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Kyoto University
The family of $K3$ surfaces with a transcendental lattice $U(2)^2 \times \langle -2 \rangle^4$ for a general member

0 Introduction

Let us consider the lattice $P = D_4^3 \oplus (-2) \oplus (2) \oplus (2)$ of rank 14. A $K3$ surface $S$ is called of type $P$ when it satisfies $P \subset \text{Pic}(S)$, where $\text{Pic}(S)$ stands for the Picard lattice of $S$. In this article we show the outline of the study on the period map for the family $\mathcal{F}$ of $K3$ surfaces of type $P$.

We work always on the ground field $\mathbb{C}$. Note that the lattice $H^2(S, \mathbb{Z})$ is always isomorphic to $L = U^3 \oplus (-E_8)^2$ and $P \subset L$ is isomorphic to $U(2)^2 \oplus (-2)^4$.

In 1992 K. Matsumoto, T. Sasaki and M. Yoshida [7] studied the period mapping for a family of $K3$ surfaces of type $(3,6)$, that is the family of double covering surfaces over $\mathbb{P}^2$ branching along six lines in general position, and Matsumoto [6] gave the description of the inverse mapping in terms of theta constants. It gives the modular map for the 4 dimensional Shimura variety in the Siegel upper half space $S_4$ derived from the family of 4 dimensional abelian varieties with generalized complex multiplication by $\sqrt{-1}$ of type $(2,2)$. So we call it MSY modular mapping. The second author showed an arithmetic application of MSY modular mapping in [11].

In the case of MSY modular map the corresponding $K3$ surface is characterized by the lattice $U(2)^2 \oplus (-2)^2$ and the moduli space is a 4-dimensional type $IV$ domain. We suspect such fruitfull results of the MSY map is the consequence of the eventual coincidence of two different bounded symmetric domains $D_{IV}^3$ and $I_{2,2}$. There are a few (finite) such exceptional coincidences. The highest one is the (analytic) equivalence between $D_{IV}^3$ and $\mathbb{H}_{II}(4)$ (in terms of Lie algebra $so(2,6;\mathbb{R}) \cong so(4,\mathbb{F})$, where $\mathbb{F}$ indicates the Hamilton quaternion field) and it contains the above coincidence of MSY case. That is our situation.

The ring structure of the regular functions on the parameter space for $\mathcal{F}$ is given by the article of Koike and Ochiai in this volume, their result is the cosequence of the discussion motivated by the RIMS workshop.

1 Realization as an algebraic surface

We fix an abstract lattice $L$ and its $P$-part $D_4^3 \oplus (-2) \oplus (2)$. A $K3$ surface $S$ is called of exact type $P$ when it satisfies $P \cong \text{Pic}(S)$ via an isomorphism $\varphi : H_2(S, \mathbb{Z}) \to L$. That is a general member of $\mathcal{F}$. A surface $S$ of exact type $P$ is realized as a double covering surface over $\mathbb{P}^1 \times \mathbb{P}^1$ branching along four bidegree $(1,1)$ curves $H_1, H_2, H_3, H_4$ satisfying

\[ J_1 \right H_k \quad (k = 1, 2, 3, 4) \] is irreducible,
$J_2$) $H_k \cap H_t$ consists of two different points,
$J_3$) for any different three indices $i,j,k$ we have $H_i \cap H_j \cap H_k = \emptyset$
under some nef condition stated in Theorem 1. Such a surface is given as the complete
nonsingular model of the affine variety

$$S = S(x) : w^2 = \prod_{k=1}^{4} (x_1^{(k)}st + x_2^{(k)}s + x_3^{(k)}t + x_4^{(k)}),$$

where we use the notation

$$x_k = \begin{pmatrix} x_1^{(k)} & x_2^{(k)} \\ x_3^{(k)} & x_4^{(k)} \end{pmatrix} \in \text{M}(2, \mathbb{C}).$$

So the curve $H_k \ (k=1,2,3,4)$ is given by

$$(s,1) \begin{pmatrix} x_1^{(k)} \\ x_3^{(k)} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = 0.$$

