SK INVARIANTS FOR G-MANIFOLDS WITH BOUNDARY

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Let $G$ be a finite abelian group. A $G$-manifold means an unoriented compact smooth manifold, which may have boundary, together with a smooth action of $G$. Let $N_i (i = 1, 2)$ be $G$-manifolds with the same dimension, $L$ a codimension zero invariant submanifold of each boundary $\partial N_i$ and $\varphi, \psi : L \to L$ $G$-equivariant diffeomorphisms. Pasting along $L$, we have $G$-manifolds $M_1 = N_1 \cup_{\varphi} N_2$ and $M_2 = N_1 \cup_{\psi} N_2$. Then $M_1$ and $M_2$ are said to be obtained from each other by an equivariant cutting and pasting or a $G$-SK process. The abbreviation SK stands for Schneiden und Kleben in German.

**Definition.** Consider a map $T$ defined for all $G$-manifolds which takes its values in the ring $\mathbb{Z}$ of rational integers and is additive with respect to the disjoint union of $G$-manifolds. We call $T$ a $G$-SK invariant or simply an invariant if it is invariant under the $G$-SK process, i.e., $T(M_1) = T(M_2)$ for the above $M_1$ and $M_2$. Further, such a $T$ is said to be multiplicative if $T(M \times N) = T(M) \cdot T(N)$ for any $G$-manifolds $M$ and $N$.

As an example, $\chi^H$ given by $\chi^H(M) = \chi(M^H)$ is a multiplicative invariant, where $H \leq G$, a subgroup of $G$, and $\chi$ is the Euler characteristic.

The purpose of this note is to characterize a form of multiplicative invariants.

By a $G$-slice type, we mean a pair $\sigma = [H; V]$ of $H (\leq G)$ and an $H$-module $V$, i.e., a finite-dimensional real vector space together with a natural linear action of $H$ which satisfies that $V^G = \{0\}$. Let $St(G)$ be the set of all $G$-slice types. There exists a partial ordering on $St(G)$ as follows: $[H; V] \preceq [K; W]$ means that $H \leq K$ and $W = V \oplus W^H$ as $H$-modules. In this case, we denote $[K; W]_H = [H; V]$.

Let $SK^G(\partial)$ be an SK group resulting from equivariant cuttings and pastings of $G$-manifolds.
Proposition (cf.[1], [2]). $SK_*^G(\partial)$ is a free $SK_*^G$-module with basis $\{[G \times_H D(V)] , [G \times_H D(V \times \mathbb{R})] \mid [H;V] \in St(G)\}$, where $D(V)$ denotes the associated $H$-disk.

An invariant $T$ induces an additive homomorphism $SK_*^G(\partial) \rightarrow \mathbb{Z}$ and denote by $\mathcal{T}_*$ the set of all these homomorphisms. For $\sigma = [H;V]$, let $\chi_\sigma$ be an invariant defined by $\chi_\sigma(M) = \chi(M_\sigma)$, where $M_\sigma$ is a submanifold of $M$ consisting those points $x \in M$ whose slice types $\sigma_x$ satisfy that $\sigma \preceq \sigma_x$. Further, consider an invariant $\theta_\sigma$ as

$$\theta_\sigma(M) := |G/H|^{-1} \left\{ \chi(M_\sigma) + \sum_{H < K \leq G} n_H(K) \left( \sum_{\sigma \prec \tau = [K;W]} \chi(M_\tau) \right) \right\},$$

where an integer $n_H(K)$ for $K$ with $H \leq K \leq G$ is defined inductively as follows:

$n_H(H) = 1$ and $n_H(K) = |K/H| - \sum_{H \leq L < K} n_H(L)$ (the order of $K/H$). By evaluating $\theta_\sigma$ on the basis elements for $SK_*^G(\partial)$ in Proposition, we have the following theorem.

Theorem (cf.[3]). The class $\{\theta_\sigma \mid \sigma \in St(G)\}$ provides a basis for $\mathcal{T}_*$ as a free $\mathbb{Z}$-module.

A multiplicative invariant $T$ is considered to be a ring homomorphism $SK_*^G(\partial) \rightarrow \mathbb{Z}$.

**Definition.** Such a (non-trivial) invariant $T$ is said to be of type $\langle G/H \rangle$ if $H$ is the minimum element with respect to the inclusion $\leq$ of subgroups in the set consisting of those subgroups $K$ of $G$ such that $T(G/K) \neq 0$.

