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Stability of Generic Pseudoplanes

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Problem (Baldwin[B1]) Is there any "generic" structure that is super-stable but not $\omega$-stable?

Theorem There is no $\delta$-generic pseudoplane that is superstable but not $\omega$-stable.

1 $\delta$-Generic Pseudoplanes

Let $L = \{R(*, *)\}$ be a language of undirected graphs: It satisfies $\models \forall z (\neg R(z, z))$ and $\models \forall x \forall y (R(x, y) \rightarrow R(y, x))$. Let $\alpha$ be a positive real number. Then

- For a finite graph $A$, $\delta_\alpha(A) := |A| - \alpha|R^A|$, where $R^A = \{\{a, b\} : A \models R(a, b)\}$.
- $K_\alpha := \{A : A$ is a finite graph, $\forall B \subset A[\delta_\alpha(B) \geq 0]\}$.

Definition Let $A$ be a finite subgraph of a graph $M$
(i) We say $A$ is closed in $M$ (in symbol, $A \leq M$), if $\delta_\alpha(XA) \geq \delta_\alpha(A)$ for any finite $X \subset M - A$.
(ii) The closure of $A$ in $M$, $\text{cl}_M(A) := \bigcap\{B : A \subset B \leq M, |B| < \omega\}$.

To simplify our notation, we write $\delta(*)$ in place of $\delta_\alpha(*)$. For finite $A, B$, we write $\delta(A/B) = \delta(AB) - \delta(B)$.

Definition Let $K \subset K_\alpha$ be closed under subgraphs. Then a countable graph $M$ is said to be $(K, \leq)$-generic, if it satisfies the following:
(i) If $A$ is a finite subset of $M$, then $A \in K$;
(ii) If $A \leq B \in K$ and $A \leq M$, then there exists $B' \leq M$ such that $B' \cong_A B$.

**Definition**  We say that a graph $M$ is $\delta$-generic, if $M$ is $(K, \leq)$-generic for some $\alpha$ and some $K \subset K_\alpha$ such that

1. $M$ has finite closures (i.e., any finite subset of $M$ has finite closures);
2. $M$ is saturated.

**Definition**  A pseudoplane $P$ is called $\delta$-generic, if there is a $\delta$-generic graph $M$ with $P = (M, M, I)$ where an incidence relation $xIy$ is defined by $R(x, y)$.

**Example**  (i) Hrushovski's pseudoplanes ([H1]) are $\delta$-generic, $\omega$-categorical and strictly stable.
(ii) Baldwin's projective planes ([B2]) are $\delta$-generic and $\aleph_1$-categorical.

**Note 1.1**  It is an open problem whether there is an $\omega$-categorical projective plane or not (for instance, see [C], [Ho]). In [I], it is proven that there is no $\delta$-generic $\omega$-categorical projective plane.

**Definition**  (i) Given a finite $A \subset M$, define $d_M(A) = \delta(\mathrm{cl}_M(A))$.
(ii) For finite $A, B$, write $d_M(A/B) = d_M(AB) - d_M(B)$. Define $d_M(A/B)$ for possibly infinite $B$ to be $\inf\{d_M(A/B') : B' \subset B, B' \text{ is finite}\}$.

**Fact 1.2**  Let $A \leq B \leq M$ and $\bar{a} \in M$. Then $\mathrm{tp}(\bar{a}/B)$ does not fork over $A$ if and only if $d_M(\bar{a}/B) = d_M(\bar{a}/A)$.

**Fact 1.3**  Let $P$ be a $\delta$-generic pseudoplane.
(i) $\mathrm{Th}(P)$ is stable;
(ii) If $\alpha$ is rational, then $\mathrm{Th}(P)$ is $\omega$-stable.

## 2 Lemmas

**Lemma 2.1**  If $\alpha > 0$ is irrational, then $\sup\{d : d = a - b\alpha < 0, a, b \in \mathbb{N}\} = 0$.

**Proof**  Let $X = \{a - b\alpha : a, b \in \mathbb{N}, a - b\alpha < 0\}$ and $Y = \{a - b\alpha : a, b \in \mathbb{Z}, a - b\alpha < 0\}$.
Claim: $\sup Y = 0$. 

