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Twisted first homology group of the automorphism group of a free group

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1 Introduction

Let F_n be a free group of rank n with basis $\mathrm{Y}=\{y_1,\ldots,y_n\}$. We denote by $\mathrm{Aut}\,\mathrm{F}_n$ and $\mathrm{Out}\,\mathrm{F}_n$ the automorphism group and outer automorphism group of F_n respectively. In this paper, we calculate the twisted first cohomology groups and homology groups of these groups with coefficients in $H_1(\mathrm{F}_n,\mathbf{Z})$ and $H^1(\mathrm{F}_n,\mathbf{Z})$.

The cohomology groups and homology groups of $\mathrm{Aut}\,\mathrm{F}_n$ and $\mathrm{Out}\,\mathrm{F}_n$ are not well known completely. In the case where the coefficients are trivial, however, there are some results. For example, S.M.Gersten [3] showed $H_2(\mathrm{Aut}\,\mathrm{F}_n,\mathbf{Z})=\mathbf{Z}/2\mathbf{Z}$ for $n\geq 5$ and A.Hatcher and K.Vogtmann [4] showed $H_4(\mathrm{Aut}\,\mathrm{F}_4,\mathbf{Q})=\mathbf{Q}$. On the other hand, the twisted cohomology groups and homology groups of $\mathrm{Aut}\,\mathrm{F}_n$ and $\mathrm{Out}\,\mathrm{F}_n$ are much more unknown.

The actions $\mathrm{Out}\,\mathrm{F}_n$ on $H_1(\mathrm{F}_n,\mathbf{Z})$ and $H^1(\mathrm{F}_n,\mathbf{Z})$ are very similar to those of M_g on $H_1(\Sigma_g,\mathbf{Z})$ and $H^1(\Sigma_g,\mathbf{Z})$. Here, Σ_g is a smooth oriented closed surface of genus g and M_g is its mapping class group. we should remark that $H_1(\Sigma_g,\mathbf{Z})$ is isomorphic to $H^1(\Sigma_g,\mathbf{Z})$ by Poincaré duality. S.Morita [8] calculated $H^1(\mathrm{M}_g,H_1(\Sigma_g,\mathbf{Z}))=0$ and $H_1(\mathrm{M}_g,H^1(\Sigma_g,\mathbf{Z}))=\mathbf{Z}/(2-2g)\mathbf{Z}$ for $g\geq 2$. Our calculation is similar to his calculation in several respects.

2 Notations

Let $\mathrm{Inn}\,\mathrm{F}_n$ be the inner automorphism group of F_n. We denote by $\mathrm{Aut}^+\mathrm{F}_n$ and $\mathrm{Out}^+\mathrm{F}_n$ the special and special outer automorphism group of F_n respectively. More precisely, $\mathrm{Aut}^+\mathrm{F}_n$ and $\mathrm{Out}^+\mathrm{F}_n$ are defined in the following way. Let F_n^{ab} be $\mathrm{F}_n/[\mathrm{F}_n,\mathrm{F}_n]$ the abelianization of F_n, which is the first homology group of F_n with trivial coefficients. Let φ be the natural epimorphism from $\operatorname{Aut} F_n$ onto $\operatorname{Aut} F_n^{ab}$ induced by the abelianizer $a : F_n \to F_n^{ab}$. Since F_n^{ab} is naturally isomorphic to $\mathbf{Z}^n$ with respect to the basis $Y = \{y_1, \ldots, y_n\}$, we can identify $\operatorname{Aut} F_n^{ab}$ with $GL(n, \mathbf{Z})$. Hence we get the epimorphism from $\operatorname{Aut} F_n$ onto $GL(n, \mathbf{Z})$. We also use φ to denote this epimorphism

$$\varphi : \operatorname{Aut} F_n \longrightarrow GL(n, \mathbf{Z}).$$

If we consider the determinant homomorphism

$$\det : GL(n, \mathbf{Z}) \longrightarrow \{\pm 1\},$$

and put $\iota = \det \circ \varphi$, then we define $\operatorname{Aut}^+ F_n$ and $\operatorname{Out}^+ F_n$ as follows:

$$\operatorname{Aut}^+ F_n = \{\sigma \in \operatorname{Aut} F_n \mid \iota(\sigma) = 1\},$$

$$\operatorname{Out}^+ F_n = \{[\sigma] \in \operatorname{Out} F_n \mid \iota(\sigma) = 1\},$$

where $[\sigma]$ is the equivalence class of σ in $\operatorname{Aut} F_n$ modulo $\operatorname{Inn} F_n$.

