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Stability of Generic Pseudoplanes

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Problem (Baldwin[B1]) Is there any "generic" structure that is superstable but not $\omega$-stable?

Theorem There is no $\delta$-generic pseudoplane that is superstable but not $\omega$-stable.

1 $\delta$-Generic Pseudoplanes

Let $L = \{R(*, *)\}$ be a language of undirected graphs. It satisfies $\models \forall x(\neg R(x, x))$ and $\models \forall x\forall y(R(x, y) \rightarrow R(y, x))$. Let $\alpha$ be a positive real number. Then

- For a finite graph $A$, $\delta_\alpha(A) := |A| - \alpha|R^A|$, where $R^A = \{\{a, b\} : A \models R(a, b)\}$.
- $K_\alpha := \{A : A$ is a finite graph, $\forall B \subset A[\delta_\alpha(B) \geq 0]\}$.

Definition Let $A$ be a finite subgraph of a graph $M$

(i) We say $A$ is closed in $M$ (in symbol, $A \leq M$), if $\delta_\alpha(XA) \geq \delta_\alpha(A)$ for any finite $X \subset M - A$.

(ii) The closure of $A$ in $M$, $cl_M(A) := \cap\{B : A \subset B \leq M, |B| < \omega\}$.

To simplify our notation, we write $\delta(*)$ in place of $\delta_\alpha(*)$. For finite $A, B$, we write $\delta(A/B) = \delta(AB) - \delta(B)$.

Definition Let $K \subset K_\alpha$ be closed under subgraphs. Then a countable graph $M$ is said to be $(K, \leq)$-generic, if it satisfies the following:

(i) If $A$ is a finite subset of $M$, then $A \in K$;
(ii) If $A \leq B \in K$ and $A \leq M$, then there exists $B' \leq M$ such that $B' \cong_A B$.

**Definition** We say that a graph $M$ is $\delta$-generic, if $M$ is $(K, \leq)$-generic for some $\alpha$ and some $K \subset K_\alpha$ such that

1. $M$ has finite closures (i.e., any finite subset of $M$ has finite closures);
2. $M$ is saturated.

**Definition** A pseudoplane $P$ is called $\delta$-generic, if there is a $\delta$-generic graph $M$ with $P = (M, M, I)$ where an incidence relation $xIy$ is defined by $R(x, y)$.

**Example** (i) Hrushovski's pseudoplanes ([H1]) are $\delta$-generic, $\omega$-categorical and strictly stable.

(ii) Baldwin's projective planes ([B2]) are $\delta$-generic and $\aleph_1$-categorical.

**Note 1.1** It is an open problem whether there is an $\omega$-categorical projective plane or not (for instance, see [C], [Ho]). In [I], it is proven that there is no $\delta$-generic $\omega$-categorical projective plane.

**Definition** (i) Given a finite $A \subset M$, define $d_M(A) = \delta(\text{cl}_M(A))$.

(ii) For finite $A, B$, write $d_M(A/B) = d_M(AB) - d_M(B)$. Define $d_M(A/B)$ for possibly infinite $B$ to be $\inf\{d_M(A/B') : B' \subset B, B' \text{ finite}\}$.

**Fact 1.2** Let $A \leq B \leq M$ and $\bar{a} \in M$. Then $\text{tp}(\bar{a}/B)$ does not fork over $A$ if and only if $d_M(\bar{a}/B) = d_M(\bar{a}/A)$.

**Fact 1.3** Let $P$ be a $\delta$-generic pseudoplane.

(i) $\text{Th}(P)$ is stable;

(ii) If $\alpha$ is rational, then $\text{Th}(P)$ is $\omega$-stable.

2 Lemmas

**Lemma 2.1** If $\alpha > 0$ is irrational, then $\sup\{d : d = a - b\alpha < 0, a, b \in \mathbb{N}\} = 0$.

**Proof** Let $X = \{a - b\alpha : a, b \in \mathbb{N}, a - b\alpha < 0\}$ and $Y = \{a - b\alpha : a, b \in \mathbb{Z}, a - b\alpha < 0\}$.

Claim: $\sup Y = 0$. 

