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Non-existence of free S^1-actions on the 17-dimensional Kervaire sphere

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Abstract. In 1970, Brumfiel calculated surgery obstructions concerning projective spaces of complex projective spaces. One of his result says that the 9-dimensional Kervaire sphere does not admit any free S^1-actions. In this note, we shall extend his calculation further to the dimension 17 and show that the 17 dimensional Kervaire sphere also does not admit any free S^1-actions. Our result is not limited to dimension 17. In fact, simpler method using mod 64 coefficients enables us to draw similar conclusions in many other higher dimensions.

1 Free S^1-actions on homotopy spheres

Let Σ^{2n+1} be an oriented homotopy sphere with a free S^1-action. Then we have a homotopy equivalence of the orbit space with the complex projective space

(1) $$f : \Sigma/S^1 \to \mathbb{C}P^n$$

and an S^1-equivariant homotopy equivalence

(2) $$\tilde{f} : \Sigma^{2n+1} \to S^{2n+1}.$$

By considering the associated D^2 bundles, we get a homotopy equivalence of $(2n+2)$-manifolds with boundary

(3) $$\bar{f} : D^2 \times_{S^1} \Sigma^{2n+1} \to D^2 \times_{S^1} S^{2n+1}.$$

Suppose in addtion that the homotopy sphere Σ^{2n+1} bounds a parallelizable manifold, then we can take a framed manifold W^{2n+2} with $\partial W = \Sigma^{2n+1}$and a framed normal map

(4) $$g : W^{2n+2} \to D^{2n+2}.$$

By splicing $\bar{f}$ and g along common boundaries, we get a normal map

(5) $F : M^{2n+2} = D^2 \times_{S^1} \Sigma^{2n+1} \cup_{\Sigma^{2n+1}} W^{2n+2} \to \mathbb{C}P^{n+1} = D^2 \times_{S^1} S^{2n+1} \cup_{S^{2n+1}} D^{2n+2}$,

where we omitted the bundle data. We know that the surgery obstruction of the normal map F coincides with the surgery obstruction of the normal map g that lies in the surgery obstructionfg group $L_{2n+2}(1)$.

In this note, we shall only deal with the case where n is even $n = 2k > 4$. Then the surgery obstruction $s_{4k+2}(F) \in L_{4k+2}(1) = \mathbb{Z}/2$ for the normal map F is equal to the Arf-Kervaire invariant $c(W)$. The surgery obstruction $s_{4k}(F|F^{-1}(\mathbb{C}P^{2k})) \in L_{4k}(1) = \mathbb{Z}$ for the codimension two surgery problem

(6) $$F|F^{-1}(\mathbb{C}P^{2k}) : F^{-1}(\mathbb{C}P^{2k}) \to \mathbb{C}P^{2k}$$

vanishes, because $F|F^{-1}(\mathbb{C}P^{2k})$ is already a homotopy equivalence f of (1). We shall call a $(4k+1)$-homotopy sphere the Kervaire sphere if it bounds a parallelizable manifold with nonzero Arf-Kervaire invariant. It is known that the Kervaire sphere is not differomorphic to the standard sphere S^{4k+1} unless $4k+4$ is a power of 2 ([1]). When $4k+4$ is a power of 2, in lower dimension where $4k+1 = 5, 13, 29$ or 61, the Kervaire sphere is diffeomorphic to the standard sphere. We shall only consider dimensions where $4k+4$ is not a power of 2. Then the Kervaire sphere, denoted by Σ_K^{4k+1} is definitely not diffeomorphic to the standard sphere.

Suppose that the Kervaire sphere Σ_K^{4k+1} admits a free S^1-action, then from the argument above, we can construct a normal map F with target space $\mathbb{C}P^{2k+1}$ such that its surgery obstruction $s_{4k+2}(F) = c(W) \in \mathbb{Z}/2$ is nonzero and its codimension 2 surgery obstruction $s_{4k}(F|F^{-1}(\mathbb{C}P^{2k}))$ is zero. Conversely, if there exists a normal map F of $\mathbb{C}P^{2k+1}$ with nonzero surgery obstruction $\mathbb{C}P^{2k+1}$ and zero codimension 2 surgery obstruction, we can first perform surgery on the codimension 2 surgery data to obtain a homotopy equivalence in codimension 2. Hence from the start, without loss of generality, we may assume that the normal map F is a homotopy equivalence in codimension 2, and the situation is exactly the same as the situation (1) in dimension $4k$ or (5) in dimension $4k+2$.