By considering the projection to the $s$-line we find that a general member $S$ of $\mathcal{F}$ is an elliptic
fibred surface with 12 singular fibres of type $I_2$ corresponding to the intersection points $H_i \cap H_j \ (i \neq j)$. Counting the Euler number we know that $S(x)$ is a K3 surface. Let $E_{ij}^{\pm}$ be the
exceptional curves obtained by the blow up processes at the intersections $H_i \cap H_j \ (i \neq j, 1 \leq i,j \leq 4)$. Let $\pi$ be the projection $S \to \mathbb{P}^1 \times \mathbb{P}^1$ and set

$$G_i = \frac{1}{2}(\pi^* H_i - \sum_{i \neq j} (E_{ij}^+ + E_{ij}^-)) \ (i=1,2,3,4)$$

$$F_s = \pi^* \{t=0\}, F_t = \pi^* \{s=0\}.$$

**Lemma 1.1** The sublattice in $H_2(S(x), \mathbb{Z})$ generated by $E_{ij}^+, E_{ij}^-, G_i, F_s, F_t$ is isomorphic to $P$. So $S(x)$ is a K3 surface of type $P$.

According to the above Lemma we can define the elements $e_{ij}^{\pm}, g_i, f_s, f_t$ in the abstract lattice
$L$ corresponding to the divisors with capital letters in $H^2(S(x), \mathbb{Z})$ via an isomorphism $\varphi : H_2(S(x), \mathbb{Z}) \to L$.

**Theorem 1** Let $S$ be a K3 surface of type $P$ with an isomorphism $\varphi$, and suppose that $\varphi^{-1}(f_s)$
and $\varphi^{-1}(f_t)$ are nef divisors. Then $S$ is realized as the form of $S(x)$.

## 2 Period map

### 2.1 Marking of a K3 surface of type $P$

**Definition 2.1** Let $S$ be a K3 surface of type $P$ with an embedding $P \to L$. We call the triple
$(S, \varphi, P)$ a $P$ marking of $S$ if we have

1. $\varphi : H_2(S, \mathbb{Z}) \to L$ is an isometry of lattices,
2. $\varphi^{-1}(f_s), \varphi^{-1}(f_t), \varphi^{-1}(e_{ij}^\pm), \varphi^{-1}(g_i)$ are effective divisors and $\varphi^{-1}(f_s)$ and $\varphi^{-1}(f_t)$ are
nef.

Let $O(L)$ be the group of isometries of the lattice $L$, and set $O(L, P) = \{ g \in O(L) : g(x) = x \text{ for } \forall x \in P \}$. Putting $T = P^\perp$ we set $G = O(T)$ and $G(2) = \{ g \in G : g \equiv I \pmod{2} \}$, where we defined $O(T)$ as same as $O(L)$.
Proposition 2.1 We have $G(2) \cong O(L, P)$.

Definition 2.2 Let $(S, \varphi, P)$ and $(S', \varphi', P)$ be $P$ markings of $S$ and $S'$, respectively. An isomorphism $\rho : S \to S'$ is called an isomorphism of these 2 markings when we have $\varphi = \varphi' \circ \rho_*$. We say 2 markings $(S, \varphi, P)$ and $(S', \varphi', P)$ are equivalent when we have a $\gamma \in O(L, P)$ such that $(S', \varphi', P)$ is isomorphic to $(S, \gamma \circ \varphi, P)$.

Remark 2.1 Equivalent markings correspond to the same configuration of the branch locus with the order of $H_1, \ldots, H_4$ of the double covering surfaces. And this relation corresponds to the base change of the K3 lattice which preserves the polarization $P$.

