DEFINABLE $G$-FIBER BUNDLES AND DEFINABLE $C^rG$-FIBER BUNDLES (Topological Transformation Groups and Related Topics)

Author(s)

Kawakami, Tomohiro

Citation

数理解析研究所講究録 2003(1343): 31-45

Issue Date

2003-10

URL

http://hdl.handle.net/2433/43496

Type

Departmental Bulletin Paper

Textversion

publisher

Kyoto University
DEFINABLE $G$-FIBER BUNDLES AND DEFINABLE $C^rG$-FIBER BUNDLES

TOMOHIRO KAWAKAMI

川上智博 (和歌山大学)

ABSTRACT. Let $G$ be a compact definable group and $f, h : X \to Y$ definable $G$-maps between definable $G$-sets. We prove that if $X$ is compact, $\eta$ is a definable $G$-fiber bundle over $Y$ and $f$ and $h$ are $G$-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably $G$-isomorphic.

Let $G$ be a compact subgroup of $GL_n(\mathbb{R})$ and $f, h : X \to Y$ definable $C^rG$ maps between definable $C^rG$ manifolds. We show that if $X$ is compact and affine, $\eta$ is a definable $C^rG$-fiber bundle over $Y$ and $f$ and $h$ are definably $C^rG$-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably $C^rG$-isomorphic.

1. INTRODUCTION

Let $\mathcal{M}$ denote an $o$-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term “definable” means “definable with parameters in $\mathcal{M}$”. In this paper, we are concerned with homotopy property of definable $G$-fiber bundles and definable $C^rG$-fiber bundles when $1 \leq r < \infty$. General references on o-minimal structures are [6], [8], see also [18]. Further properties and constructions of them are studied in [7], [9], [17]. Every definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides the semialgebraic one.

A group $G$ is a definable group if $G$ is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A definable G-set means a G-invariant definable subset of some representation of $G$. We use a definable space as in the sense of [6], and every definable set is a definable space in this sense. Throughout this paper, definable maps between definable spaces are assumed to be continuous.

Theorem 1.1. Let $G$ be a compact definable group. Suppose that $\eta = (E, p, Y, F, K)$ is a definable $G$-fiber bundle over a definable $G$ set $Y$ and $f, h : X \to Y$ are definable $G$-maps between definable $G$-sets. If $X$ is compact and $f$ and $h$ are $G$-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably $G$-isomorphic.

Two definable $G$-maps $f, h : X \to Y$ between definable $G$-sets are definably $G$-homotopic if there exists a definable $G$-map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where the action on $[0, 1]$ is trivial. By 1.2 [11], two definable $G$-maps in Theorem 1.1 are definably $G$-homotopic.

2000 Mathematics Subject Classification. 14P10, 14P20, 57R22, 57R35, 57S10, 57S15, 58A05, 58A07, 03C64.

Keywords and Phrases. Definable $G$-sets, definable $G$-fiber bundles, definable $C$-vector bundles, $o$-minimal, compact definable groups, definable $C^rG$-manifolds, definable $C^rG$-fiber bundles, definable $C^rG$-vector bundles.
In the rest of this paper except section 2, $G$ and $K$ denote compact subgroups of $GL_n(\mathbb{R})$. It is known that they are compact algebraic subgroups of $GL_n(\mathbb{R})$ (e.g. 2.2 [16]).

Let $\Omega$ be a representation of $G$ and $k \in \mathbb{N}$. Then we can consider the universal $G$-vector bundle $\gamma(\Omega, k)$ associated with $\Omega$ and $k$ (see Definition 3.1). A definable $G$-vector bundle $\eta = (E, p, X)$ over a definable $G$-set $X$ is called strongly definable if there exist a representation $\Omega$ of $G$ and a definable $G$-map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $G$-isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$. The following result is a definable version of 1.1 [3].

**Theorem 1.2.** Every definable $G$-vector bundle over a definable $G$-set is strongly definable.

Let $X$ be a definable $G$-set. Let $\text{Vect}^G_{def}(X)$ (respectively $\text{Vect}^G(X)$) denote the set of definable $G$-isomorphism (respectively $G$-isomorphism) classes of definable $G$-vector bundles (respectively $G$-vector bundles) over $X$. Then there is a canonical map $\kappa : \text{Vect}^G_{def}(X) \to \text{Vect}^G(X)$ which sends the definable $G$-isomorphism class $[\eta]^G_{def}$ of a definable $G$-vector bundle $\eta$ over $X$ to the $G$-isomorphism class $[\eta]^G$ of $\eta$.

**Theorem 1.3.** Let $X$ be a definable $G$-set. Then the map $\kappa : \text{Vect}^G_{def}(X) \to \text{Vect}^G(X)$ defined by $\kappa([\eta]^G_{def}) = [\eta]^G$ is bijective.

As a corollary of Theorem 1.3, we have the following.

**Corollary 1.4.** Let $\eta = (E, p, Y)$ be a definable $G$-vector bundle over a definable $G$-set $Y$ and $f, h : X \to Y$ definable $G$-maps between definable $G$-sets. If $f$ and $h$ are $G$-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably $G$-isomorphic.

Let $1 \leq r \leq \omega$. A definable $C^r G$-manifold is a pair $(X, \theta)$ consisting of a definable $C^r$-manifold $X$ and a group action $\theta : G \times X \to X$ which is a definable $C^r$-map. We simply write $X$ for $(X, \theta)$. A definable $C^r G$-manifold is affine if it is definably $C^r G$-diffeomorphic to a $G$-invariant definable $C^r$-submanifold of some representation of $G$.

Two definable $C^r G$-maps $f, h : X \to Y$ between definable $C^r G$-manifolds are definably $C^r G$-homotopic if there exists a definable $C^r G$-map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where $G$ acts on $[0, 1]$ trivially.

The following result is a definable $C^r G$-version of Theorem 1.1.

**Theorem 1.5.** Suppose that $\eta = (E, p, Y, F, K)$ is a definable $C^r G$-fiber bundle over a definable $C^r G$-manifold $Y$ and $1 \leq r < \infty$. Let $f, h$ be definable $C^r G$-maps from a compact affine definable $C^r G$-manifold $X$ to $Y$. If $f$ and $h$ are definably $C^r G$-homotopic and $F$ is affine, then $f^*(\eta)$ and $h^*(\eta)$ are definably $C^r G$-isomorphic.

**Corollary 1.6.** Let $f, h : X \to Y$ be definable $C^r G$-maps between definable $C^r G$-manifolds and $1 \leq r < \infty$. If $X$ is compact and affine, $\eta$ is a definable $C^r G$-vector bundle over $Y$ and $f$ is definably $C^r G$-homotopic to $h$, then $f^*(\eta)$ and $h^*(\eta)$ are definably $C^r G$-isomorphic.

