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The family of $K3$ surfaces with a transcendental lattice $U(2)^2 \times \langle -2\rangle^4$ for a general member

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0 Introduction

Let us consider the lattice $P = D_4^3 \oplus \langle -2\rangle \oplus \langle 2\rangle$ of rank 14. A $K3$ surface S is called of type P when it satisfies $P \subset \mathrm{Pic}(S)$, where $\mathrm{Pic}(S)$ stands for the Picard lattice of S. In this article we show the outline of the study on the period map for the family $\mathcal{F}$ of $K3$ surfaces of type P.

We work always on the ground field $\mathbf{C}$. Note that the lattice $H^2(S,\mathbf{Z})$ is always isomorphic to $L = U^3 \oplus (-E_8)^2$ and $P^\perp \subset L$ is isomorphic to $U(2)^2 \oplus \langle -2\rangle^4$.

In 1992 K. Matsumoto, T. Sasaki and M. Yoshida [7] studied the period mapping for a family of $K3$ surfaces of type $(3,6)$, that is the family of double covering surfaces over $\mathbf{P}^2$ branching along six lines in general position, and Matsumoto [6] gave the description of the inverse mapping in terms of theta constants. It gives the modular map for the 4 dimensional Shimura variety in the Siegel upper half space $\mathcal{S}_4$ derived from the family of 4 dimensional abelian varieties with generalized complex multiplication by $\sqrt{-1}$ of type $(2,2)$. So we call it MSY modular mapping. The second author showed an arithmetic application of MSY modular mapping in [11].

In the case of MSY modular map the corresponding $K3$ surface is characterized by the lattice $U(2)^2 \oplus \langle -2\rangle^2$ and the moduli space is a 4-dimensional type IV domain. We suspect such fruitfull results of the MSY map is the consequence of the eventual coincidence of two different bounded symmetric domains D_{IV}^4 and $I_{2,2}$. There are a few (finite) such exceptional coincidences. The highest one is the (analytic) equivalence between D_{IV}^6 and $\mathbf{H}_{II}(4)$ (in terms of Lie algebra $so(2,6;\mathbf{R}) \cong so(4,\mathbf{F})$, where $\mathbf{F}$ indicates the Hamilton quaternion field) and it contains the above coincidence of MSY case. That is our situation.

The ring structure of the regular functions on the parameter space for $\mathcal{F}$ is given by the article of Koike and Ochiai in this volume, their result is the cosequence of the discussion motivated by the RIMS workshop.

1 Realization as an algebraic surface

We fix an abstract lattice L and its P-part $D_4^3 \oplus \langle -2\rangle \oplus \langle 2\rangle$. A $K3$ surface S is called of exact type P when it satisfies $P \cong \mathrm{Pic}(S)$ via an isomorphism $\varphi : H_2(S,\mathbf{Z}) \to L$.That is a general member of $\mathcal{F}$. A surface S of exact type P is realized as a double covering surface over $\mathbf{P}^1 \times \mathbf{P}^1$ branching along four bidegree $(1,1)$ curves H_1, H_2, H_3, H_4 satisfying

$J_1)$ H_k $\quad (k = 1,2,3,4)$ is irreducible,

J_2) $H_k \cap H_\ell$ consists of two different points,

J_3) for any different three indices i, j, k we have $H_i \cap H_j \cap H_k = \emptyset$

under some nef condition stated in Theorem 1. Such a surface is given as the complete nonsingular model of the affine variety

$$S = S(x) : w^2 = \prod_{k=1}^{4} (x_1^{(k)} st + x_2^{(k)} s + x_3^{(k)} t + x_4^{(k)}),$$

where we use the notation

$$x_k = \begin{pmatrix} x_1^{(k)} & x_2^{(k)} \\ x_3^{(k)} & x_4^{(k)} \end{pmatrix} \in \mathrm{M}(2, \mathbf{C}).$$

So the curve $H_k \quad (k = 1, 2, 3, 4)$ is given by

$$(s, 1) \begin{pmatrix} x_1^{(k)} & x_2^{(k)} \\ x_3^{(k)} & x_4^{(k)} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = 0.$$

