Title

The family of $K3$ surfaces with a transcendental lattice $U(2)^2 \times <-2>^4$ for a general member (Automorphic forms on type IV symmetric domains)

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The family of $K3$ surfaces with a transcendental lattice
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0 Introduction

Let us consider the lattice $P = D_4^3 \oplus \langle -2 \rangle \oplus \langle 2 \rangle$ of rank 14. A $K3$ surface $S$ is called of type $P$ when it satisfies $P \subset \text{Pic}(S)$, where $\text{Pic}(S)$ stands for the Picard lattice of $S$. In this article we show the outline of the study on the period map for the family $\mathcal{F}$ of $K3$ surfaces of type $P$.

We work always on the ground field $\mathbb{C}$. Note that the lattice $H^2(S, \mathbb{Z})$ is always isomorphic to $L = U^3 \oplus \langle -E_8 \rangle^2$ and $P_{\perp} \subset L$ is isomorphic to $U(2)^2 \oplus \langle -2 \rangle^4$.

In 1992 K. Matsumoto, T. Sasaki and M. Yoshida [7] studied the period mapping for a family of $K3$ surfaces of type $(3, 6)$, that is the family of double covering surfaces over $\mathbb{P}^2$ branching along six lines in general position, and Matsumoto [6] gave the description of the inverse mapping in terms of theta constants. It gives the modular map for the 4 dimensional Shimura variety in the Siegel upper half space $S_4$ derived from the family of 4 dimensional abelian varieties with generalized complex multiplication by $\sqrt{-1}$ of type $(2, 2)$. So we call it MSY modular mapping. The second author showed an arithmetic application of MSY modular mapping in [11].

In the case of MSY modular map the corresponding $K3$ surface is characterized by the lattice $U(2)^2 \oplus \langle -2 \rangle^2$ and the moduli space is a 4-dimensional type $IV$ domain. We suspect such fruitfull results of the MSY map is the consequence of the eventual coincidence of two different bounded symmetric domains $D_4^{IV}$ and $I_{2,2}$. There are a few (finite) such exceptional coincidences. The highest one is the (analytic) equivalence between $D_4^{IV}$ and $H_{IV}(4)$ (in terms of Lie algebra $so(2, 6; \mathbb{R}) \cong so(4, \mathbb{F})$, where $\mathbb{F}$ indicates the Hamilton quaternion field) and it contains the above coincidence of MSY case. That is our situation.

The ring structure of the regular functions on the parameter space for $\mathcal{F}$ is given by the article of Koike and Ochiai in this volume, their result is the cosequence of the discussion motivated by the RIMS workshop.

1 Realization as an algebraic surface

We fix an abstract lattice $L$ and its $P$-part $D_4^3 \oplus \langle -2 \rangle \oplus \langle 2 \rangle$. A $K3$ surface $S$ is called of exact type $P$ when it satisfies $P \cong \text{Pic}(S)$ via an isomorphism $\varphi: H_2(S, \mathbb{Z}) \rightarrow L$. That is a general member of $\mathcal{F}$. A surface $S$ of exact type $P$ is realized as a double covering surface over $\mathbb{P}^1 \times \mathbb{P}^1$ branching along four bidegree $(1, 1)$ curves $H_1, H_2, H_3, H_4$ satisfying

$J_1) \quad H_k \quad (k = 1, 2, 3, 4)$ is irreducible,
$J_2$) $H_k \cap H_\ell$ consists of two different points, 
$J_3$) for any different three indices $i, j, k$ we have $H_i \cap H_j \cap H_k = \emptyset$
under some nef condition stated in Theorem 1. Such a surface is given as the complete 
nonsingular model of the affine variety

\[ S = S(x) : w^2 = \prod_{k=1}^{4} (x_1^{(k)} s t + x_2^{(k)} s + x_3^{(k)} t + x_4^{(k)}), \]

where we use the notation

\[ x_k = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \\ x_4^{(k)} \end{pmatrix} \in \mathbb{M}(2, \mathbb{C}). \]