Let $1 \leq r \leq \omega$. A definable $C^r G$-vector bundle $\eta = (E, p, X)$ over an affine definable $C^r G$-manifold $X$ is called strongly definable if then there exist a representation $\Omega$ of $G$ and a definable $C^r G$-map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $C^r G$-isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$. 
Theorem 1.7. Let $\eta$ be a definable $G'$-vector bundle over an affine definable $G'$-manifold $X$. If $X$ is compact and $1 \leq r < \infty$, then $\eta$ is strongly definable. Moreover if $r = \infty$ or $\omega$, then $\eta$ is strongly definable if and only if the total space of $\eta$ is affine.

This paper is organized as follows. In section 2, we give a definition of definable $G$ fiber bundles and prove Theorem 1.1. We prove Theorem 1.2, 1.3 and Corollary 1.4 in section 3 and Theorem 1.5 and 1.7 in section 4.

2. Definable $G$-fiber bundles

A group homomorphism between definable groups is a definable group homomorphism if it is a definable map. An $n$-dimensional representation of a definable group $G$ means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from $G$ to $O_n(\mathbb{R})$. A subgroup of a definable group $G$ is a definable subgroup of $G$ if it is a definable subset of $G$. A definable map (respectively A definable homeomorphism) between definable $G$-sets is a definable $G$-map (respectively a definable $G$-homeomorphism) if it is a $G$-map.

Let $G$ be a definable group. A definable set with a definable $G$-action is a pair $(X, \theta)$ consisting of a definable set $X$ and a group action $\theta : G \times X \to X$ such that $\theta$ is a definable map. We simply write $X$ instead of $(X, \theta)$. This action is not necessarily linear (orthogonal). Definable $G$-maps and definable $G$-homeomorphisms between definable sets with definable $G$-actions are defined similarly.

A definable space is an object obtained by pasting finitely many definable sets together along open definable subsets, and definable maps between definable spaces are defined similarly (see Chapter 10 [6]). Definable spaces are generalizations of semialgebraic spaces in the sense of [4].

Definition 2.1. Let $G$ be a definable group.

1. A definable $G$-space is a pair $(X, \theta)$ consisting of a definable space $X$ and a group action $\theta : G \times X \to X$ which is definable. For simplicity of notation, we write $X$ for $(X, \theta)$.

2. Let $X$ and $Y$ be definable $G$-spaces. A definable map $f : X \to Y$ is called a definable $G$-map if it is a $G$-map. We say that $X$ and $Y$ are definably $G$-homeomorphic if there exist definable $G$-maps $h : X \to Y$ and $k : Y \to X$ such that $h \circ k = id$ and $k \circ h = id$.

Note that clearly an implication "a definable $G$-set" $\Rightarrow$ "a definable set with a definable $G$-action" $\Rightarrow$ "a definable $G$-space" holds.

Definition 2.2. (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a definable fiber bundle over $X$ with fiber $F$ and structure group $K$ if the following two conditions are satisfied:

(a) The total space $E$ is a definable space, the base space $X$ is a definable set, the structure group $K$ is a definable group, the fiber $F$ is a definable set with an effective definable $K$ action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}$ of $\eta$ such that each $U_i$ is a definable open subset of $X$, $\{U_i\}_{i}$ is a finite open covering of $X$. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where
\(\pi_i\) stands for the projection \(U_i \times F \rightarrow F\). For any \(i\) and \(j\) with \(U_i \cap U_j \neq \emptyset\), the transition function \(\theta_{ij} := \phi_{j,z} \circ \phi_{i,z}^{-1} : U_i \cap U_j \rightarrow K\) is a definable map. We call these trivializations definable.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let \(\eta = (E, p, X, F, K)\) and \(\zeta = (E', p', X', F, K)\) be definable fiber bundles whose definable local trivializations are \(\{U_i, \phi_i\}_i\) and \(\{V_j, \psi_j\}_j\), respectively. A definable map \(\overline{f} : E \rightarrow E'\) is said to be a definable morphism if the following two conditions are satisfied:

(a) The map \(\overline{f}\) covers a definable map, namely there exists a definable map \(f : X \rightarrow X'\) such that \(f \circ p = p' \circ \overline{f}\).

(b) For any \(i, j\) such that \(U_i \cap f^{-1}(V_j) \neq \emptyset\) and for any \(x \in U_i \cap f^{-1}(V_j)\), the map \(f_{ij}(x) := \psi_{j,f(x)} \circ \phi_{i,x}^{-1} : F \rightarrow F\) lies in \(K\), and \(f_{ij} : U_i \cap f^{-1}(V_j) \rightarrow K\) is a definable map.

We say that a bijective definable morphism \(\overline{f} : E \rightarrow E'\) is a definable equivalence if it covers a definable homeomorphism \(f : X \rightarrow X'\) and \((\overline{f})^{-1} : E' \rightarrow E\) is a definable morphism covering \(f^{-1} : X' \rightarrow X\). A definable equivalence \(\overline{f} : E \rightarrow E'\) is called a definable isomorphism if \(X = X'\) and \(f = \text{id}_X\).

(3) A continuous section \(s : X \rightarrow E\) of a definable fiber bundle \(\eta = (E, p, X, F, K)\) is a definable section if for any \(i\), the map \(\phi_i \circ s|U_i : U_i \rightarrow U_i \times F\) is a definable map.

(4) We say that a definable fiber bundle \(\eta = (E, p, X, F, K)\) is a principal definable fiber bundle if \(F = K\) and the \(K\)-action on \(F\) is defined by the multiplication of \(K\). We write \((E, p, X, K)\) for \((E, p, X, F, K)\).

**Definition 2.3.** Let \(G\) be a definable group.

(1) A definable fiber bundle \((E, p, X, F, K)\) (respectively A principal definable fiber bundle \((E, p, X, K)\)) is called a definable \(G\)-fiber bundle (respectively a principal definable \(G\)-fiber bundle) if the total space \(E\) is a definable \(G\)-space such that \(G\) acts on \(E\) through definable equivalences, the base space \(X\) is a definable set with a definable \(G\)-action and the projection \(p\) is a definable \(G\)-map.

(2) A definable morphism (respectively A definable equivalence, A definable isomorphism) between definable \(G\)-fiber bundles is a definable \(G\)-morphism (respectively a definable \(G\)-equivalence, a definable \(G\)-isomorphism) if it is a \(G\)-map.

(3) A definable \(G\)-section of a definable \(G\)-fiber bundle means a definable section which is a \(G\)-map.

Let \(f : X \rightarrow Y\) be a definable map between definable sets. We say that \(f\) is proper if for any compact subset \(C\) of \(Y\), \(f^{-1}(C)\) is compact.

Let \(E\) be an equivalence relation on a definable set \(X\). We call \(E\) proper if \(E\) is a definable subset of \(X \times X\) and the projection \(E \rightarrow X\) defined by \((x, y) \mapsto x\) is proper.