Proof: For each $k \in \mathbb{Z}$, let $f(k) = k\alpha - \max\{m \in \mathbb{Z} : m \leq k\alpha\}$. Take any $\epsilon > 0$. Since $\alpha$ is irrational, we have $f(k) \neq f(l)$ for any distinct $k, l \in \mathbb{Z}$. So there are distinct $i, j < \omega$ with $0 > f(i) - f(j) > -\epsilon$. Let $d = f(i) - f(j)$. Then we have $d \in Y$. Hence we have $\sup Y = 0$. (End of Proof of Claim)

We assume by way of contradiction that $\sup X = \epsilon < 0$. By the claim, there is a strictly monotone increasing sequence $\{d_n\}_{n<\omega}$ of elements of $Y$ such that $\lim_{n \to \infty} d_n = 0$ and $d_n > \epsilon$ for each $n < \omega$. Then, for each $n < \omega$, $d_n \notin X$, and therefore we can write $d_n = b_n\alpha - a_n$ where $a_n, b_n \in \mathbb{N}$. Since $\{d_n\}_{n<\omega}$ is strictly monotone increasing, there is $m < \omega$ such that $b_{m+1} > b_m$. Now we have $0 > d_m - d_{m+1} > \epsilon$. On the other hand, since $b_{m+1} - b_m \in \mathbb{N}$, we have $d_m - d_{m+1} = (a_{m+1} - a_m) - (b_{m+1} - b_m)\alpha \in X$. This contradicts $\inf X = \epsilon$.

**Lemma 2.2** If $\alpha$ is irrational with $0 < \alpha < 1$, then for any $\epsilon > 0$ there exists a sequence $\{q_n\}_{1 \leq n \leq p}$ of $\mathbb{N}$ such that

(1) $0 > p - q_p\alpha > -\epsilon$;
(2) If $0 < n < p$ then $n - q_n\alpha > 0$;
(3) If $0 < n \leq m \leq p$ then $(q_m - q_n - 1)\alpha < m - n$.

**Proof:** By 2.1, for any $\epsilon > 0$ there are $p, q < \omega$ with $0 > p - q\alpha > -\epsilon$. Let

$$q_n = \begin{cases} \max\{k \in \mathbb{N} : \alpha \leq \frac{n}{k}\} & \text{if } 0 < n < p \\ q & \text{if } n = p \end{cases}$$

By the definition of $q_n$, it is clear that (1) and (2) hold. To see (3), we prove two claims.

**Claim 1:** For any $n, m$ with $0 < n < m \leq p$, $q_m - q_n - 1 \geq 0$.

**Proof:** By the definition of $q_m$, we have $\frac{m}{q_m+1} < \alpha$, so $q_m > \frac{n}{\alpha} - 1$. By the definition of $q_n$, we have $\alpha < \frac{n}{q_n}$, so $q_n < \frac{n}{\alpha}$. By our assumption, we have $0 < \alpha < 1$. It follows that $q_m - q_n - 1 > (\frac{m}{\alpha} - 1) - \frac{n}{\alpha} - 1 = \frac{m-n}{\alpha} - 2 > (m-n) - 2 \geq 1 - 2 = -1$. Hence $q_m - q_n - 1 \geq 0$.

**Claim 2:** For any $n, m$ with $0 < n < m \leq p$, $(q_m - q_n - 1)\alpha < m - n$.

**Proof:** If $q_m - q_n - 1 = 0$ then clearly $(q_m - q_n - 1)\alpha < m - n$. So, by claim 1, we can assume that $q_m - q_n - 1 > 0$. By the definition of $q_n$ and $q_m$, we have $\frac{n}{q_n+1} < \alpha < \frac{m}{q_m}$, so $mq_n - nq_m + m > 0$. Then we have $\frac{m-n}{q_m-q_n-1} - \frac{m}{q_m} = \frac{mq_n-nq_m+m}{(q_m-q_n-1)q_m} > 0$. So $\frac{m-n}{q_m-q_n-1} > \frac{m}{q_m} > \alpha$. Hence $(q_m - q_n - 1)\alpha < m - n$.

**Definition** Let $AB \in K_\alpha$ with $A \cap B = \emptyset$. Then we say that a pair $(B, A)$ is *biminimal* if it satisfies the following:
\[(i) \delta(B/A) < 0;\]
\[(ii) \delta(X/A) \geq 0 \text{ for any nonempty proper subset of } B;\]
\[(iii) \delta(B/Y) \geq 0 \text{ for any nonempty proper subset of } A.\]

We say that a graph \(A\) has no loops, if for each \(n > 2\) there do not exist distinct \(b_1, b_2, \ldots, b_n \in A\) with \(R(b_1, b_2), R(b_2, b_3), \ldots, R(b_{n-1}, b_n)\) and \(R(b_n, b_1)\).