The group $\operatorname{Aut} F_n$ acts on F_n^{ab} by the epimorphism φ. We denote $\sigma \cdot x$ by the action of σ on x, where $\sigma \in \operatorname{Aut} F_n$ and $x \in \mathbf{Z}^n$. If we identify F_n^{ab} with $\mathbf{Z}^n$ in the above sense and define $\overline{\sigma}$ to be $\varphi(\sigma)$, then the action of σ on $x \in F_n^{ab}$ is considered as the matrix action of $\overline{\sigma}$ on $x \in \mathbf{Z}^n$. Since $\operatorname{Aut} F_n$ acts on F_n^{ab}, the group $\operatorname{Aut} F_n$ also acts on $F_n^{ab} \otimes_{\mathbf{Z}} (\mathbf{Z}/q\mathbf{Z})$ and $(F_n^{ab})^*$ in the natural way. Here, $(F_n^{ab})^*$ is the group $\operatorname{Hom}_{\mathbf{Z}}(F_n^{ab}; \mathbf{Z})$ of all homomorphisms from F_n^{ab} to $\mathbf{Z}$, which is the first cohomorogy group of F_n with trivial coefficients. If we identify $F_n^{ab} \otimes_{\mathbf{Z}} (\mathbf{Z}/q\mathbf{Z})$ with $(\mathbf{Z}/q\mathbf{Z})^n$, then the action of $\operatorname{Aut} F_n$ on $(\mathbf{Z}/q\mathbf{Z})^n$ is given by

$$\sigma \cdot x = \sigma'(x),$$

where $\sigma \in \operatorname{Aut} F_n$ and $x \in (\mathbf{Z}/q\mathbf{Z})^n$ and σ' means $\overline{\sigma}$ modulo $q\mathbf{Z}$. On the other hand, the action of $\operatorname{Aut} F_n$ on $(F_n^{ab})^*$ is also given by

$$(\sigma \cdot f)(x) = f(\overline{\sigma}^{-1}(x)),$$

where $\sigma \in \operatorname{Aut} F_n$, $f \in (F_n^{ab})^*$ and $x \in \mathbf{Z}^n$.

The group $\operatorname{Out} F_n$ acts on F_n^{ab}, $F_n^{ab} \otimes_{\mathbf{Z}} (\mathbf{Z}/q\mathbf{Z})$ and $(F_n^{ab})^*$ in the same way. In fact, these actions are given by

$$[\sigma] \cdot x = \overline{\sigma}(x), \ x \in \mathbf{Z}^n,$$

$$[\sigma] \cdot x = \sigma'(x), \ x \in (\mathbf{Z}/q\mathbf{Z})^n,$$

$$([\sigma] \cdot f)(x) = \varphi(\overline{\sigma}^{-1}(x)), \ f \in (F_n^{ab})^*, \, x \in \mathbf{Z}^n.$$

These above three actions are well-defined because $\mathrm{Inn}\,\mathrm{F}_n$ acts on F_n^{ab} trivially.

We should remark that all maps in our calculation are composed right to left. Namely, the composition of two maps f and g is defined by

$$(f \circ g)(x) = f(g(x)).$$

Furthermore we also define an expansion of commutator $[x, y]$ by

$$[x, y] = y^{-1}x^{-1}yx.$$

3 Main results

We calculate the first cohomology and homology groups of $\mathrm{Aut}\,\mathrm{F}_n$, $\mathrm{Aut}^+\,\mathrm{F}_n$, $\mathrm{Out}\,\mathrm{F}_n$ and $\mathrm{Out}^+\,\mathrm{F}_n$ with coefficients in F_n^{ab}, $\mathrm{F}_n^{ab}\otimes_{\mathbf{Z}}(\mathbf{Z}/q\mathbf{Z})$ and $(\mathrm{F}_n^{ab})^*$ with respect to the above actions. The following statements are our main results.

Theorem 1 *For any $n \geq 2$, the first cohomology group of Γ with coefficients in $\mathrm{H} = \mathrm{F}_n^{ab}$ is given by*

$$H^1(\Gamma, \mathrm{H}) = \begin{cases} \mathbf{Z} & \text{if } \Gamma = \mathrm{Aut}\,\mathrm{F}_n \text{ or } \mathrm{Aut}^+\,\mathrm{F}_n, \\ 0 & \text{if } \Gamma = \mathrm{Out}\,\mathrm{F}_n \text{ or } \mathrm{Out}^+\,\mathrm{F}_n. \end{cases}$$

Theorem 2 *Let q be a positive integer greater than 1 and e be a positive integer. The first cohomology group of Γ with coefficients in $\mathrm{A}_q^n = \mathrm{F}_n^{ab}\otimes_{\mathbf{Z}}(\mathbf{Z}/q\mathbf{Z})$ is given by*

(1) $\Gamma = \mathrm{Aut}\,\mathrm{F}_n$ *or* $\mathrm{Aut}^+\,\mathrm{F}_n$ *for* $n \geq 2$

$$H^1(\Gamma, \mathrm{A}_q^n) = \begin{cases} \mathbf{Z}/q\mathbf{Z} & \text{if } n \geq 4, \\ \mathbf{Z}/q\mathbf{Z} & \text{if } n = 2, 3 \text{ and } (q, 2) = 1, \\ \mathbf{Z}/q\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2, 3 \text{ and } q = 2^e, \end{cases}$$

(2) $\Gamma = \mathrm{Out}\,\mathrm{F}_n$ *or* $\mathrm{Out}^+\,\mathrm{F}_n$ *for* $n \geq 4$

$$H^1(\Gamma, \mathrm{A}_q^n) = \begin{cases} 0 & \text{if } (q, n-1) = 1, \\ \mathbf{Z}/q\mathbf{Z} & \text{if } q \mid (n-1), \\ \mathbf{Z}/(n-1)\mathbf{Z} & \text{if } (n-1) \mid q, \end{cases}$$