So the curve $H_k$ $(k = 1, 2, 3, 4)$ is given by

\[ (s, 1) \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \\ x_4^{(k)} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = 0. \]

By considering the projection to the $s$-line we find that a general member $S$ of $\mathcal{F}$ is an elliptic 
blow up process at the intersections $H_i \cap H_j$ $(i \neq j, 1 \leq i, j \leq 4)$. Let $\pi$ be the projection $S \to \mathbb{P}^1 \times \mathbb{P}^1$ and set

\[ G_i = \frac{1}{2} (\pi^* H_i - \sum_{i \neq j} (E_{ij}^+ + E_{ij}^-)) \quad (i = 1, 2, 3, 4) \]

\[ F_s = \pi^* \{ t = 0 \}, \quad F_t = \pi^* \{ s = 0 \}. \]

Lemma 1.1 The sublattice in $H_2(S(x), \mathbb{Z})$ generated by $E_{ij}^+, E_{ij}^-, G_i, F_s, F_t$ is isomorphic to $P.$
So $S(x)$ is a K3 surface of type $P$.

According to the above Lemma we can define the elements $e_{ij}^\pm, g_i, f_s, f_t$ in the abstract lattice $L$ corresponding to the divisors with capital letters in $H^2(S(x), \mathbb{Z})$ via an isomorphism $\varphi : H_2(S(x), \mathbb{Z}) \to L$.

Theorem 1 Let $S$ be a K3 surface of type $P$ with an isomorphism $\varphi$, and suppose that $\varphi^{-1}(f_s)$ 
and $\varphi^{-1}(f_t)$ are nef divisors. Then $S$ is realized as the form of $S(x)$.

2 Period map

2.1 Marking of a K3 surface of type $P$

Definition 2.1 Let $S$ be a K3 surface of type $P$ with an embedding $P \to L$. We call the triple 
$(S, \varphi, P)$ a $P$ marking of $S$ if we have

1) $\varphi : H_2(S, \mathbb{Z}) \to L$ is an isometry of lattices, 
2) $\varphi^{-1}(f_s), \varphi^{-1}(f_t), \varphi^{-1}(e_{ij}^\pm), \varphi^{-1}(g_i)$ are effective divisors and $\varphi^{-1}(f_s)$ and $\varphi^{-1}(f_t)$ are 
nef.

Let $O(L)$ be the group of isometries of the lattice $L$, and set $O(L, P) = \{ g \in O(L) : g(x) = x \text{ for } \forall x \in P \}$. Putting $T = P^\perp$ we set $G = O(T)$ and $G(2) = \{ g \in G : g \equiv I \pmod{2} \}$, 
where we defined $O(T)$ as same as $O(L)$. 

\[ j \]
Proposition 2.1 We have $G(2) \cong O(L, P)$.

Definition 2.2 Let $(S, \varphi, P)$ and $(S', \varphi', P)$ be $P$ markings of $S$ and $S'$, respectively. An isomorphism $\rho : S \rightarrow S'$ is called an isomorphism of these 2 markings when we have $\varphi = \varphi' \circ \rho_*$. We say 2 markings $(S, \varphi, P)$ and $(S', \varphi', P)$ are equivalent when we have a $\gamma \in O(L, P)$ such that $(S', \varphi', P)$ is isomorphic to $(S, \gamma \circ \varphi, P)$.

Remark 2.1 Equivalent markings correspond to the same configuration of the branch locus with the order of $H_1, \ldots, H_4$ of the double covering surfaces. And this relation corresponds to the base change of the K3 lattice which preserves the polarization $P$.

2.2 Parameter space

From the above argument we can take the configuration space for the family of 4 bidegree $(1, 1)$ curves as a parameters space of our family of $P$ marked K3 surfaces. The presentation of $H_k$ has an ambiguity of a constant factor. By considering the projective coordinate transformation of $s$-space and $t$-space our parameter space is given by

$$X = \left( PGL(2, \mathbb{C}) \backslash \{(z_1, \ldots, z_4)\} = (GL(2, \mathbb{C}))^4 / PGL(2, \mathbb{C}) \right) / (\mathbb{C}^*)^4.$$

For further investigation we need a realization of $X$ as a projective variety. That is suggested in the article of Koike and Ochiai on this same volume.

