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INDUCTION THEORY OF
EQUIVARIANT-SURGERY-OBSTRUCTION GROUPS

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Abstract

In the present article, we recall the definitions of the Hermitian-representation ring
$G_1(R, G)$, the Grothendieck-Witt rings $\text{GW}(G, R)$ and $\text{GW}_0(R, G)$, the Wall groups
$L_n^h(R[G], w)$, and the Bak groups $L_n^h(R[G], \Lambda, w)$ of a finite group $G$, and we discuss
induction theory concerned with these rings and groups using the notion of $w$-Mackey
functor.

1. INTRODUCTION

Throughout this article, let $G$ be a finite group.

After works on surgery by J. Milnor, S. P. Novikov, W. Browder, and etc., C. T. C.
Wall [18], [19] formulated the surgery-obstruction groups $L_n^h(\mathbb{Z}[G], w)$ using quadratic
modules and automorphisms. In the case where the orientation homomorphism $w$ is
trivial, C. B. Thomas [17, Theorems 1, 3] in 1971 proved that $L_n^h(\mathbb{Z}[G], w)$ is a module

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over the Hermitian-representation ring $G_1(\mathbb{Z}, G)$, and moreover the pairing of functors

$$G_1(\mathbb{Z}, -) \times L_n^h(\mathbb{Z}[-], w|_-) \rightarrow L_n^h(\mathbb{Z}[-], w|_-)$$

is a Frobenius pairing (see Section 3). The Grothendieck-Witt ring $GW_0(\mathbb{Z}, G)$ defined in [7], [15] is the quotient ring of $G_1(\mathbb{Z}, G)$ with respect to the Quillen relation. We note that another Grothendieck-Witt ring $GW(G, \mathbb{Z})$ is defined in [8] and the canonical homomorphism $GW(G, \mathbb{Z}) \rightarrow GW_0(\mathbb{Z}, G)$ is an isomorphism. It is a folklore since 1970's, perhaps regarded as a corollary to [17, Theorems 1, 3], that if $w$ is trivial, then $L_n^h(\mathbb{Z}[G], w)$ is a module over the ring $GW_0(\mathbb{Z}, G)$ and

$$GW_0(\mathbb{Z}, -) \times L_n^h(\mathbb{Z}[-], w|_-) \rightarrow L_n^h(\mathbb{Z}[-], w|_-)$$

is a Frobenius pairing. This was a main motivation of the study of $GW_0(\mathbb{Z}, G)$ and $GW(G, \mathbb{Z})$ by A. Dress [6], [7], [8] in the respect of induction and restriction. By using the Frobenius structure above and the induction theory of $GW_0(\mathbb{Z}, -)$, various authors computed $L_n(\mathbb{Z}[G], w)$ for many finite groups $G$ (cf. [9]). In addition, A. Bak [1] introduced the notion of form parameter $\Lambda$ and defined various $K$-theoretic groups for the category of quadratic modules with form parameter (see Section 5). We [11], [12] and [13] showed that certain Bak groups $W_n(\mathbb{Z}[G], \Lambda; w)$ are equivariant-surgery-obstruction groups, as the groups $L_n^h(\mathbb{Z}[G], w)$ are surgery-obstruction groups. The groups $W_n(\mathbb{Z}[G], \Lambda; w)$ are denoted by $L_n^h(\mathbb{Z}[G], \Lambda, w)$ in the current paper. In the case where $\Lambda$ is the minimal form parameter $\text{min}$, the group $L_n^h(\mathbb{Z}[G], \Lambda, w)$ coincides with the Wall group $L_n^h(G, \mathbb{Z})$. It is important to ask whether the Bak-group functor $L_n^h(\mathbb{Z}[-], \Lambda; w|_-)$ is a Frobenius module over the Grothendieck-Witt-ring functor $GW_0(\mathbb{Z}, -)$. We have an affirmative answer as in the theorem below. Particularly if $n$ is an even integer, the answer was obtained in [15].