By considering the projection to the s-line we find that a general member S of $\mathcal{F}$ is an elliptic fibred surface with 12 singular fibres of type I_2 corresponding to the intersection points $H_i \cap H_j \quad (i \neq j)$. Counting the Euler number we know that $S(x)$ is a $K3$ surface. Let $E_{ij}^{\pm}$ be the exceptional curves obtained by the blow up processes at the intersections $H_i \cap H_j \quad (i \neq j, 1 \leq i, j \leq 4)$. Let π be the projection $S \to \mathbf{P}^1 \times \mathbf{P}^1$ and set

$$G_i = \frac{1}{2}(\pi^* H_i - \sum_{i \neq j}(E_{ij}^+ + E_{ij}^-) \quad (i = 1, 2, 3, 4)$$
$$F_s = \pi^*\{t = 0\}, F_t = \pi^*\{s = 0\}.$$

Lemma 1.1 *The sublattice in $H_2(S(x), \mathbf{Z})$ generated by $E_{ij}^+, E_{ij}^-, G_i, F_s, F_t$ is isomorphic to P. So $S(x)$ is a $K3$ surface of type P.*

According to the above Lemma we can define the elements $e_{ij}^{\pm}, g_i, f_s, f_t$ in the abstract lattice L corresponding to the divisors with capital letters in $H^2(S(x), \mathbf{Z})$ via an isomorphism $\varphi : H_2(S(x), \mathbf{Z}) \to L$.

Theorem 1 *Let S be a $K3$ surface of type P with an isomorphism φ, and suppose that $\varphi^{-1}(f_s)$ and $\varphi^{-1}(f_t)$ are nef divisors . Then S is realized as the form of $S(x)$.*

2 Period map

2.1 Marking of a $K3$ surface of type P

Definition 2.1 *Let S be a $K3$ surface of type P with an embedding $P \to L$. We call the triple (S, φ, P) a P marking of S if we have*

(1) $\varphi : H_2(S, \mathbf{Z}) \to L$ is an isometry of lattices,

(2) $\varphi^{-1}(f_s), \varphi^{-1}(f_t), \varphi^{-1}(e_{ij}^{\pm}), \varphi^{-1}(g_i)$ are effective divisors and $\varphi^{-1}(f_s)$ and $\varphi^{-1}(f_t)$ are nef.

Let $O(L)$ be the group of isometries of the lattice L, and set $O(L, P) = \{g \in O(L) : g(x) = x \text{ for } \forall x \in P\}$. Putting $T = P^{\perp}$ we set $G = O(T)$ and $G(2) = \{g \in G : g \equiv I \pmod 2\}$, where we defined $O(T)$ as same as $O(L)$.

Proposition 2.1 *We have $G(2) \cong O(L, P)$.*

Definition 2.2 *Let (S, φ, P) and (S', φ', P) be P markings of S and S', respectively. An isomorphism $\rho : S \to S'$ is called an isomorphism of these 2 markings when we have $\varphi = \varphi' \circ \rho_*$. We say 2 markings (S, φ, P) and (S', φ', P) are equivalent when we have a $\gamma \in O(L, P)$ such that (S', φ', P) is isomorphic to $(S, \gamma \circ \varphi, P)$.*

Remark 2.1 *Equivalent markings correspond to the same configuration of the branch locus with the order of $H_1, \ldots, H_4$ of the double covering surfaces. And this relation corresponds to the base change of the $K3$ lattice which preserves the polarization P.*

2.2 Parameter space

From the above argument we can take the cofiguration space for the family of 4 bidegree $(1,1)$ curves as a parameters space of our family of P marked $K3$ surfaces. The presentation of H_k has an ambiguity of a constant factor. By considering the projective coordinate transformation of s-spave and t-space our parameter space is given by

$$X = \left(PGL(2, \mathbf{C}) \backslash \{(x_1, \ldots x_4)\} = (GL(2, \mathbf{C}))^4 / PGL(2, \mathbf{C}) \right) / (\mathbf{C}^*)^4.$$

For further investigation we need a realization of X as a projective variety. That is suggested in the article of Koike and Ochiai on this same volume.