2.3 Period domain

Let $\Gamma_1, \ldots, \Gamma_8$ be a fixed basis of $T$ such that we have the intersection form

$$(\Gamma_i \cdot \Gamma_j) = A := U(2) \oplus (-2I_4).$$

For a $P$ marking $(S, \varphi, P)$ let $\Omega$ be the holomorphic 2-form on $S$ that is unique up to a constant factor. We define the period of $(S, \varphi, P)$ by

$$\eta = [\int_{\varphi^{-1}(\Gamma_1)} \Omega, \ldots, \int_{\varphi^{-1}(\Gamma_8)} \Omega] \in \mathbb{P}^7.$$

The image of the period mapping for the family of $P$ marked K3 surfaces is open dense in the 6-dimensional domain given by

$$D^+ = \{ \eta = [\eta_1, \ldots, \eta_8] \in \mathbb{P}^7 : t^\top \eta A \eta = 0, t^\top \overline{\eta} A \eta > 0, \exists (\eta_3/\eta_1) > 0 \}.$$

We get this fact by using the Riemann-Hodge relation of the period and the Torelli theorem for K3 surfaces. It is a bounded symmetric domain of type IV. We set

$$G^+ = \{ g \in G : g(D^+) = D^+ \}, \quad G(2)^+ = \{ g \in G(2) : g(D^+) = D^+ \}.$$

We can determine the modular group for the equivalence classes of the $P$ marked surfaces. Namely

Theorem 2 Let $(S, \varphi, P)$ and $(S', \varphi', P)$ be $P$ markings of K3 surfaces of type $P$. Let $\eta$ and $\eta'$ be the corresponding periods, respectively. Then these two markings are equivalent if and only if

$$g(\eta) = \eta'$$

for some $g \in G(2)^+$. 

Theorem 3 The modular group $G(2)^+$ is a reflection group.

Here a transformation

$$R_v : \lambda \mapsto v - 2(\langle vA\lambda, vAv \rangle)v, \quad \lambda \in D^+, v \in Z^8$$

is called a reflection with the root vector $v$.

2.4 Degenerate locus

3 Differential equation

We can determine the system of differential equations for the period with 16 variables $x_{ij}^k$. It becomes a holonomic system of rank 8. So our periods $\int_{\varphi^{-1}(\Gamma_i)} \Omega$ $(i = 1, \ldots, 8)$ make a basis of the space of solutions for this system defined on a domain $X' = X - V$, where $V$ is the degenerating locus corresponding to the set of $K3$ surfaces of type $P$ which violate some condition of $J_1, J_2, J_3$.

So we can consider the monodromy group $\mathcal{M}$ for this system.

Proposition 3.1 We have $G(2)^+ \subset \mathcal{M}$.

Remark 3.1 We have possibly $G(2)^+ = \mathcal{M}$. But we cannot decide it at present, because Some monodromy transformation may cause an interchange of $E_{ij}^\pm$.

Remark 3.2 The image of the degenerating locus $V$ by the period map is consists of $4, 6, 16$ hyperplanes (so in total $26$ hyperplanes) in the period domain $D^+$ which correspond to the violation of the condition $J_1, J_2, J_3$, respectively.

4 Transfer of the period domain

The type $\Pi$ domain $\mathcal{H}$ is defined by

$$\mathcal{H} = \mathcal{H}_{\Pi} = \{Z \in M(4, \mathbb{C}) : J_4Z = ^tZJ_4, \quad \frac{1}{i}(Z - Z^*) > 0\},$$

where we use the notation

$$J_{2n} = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix}$$

and the member $Z \in \mathcal{H}$ is described in the form

$$Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & t & a & c \\ -t & 0 & b & d \end{pmatrix}.$$

We define the mapping $\psi : \mathcal{H} \rightarrow \mathbb{P}^7$ by

$$\zeta = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] = [1, -ad + bc - st, a, d, b, c, -s, t].$$

As a direct translation of the above lemma, we obtain:
Proposition 4.1 The image of $\psi$ is determined in $\mathbb{P}^7$ by the following three conditions:

1. $^t \zeta (U \oplus U \oplus U \oplus U) \zeta = 0$.
2. $\zeta^* (U \oplus U \oplus U \oplus U) \zeta > 0$,

where $\zeta^* = ^t \overline{\zeta}$.
3. $\Im \left( \frac{z_3}{z_1} \right) > 0$.