Let $S(G)$ denote the set of all subgroups of $G$ and let $G(2)$ denote the set consisting of all elements $g$ in $G$ of order 2. Let $w : G \rightarrow \{1, -1\}$ be a homomorphism. For
each $H \in S(G)$, let $w_H : H \to \{1, -1\}$ denote the restriction of $w$. The group ring $\mathbb{Z}[H]$ has the involution $- : \mathbb{Z}[H] \to \mathbb{Z}[H]$ associated with $w_H$. Let $n$ be an integer and set $\lambda = (-1)^k$ and regard it as the symmetry of $\mathbb{Z}[H]$, where $k$ is the integer such that $n = 2k$ or $2k + 1$. Let $Q$ be a conjugation-invariant subset of $G(2)$ satisfying $w(g) = (-1)^{k+1}$ and set $Q_H = H \cap Q$. The form parameter $\Lambda_H$ of $\mathbb{Z}[H]$ is defined by

$$\Lambda_H = \{x - \lambda \overline{x} \mid x \in \mathbb{Z}[H]\} + \langle Q_H \rangle.$$ 

Similarly to the Wall-group functor, the bifunctor $L^h_n(\mathbb{Z}[-], \Lambda_-, w_-)$ on $S(G)$ with canonical correspondence of morphisms is not a Mackey functor if $w$ is nontrivial. However, we have

**Theorem 1.1.** The bifunctor $L^h_n(\mathbb{Z}[-], \Lambda_-, w_-)$ on $S(G)$ with canonical correspondence of morphisms is a $w$-Mackey functor (see Section 3) and furthermore a module over the Grothendieck-Witt ring functor $GW_0(\mathbb{Z}, -)$ on $S(G)$ with canonical correspondence of morphisms.

Let $\mathcal{H}_2(G)$ denote the set of all 2-hyperelementary subgroups and elementary subgroups of $G$. By [8, Theorem 1] and [1, Theorem 12.13 (a)], the Green functor $GW_0(\mathbb{Z}, -)$ on $S(G)$ is $\mathcal{H}_2(G)$-computable. By replacing the correspondence of morphisms as in [15, Proposition 2.3], the $w$-Mackey functor $L^h_n(\mathbb{Z}[-], \Lambda_-, w_-)$ on $S(G)$ is modified to a Mackey functor on $S(G)$.

**Corollary 1.2.** The modified Mackey functor $L^h_n(\mathbb{Z}[-], \Lambda_-, w_-)$ is $\mathcal{H}_2(G)$-computable (see Section 3). In particular, the restriction homomorphism

$$\text{Res} : L^h_n(\mathbb{Z}[G], \Lambda_G, w) \longrightarrow \bigoplus_{H \in \mathcal{H}_2(G)} L^h_n(\mathbb{Z}[H], \Lambda_H, w_H)$$

is injective, and the induction homomorphism

$$\text{Ind} : \bigoplus_{H \in \mathcal{H}_2(G)} L^h_n(\mathbb{Z}[H], \Lambda_H, w_H) \longrightarrow L^h_n(\mathbb{Z}[G], \Lambda_G, w)$$

is surjective.
Further results are discussed in Section 6. The other sections are organized as follows. In Section 2, we describe the definitions of the rings $G_1(R, G)$, $GW(G, R)$, and $GW_0(R, G)$. In Section 3, we give the definition of a Frobenius pairing and recall results obtained by C. B. Thomas, A. Dress and A. Bak. In Section 4, we describe the definitions of the category $\mathcal{G} (= G(G))$ and a $w$-Mackey functor given in [15] and recall relevant results. Section 5 is devoted to recalling the definitions of groups $L_n^h(R[G], \Lambda, w)$.