Using duality, we can translate this situation to homotopical language. Let the normal map F be represented by a homotopy-theoretical normal map $\varphi \in [\mathbb{C}\mathrm{P}^{2k+1}, F/O]$. Then its surgery obstruction, also denoted by $s_{4k+2}(\varphi)$, lies in $L_{4k+2}(1) = \mathbb{Z}/2$. The surgery obstruction of the restriction of the normal map φ to the codimension 2 subspace $\mathbb{C}\mathrm{P}^{2k}$ is the index obstruction $s_{4k}(\varphi|\mathbb{C}\mathrm{P}^{2k})$, which can be calculated by the virtual bundle $i_*(\varphi|\mathbb{C}\mathrm{P}^{2k}) \in \widetilde{KO}(\mathbb{C}\mathrm{P}^{2k})$.

$$
\begin{array}{ccc}
[\mathbb{C}\mathrm{P}^{2k+1}, F/O] & \xrightarrow{\quad s_{4k+2}\quad} & \mathbb{Z}/2 \\
\downarrow{\scriptstyle \mathrm{res}} & & \\
[\mathbb{C}\mathrm{P}^{2k}, F/O] & \xrightarrow{\quad s_{4k}\quad} & \mathbb{Z} \\
\downarrow{\scriptstyle i_*} & \nearrow{\scriptstyle \mathrm{index}} & \\
[\mathbb{C}\mathrm{P}^{2k}, BSO] = \widetilde{KO}(\mathbb{C}\mathrm{P}^{2k}) & &
\end{array}
$$

Thus the following two statements are equivalent unless $4k+4$ is a power of 2:

(a) The Kervaire sphere Σ_K^{4k+1} does not admit any free S^1-action.

(b) There does not exist a normal map $\varphi \in [\mathbb{C}\mathrm{P}^{2k+1}, F/O]$ such that $s_{4k+2}(\varphi) \neq 0$ and $s_{4k}(\varphi|\mathbb{C}\mathrm{P}^{2k}) = 0$.

For a normal map $\varphi \in [\mathbb{C}\mathrm{P}^{2k+1}, F/O]$, $i_*(\varphi|\mathbb{C}\mathrm{P}^{2k})$ represents a virtual vector bundle ξ over $\mathbb{C}\mathrm{P}^{2k}$, which is fiber homotopically trivial, and the surgery obstruction $s_{4k}(\varphi\mathbb{C}\mathrm{P}^{2k})$ is given by

$$
(7) \qquad s_{4k}(\varphi|\mathbb{C}\mathrm{P}^{2k}) = \left\langle (L(\xi) - 1)\left(\frac{x}{\tanh x}\right)^{2k+1}, [\mathbb{C}\mathrm{P}^{2k}] \right\rangle,
$$

where x is the generator of $H^2(\mathbb{C}\mathrm{P}^{2k}, \mathbb{Z})$ and $L(\cdot)$ is the Hirzebruch's L-class associated to the power series

$$
\frac{x}{\tanh x} = 1 + \sum_{i \geq 1} \frac{(-1)^{i+1} 2^{2i} B_i}{(2i)!} x^{2i},
$$

where B_i is the Bernoulli number. In [2], Brumfiel calculated the surgery obstruction s_8 for the image of $i_* : [\mathbb{C}\mathrm{P}^4, F/O] \to [\mathbb{C}\mathrm{P}^4, BSO]$, that is, the kernel of the J-map $\widetilde{KO}(\mathbb{C}\mathrm{P}^4) \to \tilde{J}(\mathbb{C}\mathrm{P}^4)$ and proved that for any $\varphi \in [\mathbb{C}\mathrm{P}^5, F/O]$, if the obstruction $s_8(\varphi|\mathbb{C}\mathrm{P}^4)$ vanishes, then the surgery obstruction $s_{10}(\varphi)$ should be zero.

Theorem 1.1. (Brumfiel) *The 9-dimensional Kervaire sphere does not admit any free S^1-action.*

In this note, we shall go one step further to show

Theorem 1.2. *The 17-dimensional Kervaire sphere does not admit any free S^1-action.*

After we have finished the proof of this theorem, we shall discuss a simpler method to continue the calculation to higher dimensions.