By straightforward calculation we have the following.

Theorem 4 The image $\psi(H)$ is transformed to the type IV domain $D^+_IV$ by the map:

$$\eta = P \zeta$$

with

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.$$ (4.1)

So the composite mapping $\delta = P \circ \psi$ gives the isomorphism

$$\delta : H \cong D^+_IV.$$  

Remark 4.1 The analytic equivalence of the domains $D^+_IV$ and $H$ is well known. But we are wishing to find the transfer which preserves the modular groups each other.

4.1 the Quaternion half space

Let $F$ be the Hamilton quaternion $\mathbb{R}$-algebra generated by $\{e_1, e_2, e_3, e_4\}$ with

$$e_1 = 1, \quad e_2 e_3 = e_4, \quad e_i^2 = -1.$$ 

Then the ring of integers in $F$ is given by

$$\mathcal{O}(F) = Ze_0 + Ze_1 + Ze_2 + Ze_3,$$

where $e_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$.

Proposition 4.2 The mapping

$$\varphi : M(n,F) \to M(2n, \mathbb{C})$$

deﬁned by

$$\varphi \left( \sum_i A_i e_i \right) = \begin{pmatrix}
A_1 e_1 + A_2 e_2 & A_3 e_1 + A_4 e_2 \\
-A_3 e_1 + A_4 e_2 & A_1 e_1 - A_2 e_2 \\
\end{pmatrix}$$

is an injective homomorphism of $\mathbb{R}$-algebra.
Definition 4.1 We set

\[ \text{Sym}(2, \mathbb{F}) = \{ X \in M(2, \mathbb{F}) : {}^t \overline{X} = X \} , \]
\[ \text{Pos}(2, \mathbb{F}) = \{ X \in \text{Sym}(2, \mathbb{F}) : X > 0 \} . \]

Note that we have

\[ X \in \text{Sym}(2, \mathbb{F}) \iff \varphi(X) \in \text{Sym}(4, \mathbb{C}) , \]
\[ X \in \text{Pos}(2, \mathbb{F}) \iff \varphi(X) \in \text{Pos}(4, \mathbb{C}) \]

and it holds also

\[ X \in \text{Pos}(2, \mathbb{F}) \iff X = g^* g , \quad \exists g \in \text{GL}(2, \mathbb{F}) . \]

Definition 4.2 The quaternion half space is defined by

\[ \text{H}(n, \mathbb{F}) = \{ X + \sqrt{-1}Y : X \in \text{Sym}(n, \mathbb{F}), Y \in \text{Pos}(n, \mathbb{F}) \} . \]

Remark 4.2 (1) We can define the half spaces using \( \mathbb{R} \) and \( \mathbb{C} \) instead of \( \mathbb{F} \). The half space \( \text{H}(n, \mathbb{R}) \) is a Siegel half space, and \( \text{H}(n, \mathbb{C}) \) is the bounded symmetric space of type \( I \).

(2) Two spaces \( \text{H}(2, \mathbb{F}) \) and \( \varphi(\text{H}(2, \mathbb{F})) \subset \text{H}(4, \mathbb{C}) \) are isomorphic as complex manifolds via the correspondence \( \varphi \).

Proposition 4.3 For an element \( Z \in \text{H} \) we have a decomposition \( Z = X + \sqrt{-1}Y \) with

\[ X = \frac{1}{2}(Z + {}^t \overline{Z}), Y = \frac{1}{2\sqrt{-1}}(Z - {}^t \overline{Z}) , \]

and so \( \text{H} \) is naturally embedded in the half space \( \text{H}(4, \mathbb{C}) \).