(3) $\Gamma = \mathrm{Out}\,\mathrm{F}_2$ *or* $\mathrm{Out}^+\,\mathrm{F}_2$

$$H^1(\Gamma, \mathrm{A}_q^2) = \begin{cases} 0 & \textit{if } (q,2) = 1, \\ \mathbf{Z}/2\mathbf{Z} & \textit{if } q = 2^e, \end{cases}$$

(4) $\Gamma = \mathrm{Out}\,\mathrm{F}_3$ *or* $\mathrm{Out}^+\,\mathrm{F}_3$

$$H^1(\Gamma, \mathrm{A}_q^3) = \begin{cases} 0 & \textit{if } (q,2) = 1, \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \textit{if } q = 2^e. \end{cases}$$

Theorem 3 *For any $n \geq 2$, the homology group of Γ with coefficients in $\mathrm{H}^* = (\mathrm{F}_n^{ab})^*$ is given by*

(1) $\Gamma = \mathrm{Aut}\,\mathrm{F}_n$ *or* $\mathrm{Aut}^+\,\mathrm{F}_n$

$$H_1(\Gamma, \mathrm{H}^*) = \begin{cases} \mathbf{Z} & \textit{if } n \geq 4, \\ \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \textit{if } n = 2, 3, \end{cases}$$

(2) $\Gamma = \mathrm{Out}\,\mathrm{F}_n$ *or* $\mathrm{Out}^+\,\mathrm{F}_n$

$$H_1(\Gamma, \mathrm{H}^*) = \begin{cases} \mathbf{Z}/(n-1)\mathbf{Z} & \textit{if } n \geq 4, \\ \mathbf{Z}/2\mathbf{Z} & \textit{if } n = 2, \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \textit{if } n = 3. \end{cases}$$

In order to calculate the cohomology groups, we will use presentations for $\mathrm{Aut}\,\mathrm{F}_n$, $\mathrm{Aut}^+\,\mathrm{F}_n$, $\mathrm{Out}\,\mathrm{F}_n$ and $\mathrm{Out}^+\,\mathrm{F}_n$. We determine the condition which a crossed homomorphism must satisfy by using these presentations.

Historically, the first finite presentation for $\mathrm{Aut}\,\mathrm{F}_n$ was obtained by Nielsen in 1924 and to show it, he used hyperbolic geometry. However, it is well known that his presentation is too complicated to handle. In 1974, McCool [6] gave the simplified finite presentation using Whitehead automorphisms. In 1984, Gersten [3] improved McCool's presentation. He expressed the McCool's relations, which are represented by Whitehead automorphisms, in terms of Nielsen's automorphisms. In our calculation, we improve and use the Gersten's finite presentation

Finally, we give an interpretation of the generator of $H^1(\mathrm{Aut}^+\,\mathrm{F}_n, \mathrm{H}) = \mathbf{Z}$. In general, it is well known that there is a homomorphism

$$r : \mathrm{Aut}^+\,\mathrm{F}_n \longrightarrow \mathrm{GL}(n, \mathbf{Z}[\mathrm{F}_n]),$$

which is called Magnus representation of $\mathrm{Aut}^+ \mathrm{F}_n$ (see [1]).

Let $\alpha : \mathrm{GL}(n, \mathbf{Z}[\mathrm{F}_n]) \to \mathrm{GL}(n, \mathbf{Z}[\mathrm{H}])$ be a homomorphism induced by an abelianizer $a : \mathrm{F}_n \to \mathrm{H}$ and $\det : \mathrm{GL}(n, \mathbf{Z}[\mathrm{H}]) \to \mathbf{Z}[\mathrm{H}]$ be the determinant map. Composing these maps, we obtain the map

$$f : \mathrm{Aut}^+ \mathrm{F}_n \longrightarrow \mathbf{Z}[\mathrm{H}].$$

By an easy calculation, we can show that the images of all generators of $\mathrm{Aut}^+ \mathrm{F}_n$ are contained in H. Hence, the image of f is contained in H. Moreover, f is a crossed homomorphism and the cohomology class $[f]$ of f generates $H^1(\mathrm{Aut}^+ \mathrm{F}_n, \mathrm{H})$.

The same argumaent holds in the case $H^1(\mathrm{M}_{g,1}, H_1(\Sigma_{g,1}, \mathbf{Z}))$ (see [9]). Here, $\Sigma_{g,1}$ is a smooth oriented closed surface of genus g with one fixed boundary component and $\mathrm{M}_{g,1}$ is its mapping class group. S.Morita [8] also calculated that $H^1(\mathrm{M}_{g,1}, H_1(\Sigma_{g,1}, \mathbf{Z})) = \mathbf{Z}$. The generator of $H^1(\mathrm{M}_{g,1}, H_1(\Sigma_{g,1}, \mathbf{Z}))$ is constructed by using a Magnus representation of $\mathrm{M}_{g,1}$.

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