2 Proof of the Main Theorem

First recall some facts about Adams operations $\psi_{\mathbb{F}}$ in $K\mathbb{F}$-theory ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). Adams operator ${\psi_{\mathbb{F}}}^m$ ($m = 1, 2, \cdots$) acts on $K\mathbb{F}(\cdot)$ as a ring homorphism, is natural with respect to the map between spaces and satisfies the relations:

(A1) $\quad {\psi_{\mathbb{F}}}^1(y) = y$

(A2) $\quad {\psi_{\mathbb{F}}}^m {\psi_{\mathbb{F}}}^n(y) = {\psi_{\mathbb{F}}}^{mn}(y)$

(A3) $\quad {\psi_{\mathbb{F}}}^m(\xi) = \xi^m$ (m-fold tensor product) if ξ is a line bundle

(A4) $\quad \psi_{\mathbb{C}}(y \otimes \mathbb{C}) = \psi_{\mathbb{R}}(y) \otimes \mathbb{C}$

It is well known that the $\widetilde{KO}(\mathbb{C}\mathrm{P}^{2k})$ is generated multiplicatively by $\omega = r(\eta_{\mathbb{C}} - 1_{\mathbb{C}})$ and has an additive basis ω, $\omega^2, \cdots, \omega^k$, ($\omega^{k+1} = 0$). In order to express the effect of Adams operations on ω, we introduce a polynomial $T_m(z)$ of degree m characterized by the equality $T_m(t + t^{-1} - 2) = t^m + t^{-m} - 2$. T_k can also be determined by the inductive formula

$$T_1(z) = z$$
$$T_m(z) = (z + 2)T_{m-1}(z) - T_{m-2}(z) + 2z$$

Here, we assumed $T_0(z) = 0$. In view of the properties $(1) - (4)$ of the Adams operations, it is not hard to see that the Adams operation on ω is given by ${\psi_{\mathbb{R}}}^m(\omega) = T_m(\omega)$. For small values of k, we have

$$T_1(z) = z$$
$$T_2(z) = 4z + z^2$$
$$T_3(z) = 9z + 6z + z^3$$
$$T_4(z) = 16z + 20z^2 + 8z^3 + z^4$$

In general the coefficient of z^m in $T_m(z)$ is one, and we can take $\psi^1(\omega)$, $\psi^2(\omega)$, $\cdots$, $\psi^k(\omega)$ as a basis for $\widetilde{KO}(\mathbb{C}\mathrm{P}^{2k})$. The advangage of using this basis is the convenience of expressing Pontrjagin classes.

Lemma 2.1. *The total Pontrjagin class of $\psi^m(\omega)$ is equal to $1 + m^2x^2$ where $x \in H^2(\mathbb{C}\mathrm{P}^{2k}, \mathbb{Z})$ is the generator.*

Proof. From ${\psi_{\mathbb{R}}}^m(\omega) \otimes \mathbb{C} = {\psi_{\mathbb{C}}}^m(\omega \otimes \mathbb{C}) = {\eta_{\mathbb{C}}}^m + {\eta_{\mathbb{C}}}^{-m} - 2_{\mathbb{C}}$, the total Chern class of ${\psi_{\mathbb{R}}}^m(\omega) \otimes \mathbb{C}$ is $(1 + mx)(1 - mx) = 1 - m^2x^2$. □

From the solution of the Adams conjecture, the kernel of the J-homomorphism is generated 2-locally by elements of the image of ${\psi_{\mathbb{R}}}^3 - 1$. Then it follows that the image of $[CP^{2k}, F/O] \to [CP^{2k}, BSO]$ is can be expressed as a linear combination of $({\psi_{\mathbb{R}}}^3 - 1)({\psi_{\mathbb{R}}}^j(\omega))$, $(j = 1, 2, \cdots, k)$ with $\mathbb{Z}_{(2)}$-coefficients. Here $\mathbb{Z}_{(2)}$ is the subring of $\mathbb{Q}$ composed of fractions with odd denominators. Let us now assume that $\zeta = i_*(\varphi|\mathbb{C}\mathrm{P}^{2k})$ is represented by the sum of virtual bundles as

$$\zeta = \sum_{j=1}^{k} m_j({\psi_{\mathbb{R}}}^3 - 1)({\psi_{\mathbb{R}}}^j(\omega)) \qquad (m_j \in \mathbb{Z}_{(2)}).$$

Then the index surgery obstruction is given by

$$s_{4k}(\varphi|\mathbb{C}\mathrm{P}^{2k}) = \left\langle (L(\zeta) - 1)\left(\frac{x}{\tanh x}\right)^{2k+1}, \ [\mathbb{C}\mathrm{P}^{2k}] \right\rangle,$$

where

$$L(\zeta) = \prod_{j=1}^{k} \left(\frac{3jx}{\tanh 3jx} \frac{\tanh jx}{jx}\right)^{m_j}.$$

Proof of Theorem 1.2:

When $k = 4$, we can calculate the map s_8 for the normal map $\varphi \in [\mathbb{C}\mathrm{P}^5, F/O]$ where the virtual bundle $i_*(\mathbb{C}\mathrm{P}^4)$ is given by

$$i_*(\varphi|\mathbb{C}\mathrm{P}^4) = \sum_{j=1}^{4} m_j({\psi_{\mathbb{R}}}^3 - 1)({\psi_{\mathbb{R}}}^j(\omega)).$$