**Lemma 2.3** If \(\alpha\) is irrational with \(0 < \alpha < 1\), then for any \(\epsilon > 0\) there is a finite graph \(eBC\) such that
\[(1) (C, eB)\) is biminimal;
\[(2) \delta(C/eB) > -\epsilon;\]
\[(3) eBC\) has no loops;
\[(4) eB\) has no relations.

**Proof:** Take any \(\epsilon > 0\). Then there is a sequence \(\{q_n\}_{1 \leq n \leq p}\) satisfying (1)-(3) of 2.2. Let \(q_0 = -1\). Let \(\{c_i : 1 \leq i \leq p\} \cup \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\}\) be a graph with the relations:
\[(a) R(c_1, c_2), \ldots, R(c_{n-1}, c_n);\]
\[(b) R(c_i, b_i^j) \text{ for each } i, j \text{ with } 1 \leq i \leq p \text{ and } 1 \leq j \leq q_i - q_{i-1} - 1.\]
Let \(e = b_1^1, C = \{c_i : 1 \leq i \leq p\}\) and \(B = \{b_i^j : 1 \leq i \leq p, 1 \leq j \leq q_i - q_{i-1} - 1\} - \{b_1^1\}\). Clearly \(eBC\) satisfies (3) and (4). By the definition of \(eBC\), we have
\[\delta(C/eB) = p - \left\{ (p - 1) + \sum_{i=1}^{p} (q_i - q_{i-1} - 1) \right\} \alpha = p - q_p \alpha.\]

By (1) of 2.2, we have \(0 > \delta(C/eB) > -\epsilon\), so (2) holds.

Claim: If \(X(\subset C)\) is connected with \(X \neq C\), then \(\delta(X/eB) > 0\).

Proof: Let \(X = \{c_i\}_{n \leq i < m}\) for some \(n, m\). If \(n = 0\), then \(\delta(X/eB) = m - q_m \alpha > 0\) by (2) of 2.2. If \(n > 0\), then \(\delta(X/eB) = (m - n) - (q_m - q_n - 1) \alpha > 0\) by (3) of 2.2. (End of Proof of Claim)

We show (1). Take any \(X \subset C\) with \(X \neq C\). Let \(X = \bigcup X_i\) where each \(X_i\) is connected component of \(X\). Then \(\delta(X/eB) = \sum \delta(X_i/eB) > 0\) by the claim. Hence (1) holds.

**Lemma 2.4** If \(\alpha\) is irrational with \(0 < \alpha < 1\), then for any \(\epsilon > 0\) there is a sequence \(\{eB_iC_i\}_{i< \omega}\) of finite graphs such that
\[(1) D\) has no loops;
\[(2) B_n^* \leq eB_n^* C_n^* \leq D\) for each \(n < \omega;\)
\[(3) (C_n, eB_n)\) is biminimal for each \(n < \omega;\)
(4) \(eB^*\) has no relations;
(5) For each \(i, j < \omega\) there is no relation between \(B_iC_i\) and \(B_jC_j\),
where \(B_n^* = \bigcup_{i \leq n} B_i, C_n^* = \bigcup_{i \leq n} C_i, B^* = \bigcup_{i < \omega} B_i, C^* = \bigcup_{i < \omega} C_i\) and 
\(D = eB^*C^*\).

**Proof** For each \(i < \omega\) there is \(eC_iB_i\) that satisfies \(\delta(C_i/eB_i) > -\frac{1}{2}\) and (1)-(4) of 2.3. We can assume that (5) holds. Then (1), (3) and (4) hold. To see (2), we prove two claims. Let \(X_E\) denote \(X \cap E\) for each \(X\) and \(E\).

**Claim 1:** \(eB_n^*C_n^* \leq D\).
**Proof:** Take any \(X \subset D - eB_n^*C_n^*\). Then \(\delta(X/eB_n^*C_n^*) = \delta(X/e) = \delta(X_{C^*}/eX_{B^*}) + \delta(X_{B^*}/e) = \delta(X_{C^*}/eX_{B^*}) + |X_{B^*}| \geq \delta(X_{C^*}/eX_{B^*}) + 1 = \sum_i \delta(X_{C_i}/eX_{B_i}) + 1 \geq -\sum_{i=1}^\omega \frac{1}{2^i} + 1 \geq 0.