Proof: For each \( k \in \mathbb{Z} \), let \( f(k) = k\alpha - \max\{m \in \mathbb{Z} : m \leq k\alpha\} \). Take any \( \varepsilon > 0 \). Since \( \alpha \) is irrational, we have \( f(k) \neq f(l) \) for any distinct \( k, l \in \mathbb{Z} \). So there are distinct \( i, j < \omega \) with \( 0 > f(i) - f(j) > -\varepsilon \). Let \( d = f(i) - f(j) \). Then we have \( d \in \mathbb{Y} \). Hence we have \( \sup \mathbb{Y} = 0 \). (End of Proof of Claim)

We assume by way of contradiction that \( \sup \mathbb{X} = e < 0 \). By the claim, there is a strictly monotone increasing sequence \( \{d_n\}_{n<\omega} \) of elements of \( \mathbb{Y} \) such that \( \lim_{n \to \infty} d_n = 0 \) and \( d_n > e \) for each \( n < \omega \). Then, for each \( n < \omega \), \( d_n \not\in \mathbb{X} \), and therefore we can write \( d_n = b_n\alpha - a_n \) where \( a_n, b_n \in \mathbb{N} \). Since \( \{d_n\}_{n<\omega} \) is strictly monotone increasing, there is \( m < \omega \) such that \( b_{m+1} > b_m \). Now we have \( 0 > d_m - d_{m+1} > e \). On the other hand, since \( b_{m+1} - b_m \in \mathbb{N} \), we have \( d_m - d_{m+1} = (a_{m+1} - a_m) - (b_{m+1} - b_m)\alpha \in \mathbb{X} \). This contradicts \( \inf \mathbb{X} = e \).

**Lemma 2.2** If \( \alpha \) is irrational with \( 0 < \alpha < 1 \), then for any \( \varepsilon > 0 \) there exists a sequence \( \{q_n\}_{1 \leq n \leq p} \) of \( \mathbb{N} \) such that

(1) \( 0 > p - q_p\alpha > -\varepsilon \);
(2) If \( 0 < n < p \) then \( n - q_n\alpha > 0 \);
(3) If \( 0 < n < m \leq p \) then \( (q_m - q_n - 1)\alpha < m - n \).

**Proof:** By 2.1, for any \( \varepsilon > 0 \) there are \( p, q < \omega \) with \( 0 > p - q\alpha > -\varepsilon \).

Let

\[
q_n = \begin{cases} 
\max\{k \in \mathbb{N} : \alpha \leq \frac{n}{k}\} & \text{if } 0 < n < p \\
q & \text{if } n = p 
\end{cases}
\]

By the definition of \( q_n \), it is clear that (1) and (2) hold. To see (3), we prove two claims.

Claim 1: For any \( n, m \) with \( 0 < n < m \leq p \), \( q_m - q_n - 1 \geq 0 \).

**Proof:** By the definition of \( q_m \), we have \( \frac{m}{q_{m+1}} < \alpha \), so \( q_m > \frac{n}{\alpha} - 1 \). By the definition of \( q_n \), we have \( \alpha < \frac{n}{q_n} \), so \( q_n < \frac{n}{\alpha} \). By our assumption, we have \( 0 < \alpha < 1 \). It follows that \( q_m - q_n - 1 > (\frac{m}{\alpha} - 1) - \frac{n}{\alpha} = \frac{m-n}{\alpha} - 2 > (m - n) - 2 \geq 1 - 2 = -1 \). Hence \( q_m - q_n - 1 \geq 0 \).

Claim 2: For any \( n, m \) with \( 0 < n < m \leq p \), \( (q_m - q_n - 1)\alpha < m - n \).

**Proof:** If \( q_m - q_n - 1 = 0 \) then clearly \( (q_m - q_n - 1)\alpha < m - n \). So, by claim 1, we can assume that \( q_m - q_n - 1 > 0 \). By the definition of \( q_n \) and \( q_m \), we have \( \frac{n}{q_{n+1}} < \alpha < \frac{m}{q_m} \), so \( mq_n - nq_m + m > 0 \). Then we have \( \frac{m-n}{q_m-q_n-1} - \frac{m}{q_m} = \frac{mq_n - nq_m + m}{(q_m-q_n-1)q_m} > 0 \). So \( \frac{m-n}{q_m-q_n-1} > \frac{m}{q_m} > \alpha \). Hence \( (q_m - q_n - 1)\alpha < m - n \).