2.2 Parameter space

From the above argument we can take the configuration space for the family of 4 bidegree (1, 1) curves as a parameters space of our family of $P$ marked K3 surfaces. The presentation of $H_k$ has an ambiguity of a constant factor. By considering the projective coordinate transformation of $s$-space and $t$-space our parameter space is given by

$$X = \left( PGL(2, \mathbb{C}) \setminus \{(x_1, \ldots, x_4)\} = (GL(2, \mathbb{C}))^4/\text{PGL}(2, \mathbb{C}) \right) / (\mathbb{C}^*)^4.$$

For further investigation we need a realization of $X$ as a projective variety. That is suggested in the article of Koike and Ochiai on this same volume.

2.3 Period domain

Let $\Gamma_1, \ldots, \Gamma_8$ be a fixed basis of $T$ such that we have the intersection form

$$(\Gamma_i, \Gamma_j) = A := U(2) \oplus (-2I_4).$$

For a $P$ marking $(S, \varphi, P)$ let $\Omega$ be the holomorphic 2-form on $S$ that is unique up to a constant factor. We define the period of $(S, \varphi, P)$ by

$$\eta = [\int_{\varphi^{-1}(\Gamma_1)} \Omega, \ldots, \int_{\varphi^{-1}(\Gamma_8)} \Omega] \in \mathbb{P}^7.$$

The image of the period mapping for the family of $P$ marked K3 surfaces is open dense in the 6-dimensional domain given by

$$D^+ = \{ \eta = [\eta_1, \ldots, \eta_8] \in \mathbb{P}^7 : \uparrow \eta A \eta = 0, \downarrow \eta A \eta > 0, \exists (\eta_3/\eta_1) > 0 \}.$$

We get this fact by using the Riemann-Hodge relation of the period and the Torelli theorem for K3 surfaces. It is a bounded symmetric domain of type IV. We set

$$G^+ = \{ g \in G : g(D^+) = D^+ \}, \quad G(2)^+ = \{ g \in G(2) : g(D^+) = D^+ \}.$$

We can determine the modular group for the equivalence classes of the $P$ marked surfaces. Namely

Theorem 2 Let $(S, \varphi, P)$ and $(S', \varphi', P)$ be $P$ markings of K3 surfaces of type $P$. Let $\eta$ and $\eta'$ be the corresponding periods, respectively. Then these two markings are equivalent if and only if

$$g(\eta) = \eta'$$

for some $g \in G(2)^+$.
Theorem 3 The modular group $G(2)^+$ is a reflection group.

Here a transformation

$$R_v : \lambda \mapsto v - 2(^t v A \lambda / ^t v Av) v, \quad \lambda \in D^+, v \in \mathbb{Z}^8$$

is called a reflection with the root vector $v$.

2.4 Degenerate locus

3 Differential equation

We can determine the system of differential equations for the period with 16 variables $z_{ij}^\pm$.

It becomes a holonomic system of rank 8. So our periods \( \int_{\varphi^{-1}(\Gamma_i)} \Omega \) (i = 1, \ldots, 8) make a basis of the space of solutions for this system defined on a domain $X' = X - V$, where $V$ is the degenerating locus corresponding to the set of $K3$ surfaces of type $P$ which violate some condition of $J_1, J_2, J_3$.

So we can consider the monodromy group $\mathcal{M}$ for this system.

Proposition 3.1 We have $G(2)^+ \subset \mathcal{M}$.

Remark 3.1 We have possibly $G(2)^+ = \mathcal{M}$. But we cannot decide it at present, because Some monodromy transformation may cause an interchange of $E_{ij}^\pm$.

Remark 3.2 The image of the degenerating locus $V$ by the period map is consists of 4, 6, 16 hyperplanes (so in total 26 hyperplanes) in the period domain $D^+$ which correspond to the violation of the condition $J_1, J_2, J_3$, respectively.