2. The Grothendieck-Witt rings

Let $R$ be a commutative ring with 1. Let $\mathfrak{B}(G)$ denote the category of all pairs $(M, B)$ consisting of a finitely generated $R$-projective $R[G]$-module $M$ and a symmetric, $G$-invariant, nonsingular $R$-bilinear form $B : M \times M \to R$, namely

- $B(ax + a'x', by) = abB(x, y) + a'bB(x', y)$,
- $B(x, y) = B(y, x)$,
- $B(gx, gy) = B(x, y)$,

for any $a, a', b \in R$, $x, x', y \in M$, $g \in G$, and

$$M \to \text{Hom}_R(M, R); \ x \mapsto B(x, -)$$

is a bijection. The set $\text{Morph}_{\mathfrak{B}(G)}((M, B), (M', B'))$ of morphisms $(M, B) \to (M', B')$ in $\mathfrak{B}(G)$ consists of all $R$-linear maps $f : M \to M'$ compatible with forms, namely

$$B'(f(x), f(y)) = B(x, y)$$

for all $x, y \in M$. For an $R[G]$-submodule $U$ of $M$, we define the $R[G]$-submodule $U^\perp$ of $M$ by

$$U^\perp = \{x \in M \mid B(x, y) = 0 \ (\forall \ y \in U)\}.$$  

If $U$ is $R$-projective and $U = U^\perp$ then we say that $U$ is a Lagrangian. More generally, if an $R[G]$-submodule $U$ of $M$ is an $R$-direct summand of $M$ and satisfies $U \subseteq U^\perp$,
then we refer to $U$ as a Quillen submodule of $(M, B)$ (or simply, $M$). In the case where $U$ is a Quillen submodule of $(M, B)$, the pair $(U^\perp/U, B^\perp)$ defined by

$$B^\perp(x + U, y + U) = B(x, y)$$

for $x, y \in U^\perp$ is an object in $\mathfrak{B}(G)$. For a finitely generated $R$-projective $R[G]$-module $N$, the associated hyperbolic module (in $\mathfrak{B}(G)$) $H(N) = (N \oplus N^*, B_N)$ is defined so that $B_N(N, N) = 0 = B_N(N^*, N^*)$, $B_N(n, v) = v(n)$ for $n \in N$ and $v \in N^*$, where $N^* = \text{Hom}_R(N, R)$ with $(g \cdot v)(n) = v(g^{-1}n)$.

C. B. Thomas [17] defined the group

$$G_1(R, G)$$

to be the Grothendieck Group of the category $\mathfrak{B}(G)$ with respect to orthogonal sum:

$$[M_1, B_1] + [M_2, B_2] = [M_1 \oplus M_2, B_1 \perp B_2].$$

This set also has a product operation

$$([M_1, B_1], [M_2, B_2]) \mapsto [M_1, B_1] \cdot [M_2, B_2] = [M_1 \otimes_R M_2, B_1 \otimes_R B_2],$$

and is a commutative ring with 1, actually

$$1 = [R, B_0]$$

such that $R$ has the trivial $G$-action and $B_0(a, b) = ab$ for $a, b \in R$. The ring $G_1(R, G)$ is called the Hermitian-representation ring. A. Dress [8] defined a Grothendieck-Witt ring

$$\text{GW}(G, R)$$

to be the quotient $G_1(R, G)/\langle[(M, B)]\rangle$, where $(M, B)$ ranges over all objects in $\mathfrak{B}(G)$ having Lagrangians. In addition, A. Dress [7, p.472] defined the ring

$$\text{GU}_0(R, G)$$

as the quotient

$$G_1(R, G)/\langle[(M, B)] - [(U^\perp/U, B^\perp)] - [H(U)]\rangle$$
and another Grothendieck-Witt ring

$$GW_0(R, G)$$

as the quotient

$$G_1(R, G) / ( [(M, B)] - [(U^+/U, B^+)] ),$$

where $(M, B)$ and $U$ range over all objects $(M, B)$ of $\mathfrak{B}(G)$ with Quillen submodule $U$. We remark that A. Bak [1] used the same notation $GW_0(R, G)$ to denote the group $GW(G, R)$ by it. Clearly, we have the canonical ring-epimorphisms

$$G_1(R, G) \rightarrow GW(G, R) \rightarrow GW_0(R, G).$$

By [8, Theorem 5], the last arrow is an isomorphism if $R$ is a Dedekind domain and $|G|$ is invertible in its field of fractions.