Remark 4.3 We can examine the equality \( \text{H} = \varphi(\text{H}(2, \mathbb{F})) \) by direct calculation.

Set

\[ \text{Sp}(2n, \mathbb{F}) = \{ g \in \text{GL}(2n, \mathbb{F}) : g^* J_{2n} g = J_{2n} \} \]

and we define \( \text{Sp}(2n, \mathbb{C}) \) by putting \( \mathbb{C} \) instead of \( \mathbb{F} \). The following is well-known:

Proposition 4.4 The group \( \text{Sp}(4, \mathbb{F}) \) is generated by

\[ J_4, \begin{pmatrix} W & O \\ O & W^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix} , \quad \text{where } W \in \text{GL}(2, \mathbb{F}) \text{ and } S^* = S. \]

So we obtain:

Proposition 4.5

\[ \text{Image}(\Phi) = \langle J_8, \begin{pmatrix} A & O \\ O & A^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix} \rangle , \]

with

\[ A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{pmatrix} , S^* = S. \]
Proposition 4.6 ([4] p.55)
The group \( \text{Sp}(4, \mathbb{F}) \cap \text{GL}(4, \mathcal{O}(\mathbb{F})) \) is generated by
\[
J_4, \begin{pmatrix} I & S \\ O & I \end{pmatrix}, \begin{pmatrix} U & O \\ O & (U^*)^{-1} \end{pmatrix}
\]
where \( S \in \text{Sym}(2, \mathcal{O}(\mathbb{F})), U \in \text{GL}(2, \mathcal{O}(\mathbb{F})) \).

Proposition 4.7 The group \( \text{Sp}(2n, \mathbb{F}) \) is a subgroup of \( \text{Aut}(H(n, \mathbb{F})) \) via the action
\[
Z \mapsto (AZ + B)(CD + D)^{-1}
\]
for an element
\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{F}).
\]

Proposition 4.8 ([4] p.50)
We have \( \text{Aut}(H(2, \mathbb{F})) = \text{Sp}(4, \mathbb{F}) \cdot \langle \iota \rangle \),
and we have \( \text{Aut}(H(n, \mathbb{F})) = \text{Sp}(2, \mathbb{F}) \) for \( n \geq 3 \). Where \( \iota \) indicates the transposition as an element of \( \text{M}(2, \mathbb{F}) \), and \( \cdot \) means the semi direct product.

Remark 4.4 The transposition \( \iota \) acts as an automorphism of \( H(n, \mathbb{F}) \) only for \( n = 2 \). If we have \( n \geq 3 \), it does not preserve the positivity condition \( -\sqrt{-1}(Z - Z^*) > 0 \).

4.2 Relation between \( G^+(\mathbb{Z}) \) and \( \Gamma(H) \)

Definition 4.3 Set
\[
H = \begin{pmatrix} O & iE_4 \\ -iE_4 & O \end{pmatrix}, \quad S = \begin{pmatrix} O & J_4 \\ -J_4 & O \end{pmatrix}, \quad L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}
\]
\( \text{SO}^*(8, \mathbb{C}) = \{ g \in \text{GL}(8, \mathbb{C}) : g^*Hg = H, \quad ^t g S g = S \} \).

And set
\[
\Gamma(H) = \text{SO}^*(8, \mathbb{Z}[i]) \cdot \langle \iota \rangle,
\]
where \( \iota \) indicates the involution
\[
Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & s & a & c \\ -t & 0 & b & d \end{pmatrix} \mapsto Z' = \begin{pmatrix} a & b & 0 & t \\ c & d & -t & 0 \\ 0 & s & a & c \\ -s & 0 & b & d \end{pmatrix}.
\]
We can easily examine that \( \text{SO}^*(8) \) is a subgroup of \( \text{Sp}(8, \mathbb{C}) \). We obtain the following by checking the conditions for \( \text{SO}^*(8) \).