4 Transfer of the period domain

The type II domain $H$ is defined by

$$H = H_{II} = \{Z \in M(4, \mathbb{C}) : J_4Z = ^t ZJ_4, \quad \frac{1}{i}(Z - Z^*) > 0\},$$

where we use the notation

$$J_{2n} = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix}$$

and the member $Z \in H$ is described in the form

$$Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & t & a & c \\ -t & 0 & b & d \end{pmatrix}.$$ 

We define the mapping $\psi : H \to \mathbb{P}^7$ by

$$\zeta = ^t [z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8] = ^t [1, -ad + bc - st, a, d, b, c, -s, t].$$

As a direct translation of the above lemma, we obtain:
Proposition 4.1 The image of $\psi$ is determined in $P^7$ by the following three conditions:

1. $^t\zeta(U \oplus U \oplus U \oplus U)\zeta = 0$.
2. $\zeta^*(U \oplus U \oplus U \oplus U)\zeta > 0$,

where $\zeta^* = ^t\overline{\zeta}$.

3. $\Re\left(\frac{z_3}{z_1}\right) > 0$.

By straightforward calculation we have the following.

Theorem 4 The image $\psi(H)$ is transformed to the type IV domain $D_{IV}^+$ by the map:

$$\eta = P\zeta$$

with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+i}{i} & \frac{1+i}{1+i} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-i}{i} & \frac{1-i}{1+i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+i}{1+i} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-i}{1+i} \end{pmatrix}.$$ (4.1)

So the composite mapping $\delta = P \circ \psi$ gives the isomorphism

$$\delta : H \cong D_{IV}.$$

Remark 4.1 The analytic equivalence of the domains $D_{IV}^+$ and $H$ is well known. But we are wishing to find the transfer which preserves the modular groups each other.

4.1 the Quaternion half space

Let $F$ be the Hamilton quaternion $\mathbb{R}$-algebra generated by $\{e_1, e_2, e_3, e_4\}$ with

$$e_1 = 1, \quad e_2e_3 = e_4, \quad e_i^2 = -1.$$

Then the ring of integers in $F$ is given by

$$\mathcal{O}(F) = \mathbb{Z}e_0 + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3,$$

where $e_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$.

Proposition 4.2 The mapping

$$\varphi : M(n, F) \to M(2n, \mathbb{C})$$

defined by

$$\varphi\left(\sum_i A_i e_i\right) = \begin{pmatrix} A_1e_1 + A_2e_2 & A_3e_1 + A_4e_2 \\ -A_3e_1 + A_4e_2 & A_1e_1 - A_2e_2 \end{pmatrix}$$

is an injective homomorphism of $\mathbb{R}$-algebra.
4.1 We set \[ \text{Sym}(2, \mathbb{F}) = \{X \in M(2, \mathbb{F}) : {}^tX = X\}; \]
\[ \text{Pos}(2, \mathbb{F}) = \{X \in \text{Sym}(2, \mathbb{F}) : X > 0\}. \]
Note that we have \[ X \in \text{Sym}(2, \mathbb{F}) \iff \varphi(X) \in \text{Sym}(4, \mathbb{C}) \]
\[ X \in \text{Pos}(2, \mathbb{F}) \iff \varphi(X) \in \text{Pos}(4, \mathbb{C}) \]
and it holds also \[ X \in \text{Pos}(2, \mathbb{F}) \iff X = g^*g, \exists g \in \text{GL}(2, \mathbb{F}). \]

4.2 The quaternion half space is defined by
\[ \mathbb{H}(n, \mathbb{F}) = \{X + \sqrt{-1}Y : X \in \text{Sym}(n, \mathbb{F}), Y \in \text{Pos}(n, \mathbb{F})\}. \]

Remark 4.2 (1) We can define the half spaces using \( \mathbb{R} \) and \( \mathbb{C} \) instead of \( \mathbb{F} \). The half space \( \mathbb{H}(n, \mathbb{R}) \) is a Siegel half space, and \( \mathbb{H}(n, \mathbb{C}) \) is the bounded symmetric space of type \( I \).
(2) Two spaces \( \mathbb{H}(2, \mathbb{F}) \) and \( \varphi(\mathbb{H}(2, \mathbb{F})) \subset \mathbb{H}(4, \mathbb{C}) \) are isomorphic as complex manifolds via the correspondence \( \varphi \).