3. Frobenius pairing

Let $\mathfrak{F}$ be a category such that $\text{Obj}(\mathfrak{F}) = S(G)$ the set of all subgroups of $G$, let $\mathfrak{A}$ denote the category of abelian groups, and let $L, M, N : \mathfrak{F} \rightarrow \mathfrak{A}$ be bifunctors. Namely $L = (L^*, L_*)$ consists of a contravariant functor $L^* : \mathfrak{F} \rightarrow \mathfrak{A}$ and a covariant functor $L_* : \mathfrak{F} \rightarrow \mathfrak{A}$ such that $L^*(H) = L_*(H)$ for all $H \in S(G)$. So, we usually write $L(H)$ instead of $L^*(H), L_*(H)$.

We mean by a pairing $L \times M \rightarrow N$ a family of biadditive maps

$$L(H) \times M(H) \rightarrow N(H); (x, y) \mapsto x \cdot y,$$

where $H$ runs over $S(G)$. We mean by a Frobenius pairing a paring satisfying the conditions:

1. $N^*(f)(x \cdot y) = L^*(f)(x) \cdot M^*(f)(y)$ for $x \in L(H), y \in M(H), f \in \text{Morph}_\mathfrak{F}(H, K)$,

2. $x \cdot M^*(f)(y) = N_*(f)(L^*(f)(x) \cdot y)$ for $x \in L(K), y \in M(H), f \in \text{Morph}_\mathfrak{F}(H, K)$,

3. $L_*(f)(x) \cdot y = N_*(f)(x \cdot M^*(f)(y))$ for $x \in L(H), y \in M(K), f \in \text{Morph}_\mathfrak{F}(H, K)$.

Let us note the following.
(1) C. B. Thomas [17] showed that in the case where $\text{Morph}_g(H, K)$ consists of inclusions $H \rightarrow K$ and $w$ is the trivial homomorphism $G \rightarrow \{1\}$,

$$G_1(\mathbb{Z}, -) \times L_n^h(\mathbb{Z}[-], w_-) \rightarrow L_n^h(\mathbb{Z}[-], w_-)$$

is a Frobenius pairing.

(2) In the case where $\text{Morph}_g(H, K)$ consists of all monomorphisms $H \rightarrow K$, A. Dress [8, p. 292, l. 3] claimed that

$$\text{GW}(-, \mathbb{Z}) \times L_n^h(\mathbb{Z}[-], w_-) \rightarrow L_n^h(\mathbb{Z}[-], w_-)$$

is a Frobenius pairing. A similar version of quadratic forms with form parameter is given by A. Bak [1, Theorems 12.6, 12.7] where proof of the odd-dimensional case is omitted.

(3) In the case where $\text{Morph}_g(H, K)$ consists of inclusions $H \rightarrow K$, conjugations $H \rightarrow gHg^{-1}$ and their compositions and $w$ is trivial, one has perhaps regarded that

$$\text{GW}_0(\mathbb{Z}, -) \times L_n^h(\mathbb{Z}[-], w_-) \rightarrow L_n^h(\mathbb{Z}[-], w_-)$$

is a Frobenius pairing, as a corollary to [17, Theorems 1, 3]. In fact, A. Dress [8, p. 742, ll. 6–5] claimed without showing a detailed and precise proof that $\text{GU}_0(\mathbb{Z}, -)$ acts on $L_n^h(\mathbb{Z}[-], w_-)$ as a Frobenius functor.