**Definition** Let \( AB \in K_\alpha \) with \( A \cap B = \emptyset \). Then we say that a pair \( (B, A) \) is biminimal, if it satisfies the following:
(i) $\delta(B/A) < 0$;
(ii) $\delta(X/A) \geq 0$ for any nonempty proper subset of $B$;
(iii) $\delta(B/Y) \geq 0$ for any nonempty proper subset of $A$.

We say that a graph $A$ has no loops, if for each $n > 2$ there do not exist distinct $b_1, b_2, \ldots, b_n \in A$ with $R(b_1, b_2), R(b_2, b_3), \ldots, R(b_{n-1}, b_n)$ and $R(b_n, b_1)$.

Lemma 2.3 If $\alpha$ is irrational with $0 < \alpha < 1$, then for any $\epsilon > 0$ there is a finite graph $eBC$ such that

1. $(C, eB)$ is biminimal;
2. $\delta(C/eB) > -\epsilon$;
3. $eBC$ has no loops;
4. $eB$ has no relations.

Proof: Take any $\epsilon > 0$. Then there is a sequence $\{q_n\}_{1 \leq n \leq p}$ satisfying (1)–(3) of 2.2. Let $q_0 = -1$. Let $\{c_i : 1 \leq i \leq p\} \cup \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\}$ be a graph with the relations:

(a) $R(c_1, c_2), \ldots, R(c_{n-1}, c_n)$;
(b) $R(c_i, b_i^j)$ for each $i, j$ with $1 \leq i \leq p$ and $1 \leq j \leq q_i - q_{i-1} - 1$.

Let $e = b_1^1$, $C = \{c_i : 1 \leq i \leq p\}$ and $B = \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\} \setminus \{b_1^1\}$. Clearly $eBC$ satisfies (3) and (4). By the definition of $eBC$, we have

$$\delta(C/eB) = p - \left\{ (p - 1) + \sum_{i=1}^{p} (q_i - q_{i-1} - 1) \right\} \alpha = p - q_p \alpha.$$ 

By (1) of 2.2, we have $0 > \delta(C/eB) > -\epsilon$, so (2) holds.

Claim: If $X(\subset C)$ is connected with $X \neq C$, then $\delta(X/eB) > 0$.

Proof: Let $X = \{c_i\}_{n < i \leq m}$ for some $n, m$. If $n = 0$, then $\delta(X/eB) = m - q_m \alpha > 0$ by (2) of 2.2. If $n > 0$, then $\delta(X/eB) = (m-n)-(m-q_{n-1} \alpha) > 0$ by (3) of 2.2. (End of Proof of Claim)

We show (1). Take any $X \subset C$ with $X \neq C$. Let $X = \bigcup X_i$ where each $X_i$ is connected component of $X$. Then $\delta(X/eB) = \sum \delta(X_i/eB) > 0$ by the claim. Hence (1) holds.

Lemma 2.4 If $\alpha$ is irrational with $0 < \alpha < 1$, then for any $\epsilon > 0$ there is a sequence $\{eB_iC_i\}_{i<\omega}$ of finite graphs such that

1. $D$ has no loops;
2. $B_n^* \leq eB_n^* C_n^* \leq D$ for each $n < \omega$;
3. $(C_n, eB_n)$ is biminimal for each $n < \omega$;
$eB^*$ has no relations;
(5) For each $i, j < \omega$ there is no relation between $B_i C_i$ and $B_j C_j$,
where $B_i^* = \bigcup_{i \leq n} B_i, C_i^* = \bigcup_{i \leq n} C_i, B^* = \bigcup_{i < \omega} B_i, C^* = \bigcup_{i < \omega} C_i$ and
$D = eB^* C^*$.