In fact, it is seen from the multiplicative structure of $SK_*^G(\partial)$ that $H = \cap_{\lambda} K_{\lambda}$, where $\{K_\lambda\}$ is the set of all subgroups of $G$ such that $T(G/K_\lambda) \neq 0$. For example, $\chi^H$ is of type $\langle G/H \rangle$.

Theorem (cf.[4]). If $T$ is of type $\langle G \rangle$, then it is uniquely determined by the value $a = T(D^1)$ on the one-dimensional disk $D^1$ with the trivial action and has a form $T(M) = a^{\dim(H)} \chi(M)$ for any $G$-manifold $M$. Here, if $a = 0$, then $a^0$ is regarded as 1.

Let $T$ be a multiplicative invariant of type $\langle G/H \rangle$ with $H \neq \{1\}$ in general and let $\mathcal{V}_T = \{a\} \cup \{\gamma_j\}_j$ be integers given by $a = T(D^1)$ and $\gamma_j = |G/H|^{-1} T(G \times_H D(V_j))$.
on $G$-manifolds $G \times_H D(V_j)$, where $\{V_j\}$ is the complete set of non-trivial irreducible $H$-modules.

Denote by $St[H]$ the set of all $G$-slice types with $H$ as an isotropy subgroup.

**Main Theorem** (cf. [4]). Let $T$ be a multiplicative invariants of type $(G/H)$ with $H \neq \{1\}$. Then it is uniquely determined by the class of integers $\mathcal{V}_T$ and has a form

$$T(M) = \sum_{\sigma \in St[H]} a^{\dim(M \sigma)} \gamma_{\sigma} \cdot \chi(M \sigma)$$

for any $G$-manifold $M$, where $\gamma_{\sigma} = \prod_j \gamma_j^{a(j)}$ if $\sigma = [H; \prod_j V_j^{a(j)}] \in St[H]$. In case where $a$ or $\gamma_j = 0$ for some $j$, we regard $a^0$ or $\gamma_j^0$ as 1 respectively.

**Example.**

Multiplicative invariants $T$ of type $(G/H)$, $H \neq \{1\}$, with $a, \gamma_j \in \{-1, 0, 1\}$:

1. $\gamma_j = 1$ (\forall j),

$$T(M) = \begin{cases} 
\chi(M^H) & \text{if } a = 1, \\
\chi(M^H, 0) & \text{if } a = 0, \\
\chi(M^H, \text{ev}) - \chi(M^H, \text{od}) & \text{if } a = -1,
\end{cases}$$

where $M^H, 0$ is the isolated points of $M^H$ and $M^H, \text{ev}$ (or $M^H, \text{od}$) is the union of even-dimensional (or odd-dimensional) components of $M^H$ respectively.

2. $\gamma_j = -1$ (\forall j),

$$T(M) = \begin{cases} 
\chi(M^H) - \chi(M^H) & \text{if } a = 1, \\
\chi(M^H, 0) - \chi(M^H, 0) & \text{if } a = 0, \\
(-1)^{\dim M} \{\chi(M_{2,+}^H) - \chi(M_{2,-}^H)\} & \text{if } a = -1,
\end{cases}$$

where $M^H_+ = \{x \in M^H \mid l((\sigma_x)_H) \text{; even}\}$, $M^H_- = \{x \in M^H \mid l((\sigma_x)_H) \text{; odd}\}$, $((\sigma_x)_H) \preceq \sigma_x$, $l((\sigma_x)_H) = \sum_j a(j)$; the total length of $((\sigma_x)_H) = [H; \prod_j V_j^{a(j)}]$, $M_{2,+}^H = M_{e_2}^H \cap M^H, 0$ and $M_{2,-}^H = \{x \in M^H \mid l_2((\sigma_x)_H) \text{; even}\}$, $M_{2,-}^H = \{x \in M^H \mid l_2((\sigma_x)_H) \text{; odd}\}$ $(l_2((\sigma_x)_H) = \sum_j a(j)$ summing over all $j$ with $\dim(V_j) = 2$; the total length of the two-dimensional irreducible $H$-modules of $((\sigma_x)_H)$. 


\( (3) \ \gamma_j = 0 \ (\forall j), \)

\[
T(M) = \begin{cases} 
\chi(M_{\sigma^H(0)}) & \text{if } a = 1, \\
0^{\dim(M)}\chi(M^H) & \text{if } a = 0, \\
(-1)^{\dim(M)}\chi(M_{\sigma^H(0)}) & \text{if } a = -1,
\end{cases}
\]

where \( M_{\sigma^H(0)} \) is the union of the components of \( M^H \) with \( \dim(M_{\sigma^H(0)}) = \dim(M) \).

References