Thus, it would serve our convenience to describe a detailed and precise proof of the fact that

$$\text{GW}_0(\mathbb{Z}, -) \times L_n^h(\mathbb{Z}[-], \Lambda_-, w_-) \rightarrow L_n^h(\mathbb{Z}[-], \Lambda_-, w_-)$$

is a Frobenius pairing for certain form parameters $\Lambda_-$ and general $w$. For the case $n = 2k$, one can find a proof with details in [15] (cf. [15, Theorem 12.10]).
4. $w$-MACKEY FUNCTOR

We begin this section with recalling the category $\mathcal{G} = \mathcal{G}(G)$: The set $\text{Obj}(\mathcal{G})$ is same as $\mathcal{S}(G)$. For $H, K \in \mathcal{S}(G)$, $\text{Morph}_\mathcal{G}(H, K)$ is the set of all homomorphisms

$$\varphi_{(H,g,K)} : H \to K; \quad \varphi_{(H,g,K)}(h) = ghg^{-1} \quad (h \in H)$$

for $g \in G$ such that $gHg^{-1} \subseteq K$. The composition of morphisms is given by the composition of maps. Adopting the notation in [15], we also use $j_{H,K}$ and $c(H,g)$ for $\varphi_{(H,g,K)}$ and $\varphi_{(H,g,gHg^{-1})}$, respectively.

We mean by a bifunctor $M = (M^*, M_*) : \mathcal{G} \to \mathfrak{A}$ a pair consisting of a contravariant functor $M^* : \mathcal{G} \to \mathfrak{A}$ and covariant functor $M_* : \mathcal{G} \to \mathfrak{A}$ such that $M^*(H) = M_*(H)$, which will be denoted by $M(H)$, for all $H \in \mathcal{S}(G)$. By [15, Proposition 2.1], we obtain

**Proposition 4.1.** Let $M : \mathcal{G} \to \mathfrak{A}$ be a bifunctor satisfying $M_*(c_{(H,g)}gHg^{-1}) = M^*(c_{(H,g)})$ for all $H \in \mathcal{S}(G)$ and $g \in G$. The Burnside ring $\Omega(G)$ canonically acts on $M(G)$ if and only if

$$M^*(c_{(G,g)})M_*(j_{H,G})M^*(j_{H,G}) = M_*(j_{H,G})M^*(j_{H,G})M^*(c_{(G,g)})$$

for all $H \in \mathcal{S}(G)$ and $g \in G$.

Let $w : G \to \{1, -1\}$ be a homomorphism.

**Definition 4.2.** A bifunctor $M : \mathcal{G} \to \mathfrak{A}$ is called a $w$-Mackey functor if the following conditions are fulfilled:

1. $M_*(c_{(H,g)}) = M^*(c_{(gHg^{-1},g^{-1})})$ for all $H \in \mathcal{S}(G)$ and $g \in G$,
2. $M^*(c_{(H,h)}) = w(h)id_{M(H)}$ (hence $M_*(c_{(H,h)}) = w(h)id_{M(H)}$) for all $H \in \mathcal{S}(G)$ and $h \in H$,
3. $M^*(j_{K,G}) \circ M_*(j_{H,G})$ coincides with

$$\bigoplus_{KgH \in K \setminus G/H} M_*(j_{KgHg^{-1},K}) \circ (w_1M_*(c_{(gHg^{-1}Kg,H)})) \circ M^*(j_{Hg^{-1}Kg,H})$$

for any $H, K \in \mathcal{S}(G)$. 
We note that a $w$-Mackey functor for trivial $w$ is a Mackey functor.

Recall the next proposition.

**Proposition 4.3** ([15, Proposition 2.3]). Let $M : \mathcal{G} \to \mathfrak{I}$ be a $w$-Mackey functor. Then bifunctor $M^w : \mathcal{G} \to \mathfrak{I}$ given by

$$M^w(H) = M(H),$$

$$M^w_*(\varphi_{(H,g,K)}) = w(g)M_*(\varphi_{(H,g,K)}) \quad \text{and}$$

$$M^{w*}(\varphi_{(H,g,K)}) = w(g)M^{*}(\varphi_{(H,g,K)})$$

for $H, K \in S(G)$, $\varphi_{(H,g,K)} \in \text{Morph}_G(H, K)$ with $g \in G$ is a Mackey functor.