Proof For each $i < \omega$ there is $eC_i B_i$ that satisfies $\delta(C_i/eB_i) > -\frac{1}{2}$ and
(1)-(4) of 2.3. We can assume that (5) holds. Then (1), (3) and (4) hold.
To see (2), we prove two claims. Let $X_E$ denote $X \cap E$ for each $X$ and $E$.
Claim 1: $eB_n^* C_n^* \leq D$.
Proof: Take any $X \subset D - eB_n^* C_n^*$. Then $\delta(X/eB_n^* C_n^*) = \delta(X/e) =
\delta(X_{C^*}/eX_{B^*}) + \delta(X_{B^*}/e) = \delta(X_{C^*}/eX_{B^*}) + |X_{B^*}| \geq \delta(X_{C^*}/eX_{B^*}) + 1 =
\sum_i \delta(X_{C_i}/eX_{B_i}) + 1 \geq - \sum_{i=1}^\omega \frac{1}{2^n} + 1 \geq 0$.
Claim 2: $B_n^* \leq eB_n^* C_n^*$.
Proof: Take any $X \subset eC_n^*$. To show $\delta(X/B_n^*) \geq 0$ we divide into two cases.
Suppose $e \in X$. $\delta(X/B_n^*) = \delta(X/B_n^* e) + \delta(e/B_n^*) = \sum_{i=1}^n \delta(X_{C_i}/B_i e) + 1 \geq
- \sum_{i=1}^\omega \frac{1}{2^n} + 1 \geq 0$.
Suppose $e \not\in X$. By biminimality of $(C_i, B_i e)$ it can be seen that $\delta(Y/B_i) > 0$ for any $Y \subset C_i$. So $\delta(X/B_n^*) = \sum_{i=1}^n \delta(X_{C_i}/B_i) > 0$.

3 Theorem

Lemma 3.1 Let $P = (M, M, I)$ be a $\delta$-generic pseudoplane. Suppose that a finite graph $A \subset M$ has no loops. Then $A \in K$.

Proof Take any $a_0 \in A$. Let $C_0$ be a connected component of $a_0$ in $A$. As $A$ has no loops, $C_0$ can be regarded as a tree with $\text{height}(a_0) = 0$. Since $P$ is a pseudoplane, $M$ satisfies

- For any $a \in M$ there are infinitely many $b \in M$ with $R(a, b)$;
- For any distinct $a, b \in M$ there are at most finitely many $c \in M$ with $R(a, c) \land R(b, c)$.

So, we can inductively construct $C_0^* \subset M$ with $C_0^* \cong C_0$. Take any $a_1 \in A - C_0$. Let $C_1$ be a connected component of $a_1$. In the same way, we have $C_1^* \subset M$ with $C_0^* C_1^* \cong C_0 C_1$. Iterating this process, we have $A^* \subset M$ with $A^* \cong A$. Hence $A \in K$.

Lemma 3.2 Let $P = (M, M, I)$ be a $\delta$-generic pseudoplane. Then $\alpha < 1$. 
Proof Suppose that \( \alpha \geq 1 \). Take some \( a \in M \) with \( a \leq M \). Then there is no \( b \in M \) with \( R(a, b) \). This contradicts axioms of a pseudoplane. Hence \( \alpha < 1 \).

Theorem There is no \( \delta \)-generic pseudoplane that is superstable but not \( \omega \)-stable.

Proof Take any \( \delta \)-generic pseudoplane \( P = (M, \leq M, I) \). Let \( M \) be a \( (K, \leq) \)-generic graph for some \( K \subseteq K_\alpha \). By 1.3, if \( \alpha \) is rational, then \( P \) is \( \omega \)-stable. Thus it is enough to show that, if \( \alpha \) is irrational then \( P \) is not superstable. By 3.2, we have \( 0 < \alpha < 1 \). So we have a sequence \( \{eB_iC_i\}_{i<\omega} \) satisfying (1)–(5) of 2.4. Let \( D = \bigcup_{i<\omega} eB_iC_i \). Since \( D \) has no loops, any finite subset of \( D \) belongs to \( K \) by 3.1. By genericity of \( M \), we can assume that \( D \leq M \).

Claim: \( d(e/B_n^*) = \sum_{i \leq n} \delta(C_i/eB_i) + 1 \).

Proof: By (2)–(5) of 2.4, we have \( d(e/B_n^*) = d(eB_n^*) - d(B_n^*) = \delta(eC_n^*B_n^*) - \delta(B_n^*) = \delta(eC_n^*/B_n^*) = \delta(C_n^*/eB_n^*) + 1 = \sum_{i \leq n} \delta(C_i/eB_i) + 1 \). (End of Proof of Claim)

For each \( n < \omega \), \( \text{tp}(e/B_{n+1}^*) \) is a forking extension of \( \text{tp}(e/B_n^*) \), because \( d(e/B_{n+1}^*) = d(e/B_n^*) + \delta(C_{n+1}/eB_{n+1}) < d(e/B_n^*) \) by the claim. Hence \( \text{Th}(M) \) is not superstable.

Reference


[H1] E. Hrushovski, A stable \( \aleph_0 \)-categorical pseudoplane, preprint, 1988