{\omega}_0 \right) (\in \mathfrak{p}_+^* \wedge \mathfrak{p}_-^*)
\]

Then $(\mathfrak{p}_+^* \wedge \mathfrak{p}_-^*)^M$ is a two dimensional space generated by $\nu_{11}$ and the Kähler form $\tilde{\omega} = \sqrt{-1} \sum_{i=0}^{n-1} \omega_i \wedge \overline{\omega}_i$. For $\text{Re}(s) > n/s$, put

\[
\psi_s(g) = \frac{1}{4} \sum_{i,j=0}^{n-1} R_{X_i \overline{X}_j} \phi_s^{(2)}(g) \omega_i \wedge \overline{\omega}_j \quad g \in G - HK.
\]

Here are some properties of the function $\psi_s$. 
• $\psi_s$ is a $C^\infty$-function on $G - HK$ such that
\[
\psi_s(hgk) = (\text{Ad}_{p_+}^* \, \wedge \text{Ad}_{p_-}^*)(k)^{-1} \psi_s(g), \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.
\]
Here $\text{Ad}_{p_+}^*$ be the coadjoint representation of $K$ on $p_+$.

• We have $\psi_s(a_t) = f_s(t) \nu_{11}$ with
\[
f_s(t) = \left(\tanht \frac{d}{dt} - \frac{s^2 - (n/2)^2}{n}\right) \phi_s^{(2)}(a_t), \quad t > 0.
\]

• There exists a positive $\delta$ such that $f_s(t) + \frac{s^2 - (n/2)^2}{2n} \log t$ is bounded on the interval $(0, \delta)$.

• We have the estimation:
\[
f_s(t) \prec e^{-((\Re(s)+n/2)t}, \quad t \in [1, \infty).
\]

### 3. Currents defined by Poincare series

Let $\Gamma$ be as in the introduction. For $\alpha \in A(\Gamma \backslash G/K)$, we have a unique $C^\infty$-function $\tilde{\alpha} : G \to \wedge p_\mathbb{C}^*$ such that $\tilde{\alpha}(\gamma gk) = \tau(k)^{-1}\tilde{\alpha}(g)$, $(\gamma \in \Gamma, k \in K)$ and such that
\[
(\pi^*\alpha)(gK), dL_g(\xi_0)) = \langle \tilde{\alpha}(g), \xi_0 \rangle, \quad g \in G, \xi_0 \in \wedge p = \bigwedge T_o(G/K)
\]
holds. Here $L_g$ denotes the left translation on $G/K$ by the element $g$ and we identify $p$ with $T_o(G/K)$, the tangent space of $G/K$ at $o = eK$. Let $dk$ (resp. $dk_0$) be the normalized Haar measure of $K$ (resp. $K_H$) with total volume 1. Then there exists a Haar measure $dg$ (resp. $dh$) of $G$ (resp. $H$) such that $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_0}$) corresponds to the measure of the symmetric space $G/K$ (resp. $H/K_H$) determined by the invariant volume form associated to the Kähler form.

For any left $\Gamma$-invariant function $f$ on $G$, put
\[
J_H(f; g) = \int_{\Gamma_H \backslash H} f(hg) \, dh, \quad g \in G.
\]
Let $\varphi_s = \phi_s^{(2)} (\Re(s) > n/2)$ or $\psi_s (\Re(s) > n/2)$. Then the integral
\[
\int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma_H \backslash \Gamma} \|\tilde{\varphi}_s(\gamma g)\| \right) \, dg
\]
is locally bounded in $\Re(s) > n/2$ ([1, Proposition3.1.1]), and there exists a unique current $P(\varphi_s)$ on $\Gamma \backslash G/K$ such that
\[
\langle P(\varphi_s), *\tilde{\alpha} \rangle = \int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma_H \backslash \Gamma} \varphi_s(\gamma g)|\tilde{\alpha}(g)\rangle \right) \, dg
\]
\[
= \frac{\pi}{2} \int_0^\infty (\varphi_s(a_t)|J_H(\tilde{\alpha}; a_t)) \sinh(t \cot(s))^n \, dt, \quad \forall \alpha \in A_c(\Gamma \backslash G/K)
\]
Here $(\cdot, \cdot)$ is the Hermitian inner product of $p_\mathbb{C}^*$ canonically induced by the inner product $(8n)^{-1}B(X, Y)$ on $p$.

We have the current $G_s := P(\phi_s^{(2)})$ of $(0,0)$-type and the one $\Psi_s := P(\psi_s)$ of $(1,1)$-type on $\Gamma \backslash G / K$ which depends holomorphically on $\Re(s) > n/2$. 