**Theorem 2.4** (Definable quotients (e.g. 10.2.15 [6]). Let \(E\) be a proper equivalence relation on a definable set \(X\). Then \(X/E\) exists a proper quotient, namely \(X/E\) is a definable subset of some \(\mathbb{R}^n\) and the projection \(X \rightarrow X/E\) is a surjective proper definable map.
In the remainder of this section, $G$ and $K$ denote compact definable groups. The following is a corollary of Theorem 2.4.

**Corollary 2.5** (e.g. 10.2.18 [6]). Let $X$ be a definable set with a definable $G$-action. Then $X/G$ is a definable subset of some $\mathbb{R}^n$ and the orbit map $p : X \to X/G$ is a surjective proper definable map.

By similar proofs of 2.10 [14] and 2.11 [14], the standard construction of the associated principal bundle from a fiber bundle and by Theorem 2.4, we have the following.

**Proposition 2.6.** (1) Let $(E, p, X, K)$ be a principal definable $G$-fiber bundle and $F$ a definable set with an effective definable $K$-action. Then $(E \times_K F, p', X, F, K)$ is a definable $G$-fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, k]) = p(z)$.

(2) The associated principal $G$-fiber bundle of a definable $G$-fiber bundle is definable.

(3) Two definable $G$-fiber bundles having the same base space, fiber and structure group are definably $G$-isomorphic if and only if their associated principal definable $G$-fiber bundles are definably $G$-isomorphic.

Let $X$ be a definable set with a definable $G$-action and $x \in X$. A $G_x$-invariant definable subset $S$ of $X$ is a definable slice at $x$ in $X$ if $GS$ is a $G$-invariant definable open neighborhood of the orbit $G(x)$ of $x$ in $X$, $G \times_{G_x} S$ is a definable set with the standard definable $G$-action $G \times (G \times_{G_x} S) \to G \times_{G_x} S$, $(g, [g', s]) \mapsto [gg', s]$, and the map $G \times_{G_x} S \to GS \subset X$ defined by $[g, s] \mapsto gs$ is a definable $G$-homeomorphism.

**Theorem 2.7** (Definable slices). Let $X$ be a definable $G$-set and $x \in X$. Then there exists a definable slice $S$ at $x$ in $X$.

Let $Y$ be a $G$-invariant definable subset of a definable $G$-set $X$. A definable $G$-retraction from $X$ to $Y$ means a definable $G$-map $R : X \to Y$ with $R|Y = id_Y$.

For the proof of Theorem 2.7, we recall the following result.

**Theorem 2.8** (3.4 [11]). Let $Y$ be a $G$-invariant definable closed subset of a definable $G$-set $X$. Then there exist a $G$-invariant definable open neighborhood $U$ of $Y$ in $X$ and a definable $G$-retraction from $U$ to $Y$.

Proof of Theorem 2.7. Since $G(x)$ is a $G$-invariant definable closed subset of $X$ and by Theorem 2.8, we have a $G$-invariant definable open neighborhood $U$ of $G(x)$ in $X$ and a definable $G$-retraction $q$ from $U$ to $G(x)$. Let $S := q^{-1}(x)$. Then $S$ is a definable $G_x$-set and $U = GS$. By II.4.2 [2], the map $f : G \times_{G_x} S \to GS \subset X$ defined by $f([g, s]) = gs$ is a $G$-homeomorphism. On the other hand, the map $k : G \times S \to GS$ defined by $k(g, s) = gs$ and the projection $\pi : G \times S \to G \times_{G_x} S$ are definable maps. Since the graph of $f$ is the image of that of $k$ by $\pi \times id_{GS}$, $f$ is a definable $G$-homeomorphism.

**Definition 2.9.** A definable $G$-fiber bundle $\eta = (E, p, X, F, K)$ satisfies the definable Bierstone condition if for any $x \in X$, there exist a $G_x$-invariant definable open neighborhood $U_x$ of $x$ in $X$ and a definable group homomorphism $\rho_x : G_x \to K$ such that $\eta|U_x$ is definably $G_x$-isomorphic to $U_x \times F$ with the definable $G_x$-action defined by $G_x \times (U_x \times F) \to U_x \times F, (h, u, y) \mapsto (hu, \rho_x(h)y)$.
Note that a definable $G$-fiber bundle over a definable $G$-set satisfies the definable Bierstone condition if and only if the associated principal definable $G$-fiber bundle satisfies it.

Using Theorem 2.7, similar proofs of 1.4 [15] and 1.5 [15] prove the following proposition.

**Proposition 2.10.** Every definable $G$-fiber bundle over a definable $G$-set satisfies the definable Bierstone condition.

A finite definable open covering $\{U_i\}$ of a definable $G$-set is called a *finite definable open $G$-covering* if each $U_i$ is $G$-invariant. A finite definable $G$-open covering is *numerable* if there exists a definable partition of unity $\{\lambda_i\}$ subordinate to $\{U_i\}$ such that each $\lambda_i$ is $G$-invariant.

The following proposition shows existence of (non-equivariant) definable partition of unity.

**Proposition 2.11 (e.g. 6.3.7 [6]).** Let $X$ be a definable set in $\mathbb{R}^n$ and $\{U_i\}_{i=1}^n$ a finite definable open covering of $X$. Then there exists a definable partition of unity subordinate to $\{U_i\}_{i=1}^n$, namely there exist definable functions $\lambda_1, \ldots, \lambda_n : X \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U_i$ and $\sum_{i=1}^n \lambda_i = 1$.

The following is an equivariant version of Proposition 2.11.

**Proposition 2.12 (Equivariant definable partition of unity).** Every finite definable open $G$-covering of a definable $G$-set $X$ is numerable.

**Proof.** Let $\{U_i\}_{i=1}^n$ be a finite definable open $G$-covering of a definable $G$-set $X$. By Corollary 2.5, the orbit map $p : X \to X/G$ is a surjective proper definable map. Since $p : X \to X/G$ is open, $\{p(U_i)\}_{i=1}^n$ is a finite definable open covering of $X/G$. By Proposition 2.11, one can find a definable partition of unity $\{\lambda_i\}_{i=1}^n$ subordinate to $\{p(U_i)\}_{i=1}^n$. Hence $\lambda_1 := \bar{\lambda}_1 \circ p, \ldots, \lambda_n := \bar{\lambda}_n \circ p$ are $G$-invariant and subordinate to $\{U_i\}_{i=1}^n$. \qed

Note that in Proposition 2.11 and 2.12, we can replace $\sum_{i=1}^n \lambda_i = 1$ by $\max_{1 \leq i \leq n} \lambda_i = 1$.

Theorem 1.1 follows from Theorem 2.13 below.

**Theorem 2.13.** If $X$ is a compact definable $G$-set, then every definable $G$-fiber bundle $\eta = (E, p, X \times [0,1], F, K)$ is definably $G$-isomorphic to $(p^{-1}(X \times \{0\}) \times [0,1], p', X \times [0,1], F, K)$, where $G$ acts on $[0,1]$ trivially, $X \times \{0\}$ is identified with $X$ and $p' = p|_{p^{-1}(X \times \{0\})} \times \text{id}_{[0,1]}$.