For a $w$-Mackey functor $M$, we say that $M^w$ is the Mackey functor associated with $M$.

The next proposition is fundamental in geometric applications of the notion of $w$-Mackey functor.

**Proposition 4.4** ([15, Proposition 2.6]). A $w$-Mackey functor $M : \mathcal{G} \to \mathfrak{I}$ is a module over the Burnside-ring functor $\Omega : \mathcal{G} \to \mathfrak{I}$.

*Proof.* Since $M^*(c_{(G,g)}) = \pm \text{id}_{M(G)}$, the equality (1) in Proposition 4.1 obviously holds.

Thus $M(G)$ is a module over $\Omega(G)$. Similarly, $M(H)$ is a module over $\Omega(H)$. The naturalities (1)--(3) required for a Frobenius pairing in Section 3 can be checked in a straightforward way.

Let $\mathcal{F}$ be a conjugation-invariant lower-closed subset of $S(G)$, namely $gHg^{-1} \in \mathcal{F}$ and $K \in \mathcal{F}$ both hold whenever $H \in \mathcal{F}$, $g \in G$ and $K \subset H$. A Mackey functor $L : \mathcal{G} \to \mathfrak{I}$ is said to be $\mathcal{F}$-computable if

$$L(G) = \lim_{\longrightarrow \mathcal{G} \setminus \mathcal{F}} L(-) \quad \text{and} \quad L(G) = \lim_{\longrightarrow \mathcal{G} \setminus \mathcal{F}} L(-).$$
5. EQUIVARIANT-SURGERY-OBSTRUCTION GROUPS

Let $A = (A, -, \lambda, \Lambda)$ be a form ring: $A$ is a ring with 1, $-$ is an involution on $A$ such that $\overline{ab} = \overline{b}\overline{a}$, $\lambda$ is a symmetry, namely an element of Center($A$) such that $\overline{\lambda}\lambda = 1$, and $\Lambda$ is a form parameter, namely an additive subgroup satisfying

1. $\{a - \lambda\overline{a} \mid a \in A\} \subseteq \Lambda \subseteq \{a \in A \mid a = -\lambda\overline{a}\}$ and
2. $a\Lambda\overline{a} \subseteq \Lambda$ for all $a \in A$.

Let $M$ be a finitely generated $A$-module. A biadditive map $B : M \times M \to A$ is called a $\lambda$-Hermitian form if

1. $B(ax, by) = bB(x, y)\overline{a}$ and
2. $B(x, y) = \lambda\overline{B(y, x)}$

for all $a, b \in A, x, y \in M$. A map $q : M \to A/\Lambda$ is called a quadratic ‘form’ with respect to $B$ if

1. $q(x + y) - q(x) - q(y) = B(x, y)$ in $A/\Lambda$,
2. $q(ax) = aq(x)\overline{a}$ in $A/\Lambda$ and
3. $B(x, x) = \overline{q(x)} + \lambda q(x)$ in $A$

for all $a \in A, x, y \in M$, where $\overline{q(x)} \in A$ is a lifting of $q(x) \in A/\Lambda$. Such $(M, B, q)$ is referred to as an $A$-quadratic module.

Let $\mathbf{H}(A)$ denote the standard hyperbolic plane. That is, $\mathbf{H}(A)$ is the $A$-quadratic module $(M, B, q)$ consisting of an $A$-free module $M$ with basis $\{e, f\}$, a $\lambda$-Hermitian form $B : M \times M \to A$ such that

$$B(e, e) = B(f, f) = 0, B(e, f) = 1,$$

and a quadratic ‘form’ $q : M \to A/\Lambda$ such that

$$q(e) = q(f) = 0.$$
A hyperbolic module is an $A$-quadratic module isomorphic to
\[ H(A^n) = H(A) \perp \cdots \perp H(A) \]
the orthogonal sum of $n$ copies of the standard hyperbolic plane. Let $Q(A)$ denote the category of $A$-quadratic modules $(M, B, q)$ such that $M$ is a free $A$-module and $B$ is a nonsingular form, namely
\[ M \rightarrow \text{Hom}_A(M, A); \ x \mapsto B(x, -) \]
is a bijection. The set $\text{Morph}_{Q(A)}((M, B, q), (M', B', q'))$ of morphisms $(M, B, q) \rightarrow (M', B', q')$ in $Q(A)$ consists of $A$-linear maps $f : M \rightarrow M'$ satisfying $B' \circ (f \times f) = B$ and $q' \circ f = q$.