Proposition 4.3 For an element \( Z \in \mathbb{H} \) we have a decomposition \( Z = X + \sqrt{-1}Y \) with
\[ X = \frac{1}{2}(Z + {}^t\overline{Z}), Y = \frac{1}{2\sqrt{-1}}(Z - {}^t\overline{Z}), \]
and so \( \mathbb{H} \) is naturally embedded in the half space \( \mathbb{H}(4, \mathbb{C}) \).

Remark 4.3 We can examine the equality \( \mathbb{H} = \varphi(\mathbb{H}(2, \mathbb{F})) \) by direct calculation.

Proposition 4.4 The group \( \text{Sp}(4, \mathbb{F}) \) is generated by
\[ J_4, \begin{pmatrix} iW & O \\ O & -W^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix}, \text{where } W \in \text{GL}(2, \mathbb{F}) \text{ and } S^* = S. \]

So we obtain:
Proposition 4.5
\[ \text{Image}(\Phi) = \langle J_8, \begin{pmatrix} A & O \\ O & A^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ O & I \end{pmatrix} \rangle \]
with
\[ A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{pmatrix}, S^* = S. \]
Proposition 4.6 ([4] p.55)
The group $\text{Sp}(4, F) \cap \text{GL}(4, \mathcal{O}(F))$ is generated by
$$J_4, \begin{pmatrix} I & S \\ O & I \end{pmatrix}, \begin{pmatrix} U & O \\ O & (U^*)^{-1} \end{pmatrix}$$
where $S \in \text{Sym}(2, \mathcal{O}(F)), U \in \text{GL}(2, \mathcal{O}(F))$.

Proposition 4.7 The group $\text{Sp}(2n, F)$ is a subgroup of $\text{Aut}(H(n, F))$ via the action
$$Z \mapsto (AZ + B)(CD + D)^{-1}$$
for an element
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, F).$$

Proposition 4.8 ([4] p.50 )
We have
$$\text{Aut}(H(2, F)) = \text{Sp}(4, F) \cdot \langle \Pi \rangle,$$
and we have
$$\text{Aut}(H(n, F)) = \text{Sp}(2, F)$$
for $n \geq 3$. Where $\Pi$ indicates the transposition as an element of $\text{M}(2, F)$, and $\cdot$ means the semi direct product.

Remark 4.4 The transposition $\Pi$ acts as an automorphism of $H(n, F)$ only for $n = 2$. If we have $n \geq 3$, it does not preserve the positivity condition $-\sqrt{-1}(Z - Z^*) > 0$.

4.2 Relation between $G^+(Z)$ and $\Gamma(H)$

Definition 4.3 Set
$$H = \begin{pmatrix} O & iE_4 \\ -iE_4 & O \end{pmatrix}, \quad S = \begin{pmatrix} O & J_4 \\ -J_4 & O \end{pmatrix}, \quad L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}$$
$$\text{SO}^*(8, C) = \{g \in \text{GL}(8, C) : g^* H g = H, \quad ^t g S g = S\}.$$

And set
$$\Gamma(H) = \text{SO}^*(8, Z[i]) \cdot \langle \iota \rangle,$$
where $\iota$ indicates the involution
$$Z = \begin{pmatrix} a & b & 0 & s \\ c & d & -s & 0 \\ 0 & t & a & c \\ -t & 0 & b & d \end{pmatrix} \mapsto Z' = \begin{pmatrix} a & b & 0 & t \\ c & d & -t & 0 \\ 0 & s & a & c \\ -s & 0 & b & d \end{pmatrix}.$$
We can easily examine that $\text{SO}^*(8)$ is a subgroup of $\text{Sp}(8, C)$. We obtain the following by checking the conditions for $\text{SO}^*(8)$.