Proposition 4.9 We have a injective homomorphism of \( \mathbb{R} \)-algebra \( \Phi : \text{Sp}(4, \mathbb{F}) \to \text{Sp}(8, \mathbb{C}) \) by putting
\[
\Phi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{pmatrix},
\]
and the image is contained in \( \text{SO}^*(8) \).
Remark 4.5  (1) Let $\mathcal{O}'$ denote the order $Ze_1 + Ze_2 + Ze_3 + Ze_4$ of $F$. Then we have

$$\text{SO}^*(8, \mathbb{Z}[i]) \cong \text{Sp}(4, \mathcal{O}')$$

via the mapping $\varphi$.

(2) We expect that the isomorphism $\delta$ induces injective isomorphisms

$$(\delta^{-1})^* : \Gamma(H) \to G^+$$

and

$$\delta^*(G^+(2)) \subset \Gamma(H).$$

But to get them, it is necessary to proceed more detailed argument on the discrete groups on $D^+$ and $H$. We don't have these results still now.

4.3 Embedding of $H$ into the Siegel upper space $S_8$

We use the following notation:

$$S_8 = \{\Omega \in GL(8, \mathbb{C}) : \quad ^t\Omega = \Omega, \Im(\Omega) > 0\},$$

$$K = \begin{pmatrix} O & J_4 \\ J_4 & O \end{pmatrix}, L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}$$

$$S_8(q) = \{\Omega \in S_8 : \Omega J_8 = J_8 \Omega, \Omega K = K \Omega, \Omega L = L \Omega\}.$$  

Note that $\{I_8, J_8, K, L\}$ make the basis of the Hamilton quaternionic field.

Proposition 4.10 The domain $H_{II}$ is embedded in $S_8$ by the mapping

$$\rho : Z \mapsto \frac{1}{2} \begin{pmatrix} Z + ^tZ & i(Z - ^tZ) \\ -i(Z - ^tZ) & Z + ^tZ \end{pmatrix}$$

It induces the biholomorphic equivalence between $H$ and $S_8(q)$.

The group $\text{SO}^*(8)$ has an injective embedding into $\text{Sp}(16, \mathbb{R})$ by the mapping $\lambda$:

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \Re(A) & -\Im(A) & \Re(B) & -\Im(B) \\ \Im(A) & \Re(A) & \Im(B) & \Re(B) \\ \Re(C) & -\Im(C) & \Re(D) & -\Im(D) \\ \Im(C) & \Re(C) & \Im(D) & \Re(D) \end{pmatrix}.$$  

For an element $g \in \text{SO}^*(8)$ we have $\rho^{-1} \circ g \circ \rho = \lambda(g)$.

Proposition 4.11 Put

$$J_{II} = \begin{pmatrix} O & 0 & J_4 & 0 \\ 0 & O & 0 & -J_4 \\ -J_4 & 0 & 0 & 0 \\ 0 & J_4 & 0 & 0 \end{pmatrix}, \quad \tilde{J}_{16} = J_8 \oplus J_8.$$  

Then it holds

$$^1\lambda(g)J_{II}\lambda(g) = J_{II}, \quad ^1\lambda(g)\tilde{J}_{16}\lambda(g) = \tilde{J}_{16}.$$
for every $g \in \text{SO}^*(8)$. If we put
\[ \text{Sp}(q) = \{ \gamma \in \text{Sp}(16, \mathbb{R}) : \gamma \mathbf{J}_I \gamma = \mathbf{J}_I, \quad \gamma \mathbf{J}_{16} = \mathbf{J}_{16} \gamma \} \]
the mapping $\lambda$ induces the isomorphism
\[ \text{SO}^*(8) \cong \text{Sp}(q). \]

Especially the mapping $\rho$ induces an isomorphism
\[ \text{SO}^*(8, \mathbb{Z}[i]) \cong \text{Sp}(q) \cap \text{M}(16, \mathbb{Z}). \]