To prove Theorem 2.13, we need the following three results.

**Lemma 2.14.** Let $A$ be a definable $G$-set, $X_1 = A \times [a, b], X_2 = A \times [b, c]$, and $\eta = (E, p, X, F, K)$ a definable $G$-fiber bundle over $X = X_1 \cup X_2$, where $G$ acts trivially on $[a, b]$ and $[b, c]$. If $\eta|_{X_1}$ and $\eta|_{X_2}$ are definably $G$-isomorphic to $X_1 \times F$ and $X_2 \times F$, respectively, then so is $\eta$, where the action on $F$ is induced by a definable group homomorphism from $G$ to $K$.

**Proof.** Let $u_i : X_1 \times F \to p^{-1}(X_i), (i = 1, 2)$, be definable $G$-isomorphisms and $w_i := u_i|_{(X_1 \cap X_2) \times F}, (i = 1, 2)$. Then $h := w_2^{-1} \circ w_1 : (X_1 \cap X_2) \times F \to (X_1 \cap X_2) \times F$...
is a definable $G$-isomorphism. Hence there exists a definable map $l : X_1 \cap X_2 \to K$ such that $h(x, y) = (x, l(x)y)$, where $(x, y) \in (X_1 \cap X_2) \times F$. Let $i_A : A \to K, i_A(a) = l(a, b)$. Then we can extend $h$ to a definable $G$-isomorphism

$$\tilde{h} : X_2 \times F \to X_2 \times F, \tilde{h}(x_1, x_2, y) = (x_1, x_2, i_A(x_1)y).$$

Since two definable $G$-isomorphisms $u_1 : X_1 \times F \to p^{-1}(X_1)$ and $u_2 \circ \tilde{h} : X_2 \times F \to p^{-1}(X_2)$ coincide on $(X_1 \cap X_2) \times F$ and $X_1 \times F$ and $X_2 \times F$ are closed in $(X_1 \cup X_2) \times F = X \times F$, the gluing map provides the required definable $G$-isomorphism.

Let $H$ be a definable subgroup of $G$, $\rho : H \to K$ a definable group homomorphism between definable groups, and $F$ a definable set with an effective definable $K$-action. For any definable $H$-set $S$, we define a definable $G$-fiber bundle $\epsilon^\rho(S)$ by $(G \times_H (S \times F), p, G \times F, S, F, K)$, where $p : G \times_H (S \times F) \to G \times_H S, p([g, (s, y)]) = [g, s]$ and $H$ acts on $F$ via $\rho$.

**Lemma 2.15.** Let $X$ be a compact definable $G$-set and $\eta = (E, p, X \times [0, 1], F, K)$ a definable $G$-fiber bundle over $X \times [0, 1]$. Then there exist finitely many points $x_1, \ldots, x_n$ with definable slices $S_{x_1}, \ldots, S_{x_n}$ and definable group homomorphisms $\{\rho_i : G_{x_i} \to K\}_{i=1}^n$ such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open $G$-covering of $X$ and each $\eta|([GS_{x_i} \times [0, 1])$ is definably $G$-equivalent to $\epsilon^\rho(S_{x_i}) \times [0, 1]$.

**Proof.** By Proposition 2.10, for any $(x, t) \in X \times [0, 1]$, there exist a $G_x$-invariant definable open neighborhood $U_x$ of $x$ in $X$ and $\delta > 0$ such that $\eta|U_x \times [t-\delta, t+\delta]$ is definably $G_x$-isomorphic to $(U_x \times [t-\delta, t+\delta]) \times F$, where the action on $F$ is induced by a definable group homomorphism $\rho_x : G_x \to K$. Since $[0, 1]$ is compact and by Lemma 2.14, we have a $G_x$-invariant definable open neighborhood $V_x$ of $x$ in $X$ such that $\eta|V_x \times [0, 1]$ is definably $G_x$-isomorphic to $(V_x \times [0, 1]) \times F$. By Theorem 2.7, we have a definable slice $S_x$ at $x$ with $S_x \subset V_x$. Hence there exists a definable $G_x$-isomorphism $l_x : S_x \times [0, 1] \times F \to \eta|S_x \times [0, 1]$. Thus $h_x : G \times G_x (S_x \times [0, 1] \times F) = \epsilon^\rho(S_x) \times [0, 1] \to \eta|GS_x \times [0, 1]$ defined by $h_x([g, (s, t, f)]) = g l_x(s, t, f)$ is a definable $G$-equivalence. Since $X$ is compact, there exist finitely many points $x_1, \ldots, x_n$ of $X$ such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open $G$-covering of $X$.

**Theorem 2.16.** Let $X$ be a compact definable $G$-set, $r : X \times [0, 1] \to X \times [0, 1], r(x, t) = (x, 1)$ and $\eta = (E, p, X \times [0, 1], F, K)$ a definable $G$-fiber bundle over $X \times [0, 1]$. Then there exists a definable $G$-isomorphism $\phi : E \to E$ covering $r$.

**Proof.** By Lemma 2.15, we can find finitely many points $x_1, \ldots, x_n$ with definable slices $S_{x_1}, \ldots, S_{x_n}$ and definable group homomorphisms $\{\rho_i : G_{x_i} \to K\}_{i=1}^n$ such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open $G$-covering of $X$ and each $\eta|[GS_{x_i} \times [0, 1])$ is definably $G$-equivalent to $\epsilon^\rho_i(S_{x_i}) \times [0, 1]$. By Proposition 2.12, there exist $G$-invariant definable functions $l_1, \ldots, l_n : X \to [0, 1]$ such that:

(a) The support of each $l_i$ is contained in $GS_{x_i}$.

(b) $\max_{1 \leq i \leq n} l_i(x) = 1$ for all $x \in X$.

Let $h_x : (G \times G_{x_i} (S_{x_i} \times F)) \times [0, 1] \to p^{-1}(GS_{x_i} \times [0, 1])$ be a definable $G$-equivalence covering a definable $G$-homeomorphism $f_{x_i} \times id_{[0, 1]} : (G \times G_{x_i} S_{x_i}) \times [0, 1] \to GS_{x_i} \times [0, 1]$. 


Define
\[(u_i, r_i) : (E, X \times [0, 1]) \to (E, X \times [0, 1]), 1 \leq i \leq n,\]
\[r_i(x, t) = \begin{cases} (x, \max(l_i(f_{x_i}([g, s])), t)), & ([g, s], t) \in (G \times g_x, S_{x_i}) \times [0, 1] \\ (x, t), & \text{otherwise} \end{cases},\]
\[u_i(h_{x_i}([g, (s, f)], t)) = h_{x_i}([g, (s, f)], \max(l_i(f_{x_i}([g, s])), t)), \]
for any \([([g, (s, f)], t) \in (G \times g_x, (S_{x_i} \times F)) \times [0, 1],\]
\[u_i\text{ is the identity outside } p^{-1}(GS_{x_i} \times [0, 1]).\]
Then \(r = r_n \circ \cdots \circ r_1.\) Therefore \(\phi = u_n \circ \cdots \circ u_1 : E \to E\) is the required definable G-morphism.