We define $K_{Q_0}(A)_{\text{free}}$ to be the Grothendieck Group of the category $Q(A)$ with respect to orthogonal sum. Let $W_{Q_0}(A)_{\text{free}}$ denote the quotient group $K_{Q_0}(A)_{\text{free}}/(H(A))$.

Let $R$ be a commutative ring with 1, let $w : G \rightarrow \{1, -1\}$ be a homomorphism, let $-$ denote the involution on $R[G]$ associated to $w$, let $n$ be an integer, and set $\lambda = (-1)^k$, where $k \in \mathbb{Z}$ with $n = 2k$ or $2k + 1$. The involution $-$ on $R[G]$ associated with $w$ is the map
\[ \sum_{g \in G} r_g g \mapsto \sum_{g \in G} w(g) r_g g^{-1}, \]
where $r_g \in R$.

First, consider the case where $n = 2k$ is an even integer. Given a form parameter $\Lambda$ of $(R[G], -, \lambda)$, we define the group $L^h_n(R[G], \Lambda, w)$ by
\[ L^h_n(R[G], \Lambda, w) = W_{Q_0}(A)_{\text{free}}. \]
Thus in particular, Wall's group $L^h_n(R[G], w)$ is $L^h_n(R[G], \text{min}, w)$, where
\[ \text{min} = \{ x - \lambda \bar{x} \mid x \in R[G] \}. \]

For defining $L^h_n(R[G], \text{min}, w)$ with $n$ odd, we use notation below. Let $\text{SU}_m(A, \Lambda)$ denote the subgroup of $\text{GL}_{2m}(A)$ corresponding to $\text{Aut}(H(A^m))$, let $\text{EU}_m(A, \Lambda)$ denote
the subgroup of $\text{SU}_m(A, \Lambda)$ consisting of elementary $\Lambda$-quadratic matrices, and let $\text{TU}_m(A, \Lambda)$ denote the subgroup of $\text{SU}_m(A, \Lambda)$ corresponding to the group consisting of $\alpha \in \text{Aut}(\mathcal{H}(A^m))$ such that

$$\alpha((e_1, \ldots , e_m)) = (e_1, \ldots , e_m),$$

where $(e_1, \ldots , e_m)$ is the canonical Lagrangian of $\mathcal{H}(A^m)$. Let

$$\sigma \in \text{SU}_1(A, \Lambda)$$

denote the matrix corresponding to $\alpha \in \text{Aut}(\mathcal{H}(A))$ such that $\alpha(e) = f$ and $\alpha(f) = \overline{\lambda}e$.

We set

$$\text{RU}_m(A, \Lambda) = (\text{TU}_m(A, \Lambda), \sigma).$$

Then, $\text{SU}(A, \Lambda)$ is defined to be the direct limit $\varprojlim \text{SU}_m(A, \Lambda)$ in a canonical way; moreover $\text{EU}(A, \Lambda)$, $\text{TU}(A, \Lambda)$, and $\text{RU}(A, \Lambda)$ are similarly defined.

We obtain the next lemma by using 3.5 (the Whitehead Lemma) and Corollary 3.9 of [1].

**Lemma 5.1.** If a subgroup $K$ of $\text{SU}(A, \Lambda)$ contains $\text{EU}(A, \Lambda)$, then $[K, K] = \text{EU}(A, \Lambda)$.