Symbolic calculation using a computer yields :

$$\begin{aligned}
s_8(\varphi|\mathbb{C}\mathrm{P}^4) = {} & (33554432m_4 + (75497472m_3 + 33554432m_2 + 8388608m_1 - 342884352)m_4^3 \\
& + (63700992m_3^2 + (56623104m_2 + 14155776m_1 - 491913216)m_3 \\
& + 12582912m_2^2 + (6291456m_1 - 191102976)m_2 + 786432m_1^2 - 43646976m_1 \\
& + 935698432)m_4^2 + (23887872m_3^3 + (31850496m_2 + 7962624m_1 - 227930112)m_3^2 \\
& + (14155776m_2^2 + (7077888m_1 - 171638784)m_2 + 884736m_1^2 - 38264832m_1 \\
& + 653137920)m_3 + 2097152m_2^3 + (1572864m_1 - 31260672)m_2^2 \\
& + (393216m_1^2 - 13565952m_1 + 203067392)m_2 + 32768m_1^3 - 1437696m_1^2 \\
& + 41646080m_1 - 655895808)m_4 + 3359232m_3^4 \\
& + (5971968m_2 + 1492992m_1 - 33592320)m_3^3 \\
& + (3981312m_2^2 + (1990656m_1 - 36080640)m_2 + 248832m_1^2 - 7713792m_1 \\
& + 89999424)m_3^2 + (1179648m_2^3 + (884736m_1 - 12165120)m_2^2 \\
& + (221184m_1^2 - 4921344m_1 + 42794496)m_2 + 18432m_1^3 - 470016m_1^2 \\
& + 7346304m_1 - \underline{62267616})m_3 + 131072m_2^4 + (131072m_1 - 1228800)m_2^3 \\
& + (49152m_1^2 - 663552m_1 + 3123712)m_2^2 \\
& + (8192m_1^3 - 101376m_1^2 + 636416m_1 - 2084544)m_2 + 512m_1^4 - 3072m_1^3 \\
& + 6208m_1^2 - \underline{3168}m_1)/243 \ .
\end{aligned}$$

We should remark that all the coefficients are in $\mathbb{Z}_{(2)}$ and has 2-order greater than 5 except for the two underlined coefficients whose 2-orders are 5. Thus, if this obstruction vanishes then it follows that $m_1 + m_3$ is even. We also know that the total Pontjagin class of φ is

$$p(\varphi) = \prod_{j=1}^{4} \left(\frac{1 + 9j^2x^2}{1 + j^2x^2} \right)^{m_j}$$

and the first Pontrjagin class is

$$p_1(\varphi) = \sum_{j=1}^{4} 8m_j(j^2 - 1)x^2.$$

From Sullivan's result (see Wall [4], Chap.14C), we know that $p_1(\varphi)/8$ reduced mod 2 is equal to $\varphi^*(k_2^2)$, where $k_2 \in H^2(F/O; \mathbb{Z}/2)$ is the universal Kervaire class of degree 2. Therefore $p_1(\varphi)/8$ is an even class if and only if $\varphi^*(k_2)$ is zero. Whereas the surgery obstruction $s_{10}(\varphi)$ vanishes if and only if $\varphi^*(k_2) = 0$ from Rourke-Sullivan's formula [3]. Hence, if $s_10(\varphi|\mathbb{C}P^4) = 0$, then $m_1 + m_3$ is even, and we have $\varphi^*(k_2) = 0$. This implies that $s_10(\varphi) = 0$. This completes the proof of Theorem 1.2.

3 Further calculation continues

As we have seen in the computation of $s_8(\varphi)$, for general values of k, we can similarly express the surgery obstruction $s_{4k}(\varphi|\mathbb{C}P^{2k})$ as a polynomial $q(m_1, m_2, \cdots, m_k)$. Close examination of the 2-order of coefficients of $x/\tanh(x)$ leads us to prove that all the coefficients of $q(m_1, m_2, \cdots, m_k)$ belongs to $\mathbb{Z}_{(2)}$, more than that, divisible by 8, and that the 2-order of coefficients of non-linear terms in $q(m_1, m_2, \cdots, m_k)$ are divisible by 64. So considering the polynomial $q(m_1, m_2, \cdots, m_k)$ mod 64 simplifies the polynomial into a linear combination of m_1, $m_2, \cdots$, m_k. In fact, we are able to prove results similar to our present Theorem 1.2: for $k = 5, 6, 9, 10, 11, 12, 13, 14, 17 \cdots$. For more general results, the case where k is odd ($\neq 2^r - 1$) can be settled easily. But as the 2-order of k itself increases, the solution of this problem grows harder.

References

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