Proposition 4.9 We have a injective homomorphism of $\mathbb{R}$-algebra $\Phi : \text{Sp}(4, F) \to \text{Sp}(8, C)$ by putting
$$\Phi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{pmatrix},$$
and the image is contained in $\text{SO}^*(8)$. 
Remark 4.5 (1) Let \( \mathcal{O}' \) denote the order \( \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4 \) of \( F \). Then we have

\[ \mathrm{SO}^*(8, \mathbb{Z}[i]) \cong \mathrm{Sp}(4, \mathcal{O}') \]

via the mapping \( \varphi \).

(2) We expect that the isomorphism \( \delta \) induces injective isomorphisms

\[ (\delta^{-1})^* : \Gamma(H) \to G^+ \]

and

\[ \delta^*(G^+(2)) \subset \Gamma(H). \]

But to get them, it is necessary to proceed more detailed argument on the discrete groups on \( D^+ \) and \( H \). We don't have these results still now.

4.3 Embedding of \( H \) into the Siegel upper space \( S_8 \)

We use the following notation:

\[ S_8 = \{ \Omega \in \mathrm{GL}(8, \mathbb{C}) : \Omega^t = \Omega, \Re(\Omega) > 0 \}, \]
\[ K = \begin{pmatrix} O & J_4 \\ J_4 & O \end{pmatrix}, L = \begin{pmatrix} J_4 & O \\ O & -J_4 \end{pmatrix}, \quad J_{2m} = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix} \]
\[ S_8(q) = \{ \Omega \in S_8 : \Omega J_8 = J_8 \Omega, \Omega K = K \Omega, \Omega L = L \Omega \}. \]

Note that \( \{I_8, J_8, K, L\} \) make the basis of the Hamilton quaternionic field.

Proposition 4.10 The domain \( H_{II} \) is embedded in \( S_8 \) by the mapping

\[ \rho : Z \mapsto \begin{pmatrix} Z + Z^t \\ -i(Z - Z^t) \end{pmatrix} \]

It induces the biholomorphic equivalence between \( H \) and \( S_8(q) \).

The group \( \mathrm{SO}^*(8) \) has an injective embedding into \( \mathrm{Sp}(16, \mathbb{R}) \) by the mapping \( \lambda \):

\[ Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \Re(A) & -\Im(A) & \Re(B) & -\Im(B) \\ \Im(A) & \Re(A) & \Im(B) & \Re(B) \\ \Re(C) & -\Im(C) & \Re(D) & -\Im(D) \\ \Im(C) & \Re(C) & \Im(D) & \Re(D) \end{pmatrix}. \]

For an element \( g \in \mathrm{SO}^*(8) \) we have \( \rho^{-1} \circ g \circ \rho = \lambda(g) \).

Proposition 4.11 Put

\[ J_{II} = \begin{pmatrix} O & O & J_4 & O \\ O & O & O & -J_4 \\ -J_4 & O & O & O \\ O & J_4 & O & O \end{pmatrix}, \quad \tilde{J}_{16} = J_8 \oplus J_8. \]

Then it holds

\[ \text{tr}(g)J_{II} \lambda(g) = J_{II}, \quad \text{tr}(g)\tilde{J}_{16} \lambda(g) = \tilde{J}_{16} \]
for every $g \in \text{SO}^*(8)$. If we put

$$\text{Sp}(q) = \{ g \in \text{Sp}(16, \mathbb{R}) : \gamma J_{11} g = J_{11}, \quad \gamma J_{16} = J_{16} \gamma \}$$

the mapping $\lambda$ induces the isomorphism

$$\text{SO}^*(8) \cong \text{Sp}(q).$$

Especially the mapping $\rho$ induces an isomorphism

$$\text{SO}^*(8, \mathbb{Z}[i]) \cong \text{Sp}(q) \cap \text{M}(16, \mathbb{Z}).$$