2.3 Period domain

Let $\Gamma_1, \ldots, \Gamma_8$ be a fixed basis of T such that we have the intersection form

$$(\Gamma_i \cdot \Gamma_j) = A := U(2) \oplus (-2I_4).$$

For a P marking (S, φ, P) let Ω be the holomorphic 2-form on S that is unique up to a constant factor. We define the period of (S, φ, P) by

$$\eta = [\int_{\varphi^{-1}(\Gamma_1)} \Omega, \ldots, \int_{\varphi^{-1}(\Gamma_8)} \Omega] \in \mathbf{P}^7.$$

The image of the period mapping for the family of P marked $K3$ surfaces is open dense in the 6-dimensional domain given by

$$D^+ = \{\eta = [\eta_1, \cdots, \eta_8] \in \mathbf{P}^7 : {}^t\eta A \eta = 0, {}^t\overline{\eta} A \eta > 0, \Im(\eta_3/\eta_1) > 0\}.$$

We get this fact by using the Riemann-Hodge relation of the period and the Torelli theorem for $K3$ surfaces. It is a bounded symmetric domain of type IV. We set

$$G^+ = \{g \in G : \ g(D^+) = D^+\}, \ G(2)^+ = \{g \in G(2) : \ g(D^+) = D^+\}.$$

We can determine the modular group for the equivalence classes of the P marked surfaces. Namely

Theorem 2 *Let (S, φ, P) and (S', φ', P) be P markings of $K3$ surfaces of type P. Let η and η' be the corresponding periods, respectively. Then these two markings are equivalent if and only if*

$$g(\eta) = \eta'$$

for some $g \in G(2)^+$

Theorem 3 *The modular group $G(2)^+$ is a reflection group.*

Here a transformation

$$R_v : \lambda \mapsto v - 2({}^t vA\lambda/\,{}^t vAv)v, \quad \lambda \in D^+, v \in \mathbf{Z}^8$$

is called a reflection with the root vector v.

2.4 Degenerate locus

3 Differential equation

We can determine the system of differential equations for the period with 16 variables x_{ij}^k. It becomes a holonomic system of rank 8. So our periods $\int_{\varphi^{-1}(\Gamma_i)} \Omega \quad (i = 1, \ldots, 8)$ make a basis of the space of solutions for this system defined on a domain $X' = X - V$, where V is the degenerating locus corresponding to the set of $K3$ surfaces of type P which violate some condition of J_1, J_2, J_3.

So we can consider the monodromy group $\mathcal{M}$ for this system.

Proposition 3.1 *We have $G(2)^+ \subset \mathcal{M}$.*

Remark 3.1 *We have possibly $G(2)^+ = \mathcal{M}$. But we cannot decide it at present, because Some monodromy transformation may cause an interchange of $E_{ij}^{\pm}$.*

Remark 3.2 *The image of the degenerating locus V by the period map is consists of 4, 6, 16 hyperplanes (so in total 26 hyperplanes) in the period domain D^+ which correspond to the violation of the condition J_1, J_2, J_3, respectively.*

4 Transfer of the period domain

The type II domain $\mathbf{H}$ is defined by

$$\mathbf{H} = \mathbf{H}_{II} = \{Z \in M(4, \mathbf{C}) : J_4 Z =^t Z J_4, \quad \frac{1}{i}(Z - Z^*) > 0\},$$

where we use the notation

$$J_{2n} = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix}$$

and the member $Z \in \mathbf{H}$ is described in the form

$$Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & t & a & c \\ -t & 0 & b & d \end{pmatrix}.$$

We define the mapping $\psi : \mathbf{H} \to \mathbf{P}^7$ by

$$\zeta =^t [z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8] =^t [1, -ad + bc - st, a, d, b, c, -s, t].$$

As a direct translation of the above lemma, we obtain :

Proposition 4.1 *The image of ψ is determined in $\mathbf{P}^7$ by the following three conditions:*

(1)

$$^t\zeta(U \oplus U \oplus U \oplus U)\zeta = 0.$$

(2)

$$\zeta^*(U \oplus U \oplus U \oplus U)\zeta > 0,$$

where $\zeta^* = {}^t\overline{\zeta}$.

(3)

$$\Im(\frac{z_3}{z_1}) > 0.$$

By straight forward calculation we have the following.