Theorem 2.13 follows from Theorem 2.16.

3. Definable G-vector bundles and proof of Theorem 1.2, 1.3 and Corollary 1.4

We recall that \(G\) and \(K\) denote compact subgroups of \(GL_n(\mathbb{R})\) except section 2. Then remember that \(G\) is a compact algebraic subgroup of \(GL_n(\mathbb{R})\) and any closed subgroup of \(G\) is a compact algebraic subgroup of \(G\).

Note that a definable group homomorphism from \(G\) to \(O_n(\mathbb{R})\) is a definable \(C^\infty\)-map because it is a continuous group homomorphism between Lie groups.

Recall universal G-vector bundles (e.g. [12]).

Definition 3.1. Let \(\Omega\) be an \(n\)-dimensional representation of \(G\) induced by a definable group homomorphism \(B : G \to O_n(\mathbb{R})\) of \(\Omega\). Suppose that \(M(\Omega)\) denotes the vector space of \(n \times n\)-matrices with the action \((g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)\). For any positive integer \(k\), we define the vector bundle \(\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))\) as follows:
\[G(\Omega, k) = \{ A \in M(\Omega) | A^2 = A, A = A', Tr A = k \},\]
\[E(\Omega, k) = \{ (A, v) \in G(\Omega, k) \times \Omega | Av = v \},\]
\[u : E(\Omega, k) \to G(\Omega, k), u((A, v)) = A,\]
where \(A'\) denotes the transposed matrix of \(A\) and \(Tr A\) stands for the trace of \(A\). Then \(\gamma(\Omega, k)\) is an algebraic vector bundle. Since the action on \(\gamma(\Omega, k)\) is algebraic, it is an algebraic G-vector bundle. We call it the universal G-vector bundle associated with \(\Omega\) and \(k\). Remark that \(G(\Omega, k) \subset M(\Omega)\) and \(E(\Omega, k) \subset M(\Omega) \times \Omega\) are nonsingular algebraic G-sets.

Definition 3.2. (1) A definable G-vector bundle of rank \(k\) is a definable G-fiber bundle with fiber \(\mathbb{R}^k\) and structure group \(GL_k(\mathbb{R})\). We usually write \((E, p, X)\) instead of \((E, p, X, \mathbb{R}^k, GL_k(\mathbb{R}))\).

(2) Let \(\eta = (E, p, X)\) and \(\eta' = (E', p', X)\) be definable G-vector bundles. A definable G-map \(f : E \to E'\) is called a definable G-morphism if \(p = p' \circ f\) and \(f\) is linear on each fiber. A definable G-morphism \(h : E \to E'\) is said to be a definable G-isomorphism if there exists a definable G-morphism \(h' : E' \to E\) such that \(h \circ h' = id\) and \(h' \circ h = id\).

By a way similar to 3.1 [10], we have the following proposition.

**Proposition 3.3.** If $\eta$ and $\eta'$ are two definable $G$-vector bundles over a definable $G$-set $X$, then $\eta \oplus \eta'$, $\eta \otimes \eta'$, $\text{Hom}(\eta, \eta')$ and the dual bundle $\eta^\vee$ of $\eta$ are definable $G$-vector bundles over $X$.

The next result states equivalent properties of strong definability of definable $G$ vector bundles, which is obtained in a way similar to the proof of 3.6 [3].

**Theorem 3.4.** Let $\eta = (E,p,X)$ be a definable $G$-vector bundle of rank $k$ over a definable $G$-set $X$. Then the following five properties are equivalent:

1. The bundle $\eta$ is strongly definable.
2. There exists a surjective definable $G$-morphism from a trivial $G$-vector bundle $X \times \Omega$ onto $\eta$ for some representation $\Omega$ of $G$.
3. There exists an injective definable $G$-morphism from $\eta$ to a trivial $G$-vector bundle $X \times \Omega$ for some representation $\Omega$ of $G$.
4. There exists a definable $G$-vector bundle $\eta'$ over $X$ such that $\eta \oplus \eta'$ is definably $G$-isomorphic to a trivial $G$-vector bundle.
5. There exist non-equivariant definable sections $s_1, \ldots, s_n : X \to E$ of $\eta$ such that:
   (a) For any $x \in X$, the vectors $s_1(x), \ldots, s_n(x)$ generate the fiber $p^{-1}(x)$ over $x$.
   (b) The sections $s_1, \ldots, s_n$ generate a finite dimensional $G$-invariant vector subspace of $\Gamma(\eta)$, where $\Gamma(\eta)$ denotes the set of all continuous sections of $\eta$ with the natural $G$-action, namely $(g \cdot s)(x) = g(s(g^{-1}x))$ for all $g \in G$ and $x \in X$.

Theorem 1.2 follows from Theorem 3.4 and Theorem 3.5 below.

**Theorem 3.5.** Every definable $G$ vector bundle over a definable $G$ set satisfies Condition (5) in Theorem 3.4.

By a way similar to the proof of 3.9 [3], we have the following proposition.

**Proposition 3.6.** Let $\eta = (E,p,X)$ be a definable $G$-vector bundle over a definable set $X$ with the trivial $G$-action and $A$ a closed definable subset of $X$ such that $\eta|A$ is strongly definable. If $A$ admits a definable retraction from $X$ to $A$, then there exists some open definable neighborhood $V$ of $A$ in $X$ such that $\eta|V$ is strongly definable.

The following is the equivariant definable version of Urysohn’s lemma, and its semialgebraic version is proved in 1.6 [5]. We use only a non-equivariant version of it to prove Theorem 3.5.

**Lemma 3.7.** Let $X$ be a definable set with a definable $G$-action and $A$ and $B$ disjoint closed definable $G$-subsets of $X$. Then there exists a $G$-invariant definable function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

**Proof.** By Corollary 2.5, $X/G$ is a definable subset of some $\mathbb{R}^n$ and the orbit map $p : X \to X/G$ is a surjective proper definable map. Hence $\pi(A)$ and $\pi(B)$ are closed definable
subsets of $X/G$. Then the function $h : X/G \to [0, 1]$ defined by $h(x) = \frac{d(x, \pi(A))}{d(x, \pi(A)) + d(x, \pi(B))}$ is a definable function such that $h^{-1}(0) = \pi(A)$ and $h^{-1}(1) = \pi(B)$, where $d(x, \pi(A))$ (respectively $d(x, \pi(B))$) denotes the distance between $x$ and $\pi(A)$ (respectively $x$ and $\pi(B)$). Therefore $f := h \circ \pi : X \to [0, 1]$ is the required $G$-invariant definable function.

\[ \square \]

**Proposition 3.8.** Let $H$ be a closed subgroup of $G$, $D$ the closed unit ball of a representation $\Omega$ of $H$. Then $G \times_H D$ is a compact affine definable $C^\infty G$ manifold with boundary. In particular, $G \times_H D$ is definably imbeddable into some representation of $G$.