**Claim 2:** \(B_n^* \leq eB_n^*C_n^*\).
**Proof:** Take any \(X \subset eC_n^*\). To show \(\delta(X/B_n^*) \geq 0\) we divide into two cases.
Suppose \(e \in X\). \(\delta(X/B_n^*) = \delta(X/B_n^*e) + \delta(e/B_n^*) = \sum_{i=1}^n \delta(X_{C_i}/B_i) + 1 \geq -\sum_{i=1}^\omega \frac{1}{2^i} + 1 \geq 0.

Suppose \(e \notin X\). By biminimality of \((C_i, B_i, e)\) it can be seen that \(\delta(Y/B_i) > 0\) for any \(Y \subset C_i\). So \(\delta(X/B_n^*) = \sum_{i=1}^n \delta(X_{C_i}/B_i) > 0\).

## 3 Theorem

**Lemma 3.1** Let \(P = (M, M, I)\) be a \(\delta\)-generic pseudoplane. Suppose that a finite graph \(A \subset M\) has no loops. Then \(A \in K\).

**Proof** Take any \(a_0 \in A\). Let \(C_0\) be a connected component of \(a_0\) in \(A\). As \(A\) has no loops, \(C_0\) can be regarded as a tree with \(\text{height}(a_0) = 0\). Since \(P\) is a pseudoplane, \(M\) satisfies

- For any \(a \in M\) there are infinitely many \(b \in M\) with \(R(a, b)\);
- For any distinct \(a, b \in M\) there are at most finitely many \(c \in M\) with \(R(a, c) \land R(b, c)\).

So, we can inductively construct \(C_0^* \subset M\) with \(C_0^* \cong C_0\). Take any \(a_1 \in A - C_0\). Let \(C_1\) be a connected component of \(a_1\). In the same way, we have \(C_1^* \subset M\) with \(C_0^*C_1^* \cong C_0C_1\). Iterating this process, we have \(A^* \subset M\) with \(A^* \cong A\). Hence \(A \in K\).

**Lemma 3.2** Let \(P = (M, M, I)\) be a \(\delta\)-generic pseudoplane. Then \(\alpha < 1\).
Proof Suppose that $\alpha \geq 1$. Take some $a \in M$ with $a \leq M$. Then there is no $b \in M$ with $R(a, b)$. This contradicts axioms of a pseudoplane. Hence $\alpha < 1$.

Theorem There is no $\delta$-generic pseudoplane that is superstable but not $\omega$-stable.

Proof Take any $\delta$-generic pseudoplane $P = (M, M, I)$. Let $M$ be a $(K, \leq)$-generic graph for some $K \subset K_\alpha$. By 1.3, if $\alpha$ is rational, then $P$ is $\omega$-stable. Thus it is enough to show that, if $\alpha$ is irrational then $P$ is not superstable. By 3.2, we have $0 < \alpha < 1$. So we have a sequence $\{eB_iC_i\}_{i<\omega}$ satisfying (1)-(5) of 2.4. Let $D = \bigcup_{i<\omega} eB_iC_i$. Since $D$ has no loops, any finite subset of $D$ belongs to $K$ by 3.1. By genericity of $M$, we can assume that $D \leq M$.

Claim: $d(e/B_n^*) = \sum_{i\leq n} \delta(C_i/eB_i) + 1$.

Proof: By (2)-(5) of 2.4, we have $d(e/B_n^*) = d(eB_n^*) - d(B_n^*) = \delta(eC_n^*B_n^*) - \delta(B_n^*) = \delta(eC_n^*/B_n^*) = \delta(C_n^*/eB_n^*) + 1 = \sum_{i\leq n} \delta(C_i/eB_i) + 1$. (End of Proof of Claim)

For each $n < \omega$, $\text{tp}(e/B_{n+1}^*)$ is a forking extension of $\text{tp}(e/B_n^*)$, because $d(e/B_{n+1}^*) = d(e/B_n^*) + \delta(C_{n+1}/eB_{n+1}) < d(e/B_n^*)$ by the claim. Hence $\text{Th}(M)$ is not superstable.

Reference


[H1] E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, preprint, 1988