Theorem 4 *The image $\psi(\mathbf{H})$ is transformed to the type IV domain D_{IV}^+ by the map :*

$$\eta = P\zeta$$

with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{1-i} & \frac{1}{1-i} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{1+i} & \frac{-1}{1+i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-i}{1-i} & \frac{1}{1-i} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-i}{1+i} & \frac{-1}{1+i} \end{pmatrix}. \tag{4.1}$$

So the composite mapping $\delta = P \circ \psi$ gives the isomorphism

$$\delta : \mathbf{H} \cong D_{IV}.$$

Remark 4.1 *The analytic equivalence of the domains D_{IV}^+ and $\mathbf{H}$ is well known. But we are wishing to find the transfer which preserves the modular groups each other.*

4.1 the Quaternion half space

Let $\mathbf{F}$ be the Hamilton quaternion $\mathbf{R}$−algebra generated by $\{e_1, e_2, e_3, e_4\}$ with

$$e_1 = 1, \quad e_2 e_3 = e_4, \quad e_i^2 = -1.$$

Then the ring of integers in $\mathbf{F}$ is given by

$$\mathcal{O}(\mathbf{F}) = \mathbf{Z}e_0 + \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3,$$

where $e_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$.

Proposition 4.2 *The mapping*

$$\varphi : \mathrm{M(n, \mathbf{F})} \to \mathrm{M(2n, \mathbf{C})}$$

defined by

$$\varphi(\sum_i A_i e_i) = \begin{pmatrix} A_1 e_1 + A_2 e_2 & A_3 e_1 + A_4 e_2 \\ -A_3 e_1 + A_4 e_2 & A_1 e_1 - A_2 e_2 \end{pmatrix}$$

is an injective homomorphism of **R***-algebra.*

Definition 4.1 *We set*

$$\mathrm{Sym}(2,\mathbf{F}) = \{X \in M(2,\mathbf{F}) :{}^t\,\overline{X} = X\},$$

$$\mathrm{Pos}(2,\mathbf{F}) = \{X \in \mathrm{Sym}(2,\mathbf{F}) : X > 0\}.$$

Note that we have

$$X \in \mathrm{Sym}(2,\mathbf{F}) \Longleftrightarrow \varphi(X) \in \mathrm{Sym}(4,\mathbf{C}),$$
$$X \in \mathrm{Pos}(2,\mathbf{F}) \Longleftrightarrow \varphi(X) \in \mathrm{Pos}(4,\mathbf{C})$$

and it holds also

$$X \in \mathrm{Pos}(2,\mathbf{F}) \Longleftrightarrow X = g^*g, \quad \exists g \in \mathrm{GL}(2,\mathbf{F}).$$

Definition 4.2 *The quaternion half space is defined by*

$$\mathrm{H}(n,\mathbf{F}) = \{X + \sqrt{-1}Y : X \in \mathrm{Sym}(n,\mathbf{F}), Y \in \mathrm{Pos}(n,\mathbf{F})\}.$$

Remark 4.2 *(1) We can define the half spaces using* $\mathbf{R}$ *and* $\mathbf{C}$ *instead of* $\mathbf{F}$. *The half space* $\mathrm{H}(n,\mathbf{R})$ *is a Siegel half space, and* $\mathrm{H}(n,\mathbf{C})$ *is the bounded symmetric space of type I.*

(2) Two spaces $H(2,\mathbf{F})$ *and* $\varphi(H(2,\mathbf{F})) \subset H(4,\mathbf{C})$ *are isomorphic as complex manifolds via the correspondence* φ.

Proposition 4.3 *For an element* $Z \in \mathbf{H}$ *we have a decomposition* $Z = X + \sqrt{-1}Y$ *with*

$$X = \frac{1}{2}(Z +{}^t\,\overline{Z}), Y = \frac{1}{2\sqrt{-1}}(Z -{}^t\,\overline{Z}),$$

and so $\mathbf{H}$ *is naturally embedded in the half space* $H(4,\mathbf{C})$.