**Proof.** Note that $G$ and $\Omega$ are affine definable $C^\infty H$-manifolds. Thus by 4.4 [13] and 4.5 [13], $G \times_H \Omega$ is a definable $C^\infty G$-manifold whose underlying manifold is a definable $C^\infty$-submanifold of some $\mathbb{R}^k$. Since $G \times_H D$ is compact, there exists a $C^\infty G$-imbedding $i$ from $G \times_H D$ to some representation $\Xi$ of $G$. Applying the polynomial approximation theorem to $i$ and averaging it, we have a definable $C^\infty G$-imbedding from $G \times_H D$ to $\Xi$. \[ \square \]

A **definable $G$-CW-complex** is a finite $G$-CW-complex such that the characteristic map of each $G$-cell is a definable $G$-map (see [11]).

**Theorem 3.9** (1.1 [11]). Let $X$ be a definable $G$-set and $Y$ a closed definable $G$-subset of $X$. Then there exist a definable $G$-CW-complex $Z$ in a representation $\Omega$ of $G$, a $G$-CW-subcomplex $W$ of $Z$, and a definable $G$-map $f : X \to Z$ such that:

1. The map $f$ takes $X$ and $Y$ definably-homeomorphically onto $G$-invariant definable subsets $Z_1$ and $W_1$ of $Z$ and $W$, respectively.
2. The orbit map $\pi : Z \to Z/G$ is a definable cellular map.
3. The orbit space $Z/G$ is a finite simplicial complex compatible with $\pi(Z_1)$ and $\pi(W_1)$.
4. For each open $G$-cell $c$ of $Z$, $\pi|_{\bar{c}} : \bar{c} \to \pi(\bar{c})$ has a definable section $s : \pi(\bar{c}) \to \bar{c}$, where $\bar{c}$ denotes the closure of $c$ in $Z$.

Furthermore, if $X$ is compact, then $Z = f(X)$ and $W = f(Y)$.

Using Proposition 3.6, Lemma 3.7, Proposition 3.8, Theorem 3.9, a similar proof of 3.5 [3] proves Theorem 3.5.

By Theorem 1.2 and by the proof of 4.7 [11], we have the following.

**Proposition 3.10.** Let $\eta$ a definable $G$-vector bundle over a compact definable $G$-set $X$. Then every continuous $G$-section of $\eta$ can be approximated by definable $G$-sections.

We obtain the following theorem using Proposition 3.3 and Proposition 3.10.

**Theorem 3.11.** Let $\eta$ and $\zeta$ be definable $G$-vector bundles over a compact definable $G$-set. If $\eta$ is $G$-isomorphic to $\zeta$, then they are definably $G$-isomorphic.

**Proposition 3.12** (2.11 [15]). Let $X, Y$ be definable $G$-sets. If $\eta$ is $G$-vector bundle over $Y$ and $f, h : X \to Y$ are $G$-homotopic continuous $G$-maps, then $f^*(\eta)$ is $G$-isomorphic to
DEFINABLE G-FIBER BUNDLES AND DEFINABLE C²G-FIBER BUNDLES

Proposition 3.13 ([1], [20]). Let \( X \) be a compact \( G \)-set. If \( \eta \) is a \( G \)-vector bundle over \( X \), then there exist a representation \( \Omega \) of \( G \) and a continuous \( G \)-map \( f : X \to G(\Omega, k) \) such that \( \eta \) is \( G \)-isomorphic to \( f^*(\gamma(\Omega, k)) \). where \( k \) denotes the rank of \( \eta \).

Theorem 3.14. If \( X \) is a compact definable \( G \)-set, \( \kappa : \text{Vect}_G^\text{def}(X) \to \text{Vect}_G(X) \) is bijective.

Proof. Injectivity follows from Theorem 3.11.
Let \( \eta \) be a \( G \)-vector bundle over \( X \). Then by Proposition 3.13, there exist a representation \( \Omega \) of \( G \) and a continuous \( G \)-map \( f : X \to G(\Omega, k) \) such that \( \eta \) is \( G \)-isomorphic to \( f^*(\gamma(\Omega, k)) \), where \( k \) denotes the rank of \( \eta \). By 3.5 [11], \( f \) is \( G \)-homotopic to a definable \( G \)-map \( h : X \to G(\Omega, k) \). Hence by Proposition 3.12, \( f^*(\gamma(\Omega, k)) \) is \( G \)-isomorphic to \( h^*(\gamma(\Omega, k)) \). Therefore \( \eta \) is \( G \)-isomorphic to a definable \( G \)-vector bundle \( h^*(\gamma(\Omega, k)) \).

A \( G \)-set \( X \) is \( G \)-contractible if there exist a fixed point \( x_0 \in X \) and a continuous \( G \)-map \( F : X \times [0, 1] \to X \) such that \( F(x, 0) = x \) and \( F(x, 1) = x_0 \) for all \( x \in X \), where \( G \) acts on \([0, 1]\) trivially. We have the following as a corollary of Theorem 1.1.

Corollary 3.15. Let \( X \) be a compact \( G \)-contractible definable \( G \)-set. Then every definable \( G \)-vector bundle over \( X \) is definably \( G \)-isomorphic to a trivial \( G \)-bundle.

Theorem 3.16 (3.3 [11]). Let \( X \) be a definable \( G \)-set. Then there exists a definable \( G \)-deformation retraction \( R \) from \( X \) to a compact definable \( G \)-subset \( Y \) of \( X \).

By a way similar to the proof of 4.10 [11], we have the following proposition.

Proposition 3.17. The map \( R^* : \text{Vect}_G^\text{def}(Y) \to \text{Vect}_G^\text{def}(X) \) defined by \( \eta \mapsto R^*(\eta) \) is bijective.

Theorem 1.3 follows from Theorem 3.14 and Proposition 3.17. Corollary 1.4 follows from Theorem 1.3 and Proposition 3.12.

4. Definable \( C^r \)-Fiber Bundles and Definable \( C^r \)-Vector Bundles

Definition 4.1 ([12]). Let \( 1 \leq r \leq \omega \).

1. A definable fiber bundle \( \eta = (E, p, X, F, K) \) is a definable \( C^r \)-fiber bundle if the total space \( E \) and the base space \( X \) are definable \( C^r \)-manifolds, the structure group \( K \) is a definable \( C^r \)-group, the fiber \( F \) is a definable \( C^r \)-manifold with an effective action, the projection \( p \) is a definable \( C^r \)-map and all transition functions of \( \eta \) are definable \( C^r \)-maps. A principal definable \( C^r \)-fiber bundle is defined similarly.