Define

$$\text{KQ}_1(A, \Lambda) = \text{SU}(A, \Lambda) / \text{EU}(A, \Lambda)$$

and

$$\text{WQ}_1(A, \Lambda) = \text{KQ}_1(A, \Lambda) / \langle \text{hyperbolic matrices} \rangle,$$

where we mean by a hyperbolic matrix a matrix in $\text{SU}_m(A, \Lambda)$, for some $m$, of the form

$$\mathcal{H}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}$$

with $\alpha \in \text{GL}_m(A)$. It follows from arguments in [1, p. 27] that $\text{WQ}_1(A, \Lambda)$ coincides with

$$\text{KQ}_1(A, \Lambda) / [\text{TU}(A, \Lambda)].$$
Now we consider the case where $n = 2k + 1$ is an odd integer. Since $\text{RU}(A, \Lambda) \supseteq \text{EU}(A, \Lambda)$ (cf. [13, Proposition 2.7]), the quotient

$$L_n^h(\mathbb{Z}[G], \Lambda, w) = \text{SU}(A, \Lambda)/\text{RU}(A, \Lambda)$$

is an abelian group and coincides with

$$\text{WQ}_1(A, \Lambda)/\langle \sigma \rangle.$$

In particular, the Wall group $L_n^h(R[G], w)$ is $L_n^h(R[G], \text{min}, w)$.

6. Results

Let $G$ be a finite group, $w : G \rightarrow \{1, -1\}$ a homomorphism, $n$ an integer, $Q$ an involution invariant subset of $G(2)$ satisfying $w(g) = -(-1)^k$ for all $g \in Q$, where $k$ is an integer with $n = 2k$ or $2k + 1$. For $H \leq G$, we set $Q_H = Q \cap H$, $w_H = w|_H$, and

$$\Lambda_H = \{x - (-1)^k x \mid x \in R[H]\} + \langle Q_H \rangle_R.$$

Then, our main result is

**Theorem 6.1.** The bifunctor $L_n^h(R[-], \Lambda_-, w_-) : \mathcal{G}(G) \rightarrow \mathfrak{A}$ is a $w$-Mackey functor and moreover a module over the Grothendieck-Witt-ring functor $\text{GW}_0(\mathbb{Z}, -) : \mathcal{G}(G) \rightarrow \mathfrak{A}$.

The assertion for the case $n = 2k$ follows from arguments in [15]. A detailed proof for the case $n = 2k + 1$ will be given in a forthcoming paper.

Let $H_2(G)$ denote the set of all 2-hyperelementary subgroups and elementary subgroups of $G$.

**Corollary 6.2.** With respect to the associated-Mackey-functor structure, the bifunctor $L_n^h(R[-], \Lambda_-, w_-) : \mathcal{G}(G) \rightarrow \mathfrak{A}$ is $H_2(G)$-computable. In particular, the restriction homomorphism

$$\text{Res} : L_n^h(R[G], \Lambda_G, w) \longrightarrow \bigoplus_{H \in H_2(G)} L_n^h(R[H], \Lambda_H, w_H)$$
is injective, and the induction homomorphism

$$\text{Ind} : \bigoplus_{H \in \mathcal{H}_2(G)} L^h_n(R[H], \Lambda_H, w_H) \rightarrow L^h_n(R[G], \Lambda_G, w)$$

is surjective.

This follows from [8, Theorem 1] and [1, Theorem 12.13 (a)].

**Corollary 6.3.** Let $\beta$ be an element in the Burnside ring $\Omega(G)$ such that $\chi_H(\beta) = 0$ for all $H \in \mathcal{H}_2(G)$ (resp. cyclic subgroup $H$ of $G$). Then one has

$$\beta L^h_n(R[G], \Lambda_G, w) = 0 \quad (\text{resp. } \beta^{2(a+1)} L^h_n(R[G], \Lambda_G, w) = 0),$$

where $a$ is the integer such that $|G| = 2^a m$ with odd integer $m$.

This follows from [7, Theorems 1, 3 (iii)] and [10, Proposition 6.3].

Finally we remark that the construction of smooth actions on spheres of finite groups in [16] is a geometric application of the induction theory above.

**References**


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