Let $\Omega$ be a point on $S_8$, and set $\Lambda = \Lambda = \mathbb{Z}^8 + \mathbb{Z}^8 \Omega$. Let $V_\Omega$ denote the abelian variety $\mathbb{C}^8/\Lambda_\Omega$. So we regard $S_8$ as the coarse moduli space of principally polarized abelian varieties $V_\Omega$. Note that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ belong to $\text{Sp}(16, \mathbb{Z})$. We can check that $I_{16}, J_8 \oplus J_8, K \oplus K$ and $L \oplus L$ are contained in the algebra of endomorphisms of $\Lambda$ provided $\Omega \in S_8(q)$. Let

$$\langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle$$

be a $\mathbb{Q}$-algebra generated by $I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L$. Then we have

$$F_\mathbb{Q} \cong \langle I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L \rangle \subset \text{End}(V_\Omega) \quad \text{for} \quad \Omega \in S_8(q),$$

where $F_\mathbb{Q}$ indicates the Hamilton quaternion algebra over $\mathbb{Q}$.

**Proposition 4.12** The space $S_8(q)$ is the coarse moduli space for the family of 8-dimensional abelian variety $V$ with the property

$$F_\mathbb{Q} \cong < I_{16}, J_8 \oplus J_8, K \oplus K, L \oplus L > \subset \text{End}(V_\Omega).$$

In this sense we can call $S_8(q)$ the Shimura variety for the Hamilton quaternion endomorphism algebra $F_\mathbb{Q}$.

## 5 Corresponding Kuga-Satake varieties

We use the method developed in [12] and [10]. The detailed calculation and argument are exposed in [2] also.

Let us consider the lattice $T$ defined by the intersection matrix $A = U(2) \oplus U(2) \oplus (-2I_4)$ and $V_k = T \oplus k$ ($k = \mathbb{R}$ or $\mathbb{Q}$). Let $Q(x)$ denote the quadratic form on $T$ and at the same time on $V_k$. Let $\text{Tens}(T)$ and $\text{Tens}(V_k)$ be the corresponding tensor algebras. And we let $\text{Tens}^+(T)$ and $\text{Tens}^+(V_k)$ denote the subalgebras composed of the parts with even degree in $\text{Tens}(T)$ and $\text{Tens}(V_k)$, respectively. We consider the two-sided ideal $I$ in $\text{Tens}^+(V_k)$ generated by elements $x \otimes x - Q(x)$ for $x \in V_k$, and the ideal $I_2$ in $\text{Tens}(T)$ is defined by the same manner. The corresponding even Clifford algebra is defined by

$$C^+(V_k, Q) = \text{Tens}^+(V_k)/I.$$  

By the same manner, we define

$$C^+(T, Q) = \text{Tens}^+(T)/I_2.$$  

We note that $C^+(V_{\mathbb{R}}, Q)$ is a 128 dimensional real vector space and $C^+(T, Q)$ is a lattice in it. So we obtain a real torus

$$T_\mathbb{R} = C^+(V_{\mathbb{R}}, Q)/C^+(T, Q).$$  

Let $F$ denote the quaternion algebra

$$\mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$$  

with $i^2 = j^2 = -1$. By some routine calculations of the Clifford algebra we obtain the following.
Proposition 5.1 We have an isomorphism of algebras $C^+(V_Q, Q) \cong M(4, F) \oplus M(4, F)$.

Let a complex vector $\eta = (\eta_1, \cdots, \eta_8)$ be a representative of a point $\eta = [\eta_1, \cdots, \eta_8] \in D^+$. So it has an ambiguity of the multiplication by a non zero complex number. Put $\eta = s + it$ ($s, t \in \mathbb{R}^8$). If we impose the condition $(st)^2 = -1$ in $C^+(V_{\mathbb{R}}, Q)$, the representative is uniquely determined up to a multiplication by a complex unit. We denote it by

$$\eta = m_1(\eta) + im_2(\eta).$$

Put

$$m(\eta) = m_1(\eta)m_2(\eta).$$

It is uniquely determined by $\eta$ without any ambiguity. According to the imposed condition, the element $m(\eta) \in C^+(V_{\mathbb{R}}, Q)$ defines a complex structure on $C^+(V_{\mathbb{R}}, Q)$ by the left action. It induces a complex structure on the real torus $\mathcal{T}_{\mathbb{R}}$ also.