2. Definable \( C^r \)-morphisms, definable \( C^r \)-equivalences, definable \( C^r \)-isomorphisms between definable \( C^r \)-fiber bundles and definable \( C^r \)-sections of a definable \( C^r \) fiber bundle are defined similarly.

3. A definable \( C^r \)-fiber bundle \( \eta = (E, p, X, F, K) \) is a definable \( C^r \)-G-fiber bundle if the total space \( E \) and the base space \( X \) are definable \( C^r \)-manifolds, the projection \( p \) is a definable \( C^r \)-map and \( G \) acts on \( E \) through definable \( C^r \)-equivalences. A principal definable \( C^r \)-G-fiber bundle is defined similarly.
(4) A definable $C^r$-morphism (resp. a definable $C^r$-equivalence, a definable $C^r$-isomorphism, a definable $C^r$-section) is a definable $C^rG$-morphism (resp. a definable $C^rG$-equivalence, a definable $C^rG$-isomorphism, a definable $C^rG$-section) if it is a $G$-map.

The following is a definable $C^rG$-version of Proposition 2.6, which is obtained similarly.

**Proposition 4.2.** Suppose that $1 \leq r \leq \omega$.

1. Let $(E, p, X, K)$ be a principal definable $C^rG$-fiber bundle and $F$ an affine definable $C^rK$-manifolds with an effective action. Then $(E \times_K F, p', X, F, K)$ is a definable $C^rG'$-fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, k]) = p(z)$.

2. The associated principal $G$-fiber bundle of a definable $C^rG$-fiber bundle is a principal definable $C^rG$-fiber bundle.

3. Two definable $C^rG$-fiber bundles having the same base space, fiber and structure group are definably $C^rG$-isomorphic if and only if their associated principal definable $C^rG$-fiber bundles are definably $C^rG$-isomorphic.

**Proposition 4.3.** Let $X$ be a definable $C^rG$-submanifold of a representation $\Omega$ of $G$ and $1 \leq r < \infty$. Then for any $x \in X$, there exists a linear definable $C^r$-slice at $x$ in $X$, namely, there exists a definable $C^rG_x$-imbedding $i$ from a representation $\Xi$ of $G_x$ into $X$ such that $i(0) = x$, $G \times_{G_x} \Xi$ is a definable $C^rG$-manifold with the standard action $(g, [g', x]) \mapsto [gg', x]$ and the map $\mu : G \times_{G_x} \Xi \to X$ defined by $[g, x] \mapsto gi(x)$ is a definable $C^rG$-diffeomorphism onto some $G$-invariant definable open neighborhood of $G(x)$ in $X$.

**Proof.** Since $G$ is a compact algebraic subgroup of $GL_n(\mathbb{R})$ and by 4.1 [13], for any $x \in X$, there exists a linear definable $C^\infty$ slice at $x$ in $\Omega$, namely, we have a representation $\Xi'$ of $G_x$ and a definable $C^\infty$ imbedding $j : \Xi' \to \Omega$ such that $j(0) = x$, $G \times_{G_x} \Xi'$ is a definable $C^\infty$ manifold and the map $\mu' : G \times_{G_x} \Xi' \to \Omega$ defined by $\mu'([g, x]) = gj(x)$ is a definable $C^\inftyG$ dffeomorphism onto a $G$ invariant definable open neighborhood $Gj(\Xi')$ of $G(x)$ in $\Omega$. Then $j^{-1}(X)$ is a definable $C^rG_x$ submanifold of $\Xi'$ and $j|j^{-1}(X) : j^{-1}(X) \to X$ is a definable $C^rG_x$ imbedding. Hence there exists a sufficiently small $G_x$ invariant definable open neighborhood $U$ of $0$ in $j^{-1}(X)$ such that $U$ is definably $C^rG_x$ diffeomorphic to a representation $\Xi$ of $G_x$. Take a definable $C^rG_x$ diffeomorphism $l : \Xi \to U$ with $l(0) = 0$ and let $i = j \circ l$. Then $i$ is a definable $C^rG_x$ imbedding from $\Xi$ to $X$ and the map $\mu : G \times_{G_x} \Xi \to X$ defined by $\mu([g, x]) = gi(x)$ is a definable $C^rG$ diffeomorphism onto a $G$ invariant definable open neighborhood $Gi(\Xi) = Gj(U)$ of $G(x)$ in $X$. \qed

Note that if $r = \infty$ or $\omega$, then Proposition 4.3 is proved in 4.1 [13].

We can consider the definably $C^r$-Bierstone condition as a definable $C^rG$-version of Definition 2.9. Using Proposition 4.2 and 4.3, we have the following definable $C^r$-version of Proposition 2.10.

**Proposition 4.4.** Let $1 \leq r \leq \omega$. Then every definable $C^rG$-fiber bundle over an affine definable $C^rG$-manifold satisfies the definable $C^r$-Bierstone condition.

The proof of 4.8 [12] proves the following.
Proposition 4.5 (4.8 [12]). (Definable $C^\ast$-partition of unity). Let $X$ be a definable closed subset of $\mathbb{R}^n$, $\{U_i\}_{i=1}^n$ a finite definable open covering of $X$ and $0 \leq r < \infty$. Then there exist definable $C^\ast$ functions $\lambda_1, \ldots, \lambda_l : \mathbb{R}^n \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^n \lambda_i(x) = 1$ for any $x \in X$.

The following is a definable $C^\ast$-version of Proposition 2.12.

Proposition 4.6 (Equivariant definable $C^\ast$-partition of unity). Let $X$ be a definable $C^\ast\mathbb{G}$-submanifold closed in a representation $\Omega$ of $\mathbb{G}$ and $\{U_i\}_{i=1}^n$ a finite definable open $G$-covering of $X$ and $0 \leq r < \infty$. Then $\{U_i\}_{i=1}^n$ is numerable, namely there exist $G$-invariant definable $C^\ast$-functions $\lambda_1, \ldots, \lambda_n : X \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^n \lambda_i(x) = 1$ for any $x \in X$.

Proof. First of all, we recall the structure of the orbit space $\Omega/G$. The algebra $\mathbb{R}[\Omega]^G$ of $G$ invariant polynomials on $\Omega$ is finitely generated [21]. Let $p_1, \ldots, p_n : \Omega \to \mathbb{R}$ be $G$ invariant polynomials generating $\mathbb{R}[\Omega]^G$, and put $p : \Omega \to \mathbb{R}^n, p = (p_1, \ldots, p_n)$. Then $p$ is a proper polynomial map, and it induces a closed imbedding $j : \Omega/G \to \mathbb{R}^n$ such that $p = j \circ \pi$, where $\pi : \Omega \to \Omega/G$ denotes the orbit map. Hence we can identify $\Omega/G$ (resp. $X/G$, $\pi$) with $j(\Omega/G)$ (resp. $j(X/G)$, $p$). Thus $\{p(U_i)\}_{i=1}^l$ is a finite definable open covering of $X/G$ because $p|X : X \to X/G$ is open. Note that $p(X)$ is closed in $\mathbb{R}^n$ because $X$ is closed in $\Omega$. By Proposition 4.5, one can find a definable partition of unity $\{\lambda_i\}_{i=1}^l$ subordinate to $\{p(U_i)\}_{i=1}^l$. Hence $\lambda_1 := \lambda_1 \circ p, \ldots, \lambda_l := \lambda_l \circ p$ are the required $G$ invariant definable $C^\ast$ functions.