Remark 4.3 *We can examine the equality* $\mathbf{H} = \varphi(H(2,\mathbf{F}))$ *by direct calculation.*

Set

$$\mathrm{Sp}(2n,\mathbf{F}) = \{g \in \mathrm{GL}(2\mathrm{n},\mathbf{F}) : \mathrm{g}^*\mathrm{J}_{2\mathrm{n}}\mathrm{g} = \mathrm{J}_{2\mathrm{n}}\}$$

and we define $\mathrm{Sp}(2n,\mathbf{C})$ by putting $\mathbf{C}$ instead of $\mathbf{F}$. The following is wellknown:

Proposition 4.4 *The group* $\mathrm{Sp}(4,\mathbf{F})$ *is generated by*

$$J_4, \begin{pmatrix} {}^t\overline{W} & O \\ O & W^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix} \qquad \text{, where } W \in \mathrm{GL}(2,\mathbf{F}) \text{ and } S^* = S.$$

So we obtain :

Proposition 4.5

$$\mathrm{Image}(\Phi) =< J_8, \begin{pmatrix} A & O \\ O & {}^t\overline{A}^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix} >$$

with

$$A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{pmatrix}, S^* = S.$$

Proposition 4.6 *([4] p.55)*
The group $\mathrm{Sp}(4,\mathbf{F}) \cap \mathrm{GL}(4,\mathcal{O}(\mathbf{F}))$ *is generated by*

$$J_4, \begin{pmatrix} I & S \\ O & I \end{pmatrix}, \begin{pmatrix} U & O \\ O & (U^*)^{-1} \end{pmatrix}$$

where $S \in \mathrm{Sym}(2,\mathcal{O}(\mathbf{F})), U \in \mathrm{GL}(2,\mathcal{O}(\mathbf{F}))$.

Proposition 4.7 *The group* $\mathrm{Sp}(2n,\mathbf{F})$ *is a subgroup of* $\mathrm{Aut}(\mathrm{H}(\mathrm{n},\mathbf{F}))$ *via the action*

$$Z \mapsto (AZ+B)(CD+D)^{-1}$$

for an element

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbf{F}).$$

Proposition 4.8 *([4] p.50)*
We have

$$\mathrm{Aut}(H(2,\mathbf{F})) = \mathrm{Sp}(4,\mathbf{F}) \cdot < \Pi >,$$

and we have

$$\mathrm{Aut}(H(n,\mathbf{F})) = \mathrm{Sp}(2,\mathbf{F})$$

for $n \geq 3$*. Where* Π *indicates the transposition as an element of* $\mathrm{M}(2,\mathbf{F})$*, and* $\cdot$ *means the semi direct product.*

Remark 4.4 *The transposition* Π *acts as an automorphism of* $H(n,\mathbf{F})$ *only for* $n=2$*. If we have* $n \geq 3$*, it does not preserve the positivity condition* $-\sqrt{-1}(Z-Z^*) > 0$.

4.2 Relation between $G^+(\mathbf{Z})$ and $\Gamma(\mathbf{H})$

Definition 4.3 *Set*

$$H = \begin{pmatrix} O & iE_4 \\ -iE_4 & O \end{pmatrix}, \quad S = \begin{pmatrix} O & J_4 \\ -J_4 & O \end{pmatrix}, \quad L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}$$

$$\mathrm{SO}^*(8,\mathbf{C}) = \{\mathrm{g} \in \mathrm{GL}(8,\mathbf{C}) : \mathrm{g}^*\mathrm{Hg} = \mathrm{H}, \quad {}^t\mathrm{gSg} = \mathrm{S}\}.$$

And set

$$\Gamma(\mathbf{H}) = \mathrm{SO}^*(8,\mathbf{Z}[\mathrm{i}]) \cdot \langle \iota \rangle,$$

where ι *indicates the involution*

$$Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & t & a & c \\ -t & 0 & b & d \end{pmatrix} \mapsto Z' = \begin{pmatrix} a & b & 0 & t \\ c & d & -t & 0 \\ 0 & s & a & c \\ -s & 0 & b & d \end{pmatrix}.$$

We can easily examine that $\mathrm{SO}^*(8)$ is a subgroup of $\mathrm{Sp}(8,\mathbf{C})$. We obtain the following by checking the conditions for $\mathrm{SO}^*(8)$.

Proposition 4.9 *We have a injective homomorphism of* $\mathbf{R}$*-algebra* $\Phi : \mathrm{Sp}(4,\mathbf{F}) \to \mathrm{Sp}(8,\mathbf{C})$ *by putting*

$$\Phi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{pmatrix},$$

and the image is contained in $\mathrm{SO}^*(8)$.