Let $\Omega$ be a point on $\mathcal{S}_8$, and set $\Lambda_{\Omega} = \Lambda = \mathbb{Z}^8 + \mathbb{Z}^8 \Omega$. Let $V_{\Omega}$ denote the abelian variety $\mathbb{C}^8/\Lambda_{\Omega}$. So we regard $\mathcal{S}_8$ as the coarse moduli space of principally polarized abelian varieties $V_{\Omega}$. Note that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ belong to $\text{Sp}(16, \mathbb{Z})$. We can check that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ are contained in the algebra of endomorphisms of $\Lambda$ provided $\Omega \in \mathcal{S}_8(q)$. Let $\langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle$ be a $\mathbb{Q}$-algebra generated by $I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L$. Then we have
\[ F_{\mathbb{Q}} \cong \langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle \subset \text{End}(V_{\Omega}) \quad \text{for} \quad \Omega \in \mathcal{S}_8(q), \]
where $F_{\mathbb{Q}}$ indicates the Hamilton quaternion algebra over $\mathbb{Q}$.

**Proposition 4.12** The space $\mathcal{S}_8(q)$ is the coarse moduli space for the family of 8-dimensional abelian variety $V$ with the property
\[ F_{\mathbb{Q}} \cong \langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle \subset \text{End}(V_{\Omega}). \]
In this sense we can call $\mathcal{S}_8(q)$ the Shimura variety for the Hamilton quaternion endomorphism algebra $F_{\mathbb{Q}}$.

### 5 Corresponding Kuga-Satake varieties

We use the method developed in [12] and [10]. The detailed calculation and argument are exposed in [2] also.

Let us consider the lattice $T$ defined by the intersection matrix $A = U(2) \oplus U(2) \oplus (-2I_4)$ and $V_k = T \oplus k$ ($k = \mathbb{R}$ or $\mathbb{Q}$). Let $Q(x)$ denote the quadratic form on $T$ and at the same time on $V_k$. Let $\text{Tens}(T)$ and $\text{Tens}(V_k)$ be the corresponding tensor algebras. And we let $\text{Tens}^+(T)$ and $\text{Tens}^+(V_k)$ denote the subalgebras composed of the parts with even degree in $\text{Tens}(T)$ and $\text{Tens}(V_k)$, respectively. We consider the two sided ideal $I$ in $\text{Tens}^+(V_k)$ generated by elements $x \otimes x - Q(x)$ for $x \in V_k$, and the ideal $I_2$ in $\text{Tens}(T)$ is defined by the same manner. The corresponding even Clifford algebra is defined by
\[ C^+(V_k, Q) = \text{Tens}^+(V_k)/I. \]

By the same manner, we define
\[ C^+(T, Q) = \text{Tens}^+(T)/I_2. \]

We note that $C^+(V_{\mathbb{R}}, Q)$ is a 128 dimensional real vector space and $C^+(T, Q)$ is a lattice in it. So we obtain a real torus
\[ T_{\mathbb{R}} = C^+(V_{\mathbb{R}}, Q)/C^+(T, Q). \]

Let $\mathbb{F}$ denote the quaternion algebra
\[ \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij \]
with $i^2 = j^2 = -1$. By some routine calculations of the Clifford algebra we obtain the following.
Proposition 5.1 We have an isomorphism of algebras \( C^{+}(V, Q) \cong M(4, F) \oplus M(4, F) \).

Let a complex vector \( \eta = (\eta_1, \cdots, \eta_8) \) be a representative of a point \( \eta = [\eta_1, \cdots, \eta_8] \in D^+ \). So it has an ambiguity of the multiplication by a non-zero complex number. Put \( \eta = s + it \) \((s, t \in \mathbb{R}^8)\). If we impose the condition \((st)^2 = -1\) in \( C^{+}(V, Q) \), the representative is uniquely determined up to a multiplication by a complex unit. We denote it by

\[
\eta = m_1(\eta) + im_2(\eta).
\]

Put

\[
m(\eta) = m_1(\eta)m_2(\eta).
\]

It is uniquely determined by \( \eta \) without any ambiguity. According to the imposed condition, the element \( m(\eta) \in C^{+}(V, Q) \) defines a complex structure on \( C^{+}(V, Q) \) by the left action. It induces a complex structure on the real torus \( \mathcal{T}_R \) also.