We can replace $\sum_{i=1}^n \lambda_i = 1$ by $\max_{1 \leq i \leq n} \lambda_i = 1$ in Proposition 4.5 and 4.6.

By the proof of 2.10 [12], we may assume that an affine definable $C^\ast\mathbb{G}$-manifold is a definable $C^\ast\mathbb{G}$-submanifold closed in some representation $\Omega$ of $\mathbb{G}$. Thus similar proofs of Lemma 2.14, 2.15 and Theorem 2.16 prove the following.

Theorem 4.7. If $X$ is a compact affine definable $C^\ast\mathbb{G}$-manifold and $1 \leq r < \infty$, then every definable $C^\ast\mathbb{G}$-fiber bundle $\eta = (E, p, X \times [0, 1], F, K)$ is definably $C^\ast\mathbb{G}$-isomorphic to $(p^{-1}(X \times \{0\}) \times [0, 1], p', X \times [0, 1], F, K)$, where $G$ acts on $[0, 1]$ trivially, $X \times \{0\}$ is identified with $X$ and $p' = p|p^{-1}(X \times \{0\}) \times id_{[0, 1]}$.

Theorem 1.5 follows from Theorem 4.7.

The following result is a definable $C^\ast\mathbb{G}$-version of Theorem 3.4, which is obtained similarly.

Theorem 4.8. Let $\eta = (E, p, X)$ be a definable $C^\ast\mathbb{G}$-vector bundle of rank $k$ over an affine definable $C^\ast\mathbb{G}$-manifold $X$ and $1 \leq r < \infty$. Then the following five properties are equivalent.

1. The bundle $\eta$ is strongly definable.
2. There exists a surjective definable $C^\ast\mathbb{G}$-morphism from a trivial $G$-vector bundle $X \times \Omega$ onto $\eta$ for some representation $\Omega$ of $\mathbb{G}$.
3. There exists an injective definable $C^\ast\mathbb{G}$-morphism from $\eta$ to a trivial $G$-vector bundle $X \times \Omega$ for some representation $\Omega$ of $\mathbb{G}$.
4. There exists a definable $C^\ast\mathbb{G}$-vector bundle $\eta'$ over $X$ such that $\eta \oplus \eta'$ is definably $C^\ast\mathbb{G}$-isomorphic to a trivial $G$-vector bundle.
(5) There exist non-equivariant definable $C^r$-sections $s_1, \ldots, s_n : X \to E$ of $\eta$ such that:
(a) For any $x \in X$, the vectors $s_1(x), \ldots, s_n(x)$ generate the fiber $p^{-1}(x)$ over $x$.
(b) The sections $s_1, \ldots, s_n$ generate a finite dimensional $G$-invariant vector subspace of $\Gamma(\eta)$.

Proof of Theorem 1.7. Since $X$ is compact, a similar proof of Lemma 2.15 proves that there exist finitely many points $x_1, \ldots, x_n \in X$ with definable $C^r$-slices $S_{x_1}, \ldots, S_{x_n}$ and $\alpha$-dimensional representations $\Omega_{x_1}, \ldots, \Omega_{x_n}$ of $G_{x_1}, \ldots, G_{x_n}$, respectively, such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open $G$-covering of $X$ and each $\eta|GS_{x_i}$ is definably $C^r$-G-equivalent to $\epsilon(S_{x_i})$, where $\epsilon(S_{x_i}) = (G \times_{G_{x_i}} (S_{x_i} \times \Omega_{x_i}), p, G \times_{G_{x_i}} S_{x_i}), p : G \times_{G_{x_i}} (S_{x_i} \times \Omega_{x_i}) \to G \times_{G_{x_i}} S_{x_i}, p([g, x, y]) = [g, x]$ and $\alpha$ denotes the rank of $\eta$. Clearly each $\epsilon(S_{x_i})$ admits finitely many definable $C^r$-sections satisfying Condition (5) in Theorem 4.8. Thus every $\eta|GS_{x_i}$ admits definable $C^r$-sections $s_{i1}, \ldots, s_{it_i}$ satisfying the same condition.

By Proposition 4.6, we have an equivariant definable $C^r$-partition of unity $\{\lambda_i\}_{i=1}^n$ subordinate to $\{GS_{x_i}\}_{i=1}^n$. Let $\overline{s_{iq}} := \lambda_i s_{iq}$. Then for any $g \in G$, $g \cdot \overline{s_{iq}} = \lambda_i (g \cdot s_{iq})$. Therefore a finite family of definable $C^r$-sections $\overline{s_{11}}, \ldots, \overline{s_{it_1}}, \ldots, \overline{s_{n1}}, \ldots, \overline{s_{n1}}$ satisfies the required conditions.

Now we prove the second part of the theorem. If $\eta$ is strongly definable, then there exist a representation $\Omega$ of $G$ and a definable $C^r$-map $f$ from $X$ to $G(\Omega, \alpha)$ such that $\eta$ is definably $C^r$-isomorphic to $f^* (\gamma(\Omega, \alpha))$. Since the total space of $f^* (\gamma(\Omega, \alpha))$ is affine, $E$ is affine.

Conversely, we assume that $E$ is a definable $C^r$-submanifold of a representation $\Xi$ of $G$.

Let

\begin{align*}
F_1 : X \to M(\Xi), & F_1(x) = \text{the matrix projecting } T_x \Xi \text{ onto } T_x E, \\
F_2 : X \to M(\Xi), & F_2(x) = \text{the matrix projecting } T_x \Xi \text{ onto } T_x X.
\end{align*}

Then by a way similar to the proof of I.3.3 [19], $F_1$ and $F_2$ are definable maps. Thus they are definable $C^r$-maps. By the definition of $G$-action, they are $G$-maps. Hence they are definable $C^r$-maps. Let

\[ F : X \to G(\Xi, \alpha), F = (id - F_2)F_1. \]

Then $F$ is a definable $C^r$-map and $\eta$ is definably $C^r$-isomorphic to $F^* (\gamma(\Xi, \alpha))$. Therefore $\eta$ is strongly definable.

\[
\square
\]

References

DEFINABLE G-FIBER BUNDLES AND DEFINABLE C¹G-FIBER BUNDLES