Remark 4.5 *(1) Let $\mathcal{O}'$ denote the order $\mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$ of $\mathbf{F}$. Then we have*

$$\mathrm{SO}^*(8, \mathbf{Z}[\mathrm{i}]) \cong \mathrm{Sp}(4, \mathcal{O}')$$

via the mapping φ.

(2) We expect that the isomorphism δ induces injective isomorphisms

$$(\delta^{-1})^* : \Gamma(\mathbf{H}) \to G^+$$

and

$$\delta^*(G^+(2)) \subset \Gamma(\mathbf{H}).$$

But to get them, it is necessary to proceed more detailed argument on the discrete groups on D^+ and $\mathbf{H}$. We don't have these results still now.

4.3 Embedding of H into the Siegel upper space $\mathcal{S}_8$

We use the following notation:

$$\begin{aligned} &\mathcal{S}_8 = \{\Omega \in GL(8, \mathbf{C}) : {}^t\Omega = \Omega, \Im(\Omega) > 0\}, \\ &K = \begin{pmatrix} O & J_4 \\ J_4 & O \end{pmatrix}, L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix} \\ &\mathcal{S}_8(q) = \{\Omega \in \mathcal{S}_8 : \Omega J_8 = J_8\Omega, \Omega K = K\Omega, \Omega L = L\Omega\}. \end{aligned}$$

Note that $\{I_8, J_8, K, L\}$ make the basis of the Hamilton quaternionic field.

Proposition 4.10 *The domain $\mathbf{H}_{II}$ is embedded in $\mathcal{S}_8$ by the mapping*

$$\rho : Z \mapsto \frac{1}{2}\begin{pmatrix} Z +^t Z & i(Z -^t Z) \\ -i(Z -^t Z) & Z +^t Z \end{pmatrix}$$

It induces the biholomorphic equivalence between $\mathbf{H}$ and $\mathcal{S}_8(q)$.

The group $\mathrm{SO}^*(8)$ has an injective embedding into $\mathrm{Sp}(16, \mathbf{R})$ by the mapping λ:

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \Re(A) & -\Im(A) & \Re(B) & -\Im(B) \\ \Im(A) & \Re(A) & \Im(B) & \Re(B) \\ \Re(C) & -\Im(C) & \Re(D) & -\Im(D) \\ \Im(C) & \Re(C) & \Im(D) & \Re(D) \end{pmatrix}.$$

For an element $g \in \mathrm{SO}^*(8)$ we have $\rho^{-1} \circ g \circ \rho = \lambda(g)$.

Proposition 4.11 *Put*

$$J_{II} = \begin{pmatrix} O & O & J_4 & O \\ O & O & O & -J_4 \\ -J_4 & O & O & O \\ O & J_4 & O & O \end{pmatrix}, \quad \tilde{J}_{16} = J_8 \oplus J_8.$$

Then it holds

$${}^t\lambda(g) J_{II} \lambda(g) = J_{II}, \quad {}^t\lambda(g) \tilde{J}_{16} \lambda(g) = \tilde{J}_{16}$$

for every $g \in \mathrm{SO}^*(8)$. *If we put*

$$\mathrm{Sp(q)} = \{\gamma \in \mathrm{Sp}(16, \mathbf{R}) :^t \gamma \mathrm{J_{II}} \gamma = \mathrm{J_{II}}, \quad \gamma \tilde{\mathrm{J}}_{16} = \tilde{\mathrm{J}}_{16} \gamma\}$$

the mapping λ *induces the isomorphism*

$$\mathrm{SO}^*(8) \cong \mathrm{Sp(q)}.$$

Especially the mapping ρ *induces an isomorphism*

$$\mathrm{SO}^*(8, \mathbf{Z}[\mathrm{i}]) \cong \mathrm{Sp(q)} \cap \mathrm{M}(16, \mathbf{Z}).$$