Let \( \{e_1, \cdots, e_8\} \) be a basis of \( T \) with the intersection matrix \( U(2) \oplus U(2) \oplus (-2I_4) \), and let \( \{e_1, \cdots, e_8\} \) be an orthonormal basis of \( V \) given by

\[
(e_1, \cdots, e_8) = (e_1, \cdots, e) (\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}) \oplus (I_4)).
\]

Then the corresponding intersection matrix takes the form \( I_2 \oplus (-I_2) \oplus (-2I_4) \).

Let \( \iota \) be an involution on \( C^{+}(V, Q) \) induced from the transformation

\[
\iota : e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \mapsto e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}
\]

for the basis. Set \( \alpha = 4e_2e_1 \). According to the method in [St] we know that

\[
E(x, y) = tr(\alpha x^\iota y)
\]

determines a Riemann form. So the triple \((\mathcal{T}_R, m(\eta), E(x, y))\) determines an abelian variety. We denote it by \( A^+(\eta) \), that is so called the Kuga-Satake variety attached to the \( K3 \) surface corresponding to the period \( \eta \). In this way we can construct a family of abelian varieties

\[
A^+ = \{A^+(\eta) : \eta \in D^+ \}
\]

induced from the lattice \( T \) parameterized by the domain \( D^+ \). We can construct the "conjugate family"

\[
A^- = \{A^-(\eta) : \eta \in D^- \}
\]

parameterized by

\[
D^- = \{\eta = [\eta_1, \cdots, \eta_8] : ^t\eta A\eta = 0, ^t\eta A\eta > 0, \exists (\eta_3/\eta_1) < 0\}
\]

by the same procedure with the Riemann form \( E^-(x, y) = -tr(\alpha x^y) \). The right action of \( C^+(V, Q) \) on \( C^+(V, Q) \) commutes with the left action of \( m(\eta) \). So we have

\[
C^+(T, Q) \subset \text{End}(A^+(\eta)) \otimes Q
\]

for any \( A^+(\eta) \). For a general member \( \eta \in D^+ \), the endomorphism ring is given by

\[
\text{End}_Q(A(\eta)) = \text{End}(A(\eta)) \otimes Q \cong C^+(V, Q).
\]

According to Proposition 6.1 we obtain:
Theorem 5 For a general member $\eta \in D^+$, $A^+(\eta)$ is isogenous to a product of abelian varieties $(A_1(\eta) \times A_2(\eta))^4$ where $A_1(\eta)$ and $A_2(\eta)$ are 8-dimensional simple abelian varieties with $\operatorname{End}_Q(A_i(\eta)) = F_Q$ ($i = 1, 2$).

Remark 5.1 Here we describe the relation between $A_1(\eta)$ and $A_2(\eta)$. Now we define the linear involution $\ast$ on $V_R$ by $e_1^* = -e_1$ and $e_i^* = e_i$ ($i = 2, \ldots, 8$).

It can be extended on $C^+(V_R, Q)$ as an automorphism of algebra. We define an involution $\sigma$ on $D$:

$$\sigma : D \rightarrow D, \quad (\eta_\infty, \eta_1, \ldots, \eta_n) \mapsto (-\eta_1, -\eta_\infty, \eta_2, \ldots, \eta_n).$$

So we have $D^+_\sigma = D_-$. It is easy to check that we have $A_2(\eta) \sim A_1(\eta^\sigma)$, $A_1(\eta) \sim A_2(\eta^\sigma)$, where $\sim$ indicates the isogenous relation.

E. Freitag and C. F. Hermann [1] study a similar family of lattice $K3$ surfaces from a different viewpoint. We think that it should be clarified the exact relation between their family and our $\mathcal{F}$.

References


[2] K. Koike, The Kuga-Satake variety attached to the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along four curves of bidegree $(1, 1)$, preprint.