4. Differential equations

Let \( \text{Re}(s) > n/2 \). Then the currents \( G_s \) and \( \Psi_s \) satisfy the differential equations:

\[
\triangle G_s = -((2s)^2 - n^2) G_s - 2\pi \Lambda \delta_D,
\]

\[
\triangle \Psi_s = -((2s)^2 - n^2) \left( \Psi_s - \frac{\pi\sqrt{-1}}{4} \delta_D - \frac{\pi\sqrt{-1}}{4n} L \Lambda \delta_D \right),
\]

\[
\partial \overline{\partial} G_s + \pi\sqrt{-1} \delta_D = \frac{\sqrt{-1}}{2n} ((2s)^2 - n^2) L G_s + 4 \Psi_s.
\]

Here \( \Lambda \) is the adjoint of the Lefschets operator \( L \alpha = \omega_{\Gamma \backslash G/K} \Lambda \alpha \) (\([1, \text{Theorem 7.6.1}]\)).

5. Meromorphicity

Suppose \( \Gamma \backslash G \) is compact. Let \( \{\lambda_m\}_{m \in \mathbb{N}} \) be the increasing sequence of the eigenvalues of the negative of the Casimir operator \( -R_{\Omega} \) acting on \( L^2(\Gamma \backslash G/K) \) such that each eigenvalue occurs with its multiplicity. We fix an orthonormal basis \( \{\varphi_m\}_{m \in \mathbb{N}} \) of \( L^2(\Gamma \backslash G/K) \) consisting of automorphic forms on \( \Gamma \backslash G/K \) such that \( -R_{\Omega} \varphi_m = \lambda_m \varphi_m \) (\( \forall m \in \mathbb{N} \)). Then we have the spectral expansion of \( G_s \) (\( \text{Re}(s) > n/2 \)):

\[
\langle G_s, *\bar{\alpha} \rangle = \sum_{m=0}^{\infty} \frac{J_H(\varphi_m, e)}{(n/2)^2 - \lambda_m - s^2} \cdot \langle \varphi_m | \bar{\alpha} \rangle_{L^2}, \quad \alpha \in A_c(\Gamma \backslash G/K).
\]

Here \( \langle \cdot | \cdot \rangle_{L^2} \) is the \( L^2 \)-inner product of \( L^2(\Gamma \backslash G/K) \). The corresponding result for the 'unitary case' is proved in \([1, \text{Proposition 6.2.2}]\). The proof for the present case is pararell since we assume \( \Gamma \backslash G \) is compact. Then by an estimation similar to that in \([1, \text{Theorem 6.2.1 (1)}] \), the series (2) is absolutely convergent for an arbitrary \( s \in \{ s \in \mathbb{C} | s^2 \neq (n/2)^2 - \lambda_m (\exists m) \} \) locally uniformly. Hence the current \( s \mapsto G_s \), which is originally holomorphic only on \( \text{Re}(s) > n/2 \), has a meromorphic continuation to the whole \( s \)-plane with possible simple poles at the points \( s \in \mathbb{C} \) such that \( s^2 = (n/2)^2 - \lambda_m (\exists m) \).

6. Green current

The point \( s = n/2 \) is a simple pole of \( G_s \) with the residue

\[
\text{Res}_{s=n/2} G_s = -\frac{1}{n} \frac{\text{vol}(\Gamma_H \backslash H)}{\text{vol}(\Gamma \backslash G)},
\]

a constant function on \( \Gamma \backslash G/K \).

**Definition**

We put \( \mathcal{G} \) to be \( (-2\pi)^{-1} \) times the constant term of the Laurent expansion of \( G_s \) at \( s = n/2 \), i.e.,

\[
\mathcal{G}(x) = \frac{-1}{2\pi} \lim_{s \to n/2} \left( G_s(x) - \frac{\kappa}{s - n/2} \right)
\]

with \( \kappa = -\frac{1}{n} \frac{\text{vol}(\Gamma_H \backslash H)}{\text{vol}(\Gamma \backslash G)} \).

**Theorem**
The current-valued function $s \mapsto \Psi_s$ on $\text{Re}(s) > n/2$ has a meromorphic continuation to the whole $s$-plane. The point $s = n/2$ is a regular point of the meromorphic function $\Psi_s$ and the value $\Psi_{n/2}$ is harmonic, i.e.,
\[ \Delta \Psi_{n/2} = 0. \]

The current $G$ satisfies Green's equation:
\[ \text{dd}^c G + \delta_D = \frac{1}{\pi} (\kappa \omega_{T\backslash G/K} + 4 \Psi_{n/2}). \]

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