Let Ω be a point on $\mathcal{S}_8$, and set $\Lambda_\Omega = \Lambda = \mathbf{Z}^8 + \mathbf{Z}^8\Omega$. Let V_Ω denote the abelian variety $\mathbf{C}^8/\Lambda_\Omega$. So we regard $\mathcal{S}_8$ as the coarse moduli space of principally polarized abelian varieties V_Ω. Note that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ belong to $\mathrm{Sp}(16, \mathbf{Z})$. We can check that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ are contained in the algebra of endomorphisms of Λ provided $\Omega \in \mathcal{S}_8(q)$. Let $\langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L\rangle$ be a Q-algebra generated by $I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L$. Then we have

$$\mathbf{F_Q} \cong \langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L\rangle \subset \mathrm{End}(\mathrm{V}_\Omega) \quad \text{for} \quad \Omega \in \mathcal{S}_8(\mathrm{q}),$$

where $\mathbf{F_Q}$ indicates the Hamilton quaternion algebra over $\mathbf{Q}$.

Proposition 4.12 *The space* $\mathcal{S}_8(q)$ *is the coarse moduli space for the family of 8-dimensional abelian variety* V *with the property*

$$\mathbf{F_Q} \cong < I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L > \subset \mathrm{End}(\mathrm{V}_\Omega).$$

In this sense we can call $\mathcal{S}_8(q)$ *the Shimura variety for the Hamilton quaternion endomorphism algebra* $\mathbf{F_Q}$.

5 Corresponding Kuga-Satake varieties

We use the method developped in [12] and [10]. The detailed calculation and argument are exposed in [2] also.

Let us consider the lattice T defined by the intersection matrix $A = U(2) \oplus U(2) \oplus (-2I_4)$ and $V_k = T \otimes k$ ($k = \mathbf{R}$ or $\mathbf{Q}$). Let $Q(x)$ denote the quadratic form on T and at the same time on V_k. Let $Tens(T)$ and $Tens(V_k)$ be the corresponding tensor algebras. And we let $Tens^+(T)$ and $Tens^+(V_k)$ denote the subalgebras composed of the parts with even degree in $Tens(T)$ and $Tens(V_k)$, respectively. We consider the two sided ideal I in $Tens^+(V_k)$ generated by elements $x \otimes x - Q(x)$ for $x \in V_k$, and the ideal $I_\mathbf{Z}$ in $Tens(T)$ is defined by the same manner. The corresponding even Clifford algebra is defined by

$$C^+(V_k, Q) = Tens^+(V_k)/I.$$

By the same manner, we define

$$C^+(T, Q) = Tens^+(T)/I_\mathbf{Z}.$$

We note that $C^+(V_\mathbf{R}, Q)$ is a 128 dimensional real vector space and $C^+(T, Q)$ is a lattice in it. So we obtain a real torus

$$\mathcal{T}_\mathbf{R} = C^+(V_\mathbf{R}, Q)/C^+(T, Q).$$

Let $\mathbf{F}$ denote the quaternion algebra

$$\mathbf{Q} \oplus \mathbf{Q}i \oplus \mathbf{Q}j \oplus \mathbf{Q}ij$$

with $i^2 = j^2 = -1$. By some routine calculations of the Clifford algebra we obtain the following.

Proposition 5.1 *We have an isomorphism of algebras* $C^+(V_{\mathbf{Q}}, Q) \cong \mathrm{M}(4, \mathbf{F}) \oplus \mathrm{M}(4, \mathbf{F})$.

Let a complex vector $\underline{\eta} = (\eta_1, \cdots, \eta_8)$ be a representative of a point $\eta = [\eta_1, \cdots, \eta_8] \in D^+$. So it has an ambiguity of the multiplication by a non zero complex number. Put $\underline{\eta} = s+it$ $(s, t \in \mathbf{R}^8)$. If we impose the condition $(st)^2 = -1$ in $C^+(V_{\mathbf{R}}, Q)$, the representative is uniquely determined up to a multiplication by a complex unit. We denote it by

$$\underline{\eta} = m_1(\eta) + i m_2(\eta).$$

Put

$$m(\eta) = m_1(\eta) m_2(\eta).$$

It is uniquely determined by η without any ambiguity. According to the imposed condition, the element $m(\eta) \in C^+(V_{\mathbf{R}}, Q)$ defines a complex structure on $C^+(V_{\mathbf{R}}, Q)$ by the left action. It induces a complex structure on the real torus $\mathcal{T}_{\mathbf{R}}$ also.