Let $\{\epsilon_1, \cdots, \epsilon_8\}$ be a basis of $T$ with the intersection matrix $U(2) \oplus U(2) \oplus (-2I_4)$, and let $\{e_1, \cdots, e_8\}$ be an orthonormal basis of $V$ given by

$$(e_1, \cdots, e_8) = (\epsilon_1, \cdots, \epsilon)(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \oplus (I_4)).$$

Then the corresponding intersection matrix takes the form $I_2 \oplus (-I_2) \oplus (-2I_4)$.

Let $\iota$ be an involution on $C^+(V, Q)$ induced from the transformation

$$\iota : e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \mapsto e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}$$

for the basis. Set $\alpha = 4e_2e_1$. According to the method in [St] we know that

$$E(x, y) = tr(\alpha x^\iota y)$$

determines a Riemann form. So the triple $(\mathcal{T}_R, m(\eta), E(x, y))$ determines an abelian variety. We denote it by $A^+(\eta)$, that is so called the Kuga-Satake variety attached to the $K3$ surface corresponding to the period $\eta$. In this way we can construct a family of abelian varieties

$$A^+ = \{A^+(\eta) : \eta \in D^+\}$$

induced from the lattice $T$ parameterized by the domain $D^+$. We can construct the "conjugate family"

$$A^- = \{A^-(\eta) : \eta \in D^-\}$$

parameterized by

$$D^- = \{\eta = [\eta_1, \cdots, \eta_8] : \imath^\eta A\eta = 0, \imath^\eta A\eta > 0, \Im(\eta_3/\eta_1) < 0\}$$

by the same procedure with the Riemann form $E^-(x, y) = -tr(\alpha x^\iota y)$. The right action of $C^+(V_Q, Q)$ on $C^+(V_{\mathbb{R}}, Q)$ commutes with the left action of $m(\eta)$. So we have

$$C^+(T_Q) \subset End(A^\pm(\eta)) \otimes Q$$

for any $A^\pm(\eta)$. For a general member $\eta \in D^+$, the endomorphism ring is given by

$$\text{End}_Q(A(\eta)) = \text{End}(A(\eta)) \otimes Q \cong C^+(V_Q).$$

According to Proposition 6.1 we obtain:
Theorem 5 For a general member $\eta \in \mathcal{D}^+$, $A^+(\eta)$ is isogenous to a product of abelian varieties $(A_1(\eta) \times A_2(\eta))^4$ where $A_1(\eta)$ and $A_2(\eta)$ are 8-dimensional simple abelian varieties with $\text{End}_{\mathbb{Q}}(A_i(\eta)) = F_\mathbb{Q}$ $(i = 1, 2)$.

Remark 5.1 Here we describe the relation between $A_1(\eta)$ and $A_2(\eta)$. Now we define the linear involution $* \text{ on } V^\mathbb{R}$ by

$$e_i^* = -e_i \quad \text{and} \quad e_i^* = e_i \quad (i = 2, \ldots, 8).$$

It can be extended on $C^+(V^\mathbb{R}, Q)$ as an automorphism of algebra. We define an involution $\sigma$ on $\mathcal{D}$:

$$\sigma : \mathcal{D} \rightarrow \mathcal{D}, \quad (\eta_{\infty}, \cdots, \eta_N) \mapsto (-\eta_{\infty}, -\eta_{\infty}, \eta_2, \cdots, \eta_N).$$

So we have $\mathcal{D}^\sigma = \mathcal{D}$. It is easy to check that we have

$$A_2(\eta) \sim A_1(\eta^\sigma), \quad A_1(\eta) \sim A_2(\eta^\sigma),$$

where $\sim$ indicates the isogenous relation.

E. Freitag and C. F. Hermann [1] study a similar family of lattice $K3$ surfaces from a different viewpoint. We think that it should be clarified the exact relation between their family and our $\mathcal{F}$.

References


[2] K. Koike, The Kuga-Satake variety attached to the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along four curves of bidegree $(1, 1)$, preprint.