Let $\{\varepsilon_1, \cdots, \varepsilon_8\}$ be a basis of T with the intersection matrix $U(2) \oplus U(2) \oplus (-2I_4)$, and let $\{e_1, \cdots, e_8\}$ be a orthonormal basis of V given by

$$(e_1, \cdots, e_8) = (\varepsilon_1, \cdots, \varepsilon)(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \oplus (I_4)).$$

Then the corresponding intersection matrix takes the form $I_2 \oplus (-I_2) \oplus (-2I_4)$.

Let ι be an involution on $C^+(V, Q)$ induced from the transformation

$$\iota : e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \mapsto e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}$$

for the basis. Set $\alpha = 4e_2e_1$. According to the method in [St] we know that

$$E(x, y) = tr(\alpha x^{\iota} y)$$

determines a Riemann form. So the triple $(\mathcal{T}_{\mathbf{R}}, m(\eta), E(x, y))$ determines an abelian variety. We denote it by $A^+(\eta)$, that is so called the Kuga-Satake variety attached to the $K3$ surface corresponding to the period η. In this way we can construct a family of abelian varieties

$$\mathcal{A}^+ = \{A^+(\eta) \ : \ \eta \in D^+\}$$

induced from the lattice T parameterized by the domain D^+. We can construct the "conjugate family"

$$\mathcal{A}^- = \{A^-(\eta) \ : \ \eta \in D^-\}$$

parameterized by

$$D^- = \{\eta = [\eta_1, \cdots, \eta_8] : {}^t\eta A \eta = 0, \ {}^t\overline{\eta} A \eta > 0, \ \Im(\eta_3/\eta_1) < 0\}$$

by the same procedure with the Riemann form $E^-(x, y) = -\mathrm{tr}(\alpha x^{\iota} y)$. The right action of $C^+(V_{\mathbf{Q}}, Q)$ on $C^+(V_{\mathbf{R}}, Q)$ commutes with the left action of $m(\eta)$. So we have

$$C^+(T_{\mathbf{Q}}) \subset \mathrm{End}(\mathrm{A}^{\pm}(\eta)) \otimes \mathbf{Q}$$

for any $A^{\pm}(\eta)$. For a general member $\eta \in \mathcal{D}^+$, the endomorphism ring is given by

$$\mathrm{End}_{\mathbf{Q}}(\mathrm{A}(\eta)) = \mathrm{End}(\mathrm{A}(\eta)) \otimes \mathbf{Q} \cong \mathrm{C}^+(\mathrm{V}_{\mathbf{Q}}).$$

According to Proposition 6.1 we obtain :

Theorem 5 *For a general member $\eta \in \mathcal{D}^+$, $A^+(\eta)$ is isogenous to a product of abelian varieties $(A_1(\eta) \times A_2(\eta))^4$ where $A_1(\eta)$ and $A_2(\eta)$ are 8-dimensional simple abelian varieties with* $\mathrm{End}_{\mathbf{Q}}(\mathrm{A_i}(\eta)) = \mathbf{F_Q} \quad (\mathrm{i} = 1, 2)$.

Remark 5.1 *Here we describe the relation between $A_1(\eta)$ and $A_2(\eta)$. Now we define the linear involution $*$ on $V_{\mathbf{R}}$ by*

$$e_1^* = -e_1 \quad \text{and} \quad e_i^* = e_i \quad (i = 2, \cdots, 8).$$

It can be extended on $C^+(V_{\mathbf{R}}, Q)$ as an automorphism of algebra. We define an involution σ on $\mathcal{D}$:

$$\sigma : \mathcal{D} \longrightarrow \mathcal{D}, \quad (\eta_{\infty}, \cdots, \eta_{\forall}) \mapsto (-\eta_{\in}, -\eta_{\infty}, \eta_{\ni}, \cdots \eta_{\forall}).$$

So we have $\mathcal{D}_+^\sigma = \mathcal{D}_-$. It is easy to check that we have

$$A_2(\eta) \sim A_1(\eta^\sigma), \quad A_1(\eta) \sim A_2(\eta^\sigma),$$

where $\sim$ indicates the isogenous relation.

E. Freitag and C. F. Hermann [1] study a similar family of lattice $K3$ surfaces from a different view point. We think that it should be clarified the exact relation between their family and our $\mathcal{F}$.

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