GROUP CONFIGURATIONS IN SIMPLE THEORIES
(PART. 2) (Interaction between model theory and algebraic geometry)

Author(s)
Okamoto, Keisuke

Citation
数理解析研究所講究録 2003, 1344: 73-77

Issue Date
2003-10

URL
http://hdl.handle.net/2433/43520

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
GROUP CONFIGURATIONS IN SIMPLE THEORIES (PART 2)

岡本 幸史 KEISHI OKAMOTO
東京女子大学 (Tokyo Women's Christian University)

Our goal is to make a gradedly almost hyperdefinable group from a group configuration in a simple theory. In a stable case, we get a group and a group operation from germs and a function on germs. But, in a simple case, since our hyperdefinable multifunction is not a function, we only have a polygroup. So, by using a core equivalence relation and a blow-up construction, we construct a group from a polygroup. In the following parts of this paper, our aim is to introduce necessary tools and new technique for simple version of group configuration theorem and illustrate these ideas. For the complete proof, see [4] and [5].

1. ULTRAIMAGINARIES AND ALMOST HYPERIMAGINARIES

To make a group from a polygroup, a core equivalence relation will be needed. But core equivalence relation is not type-definable but almost type-definable. So, we define new sorts called an ultraimaginary and a almost hyperimaginary. We will show that we can define a type of ultraimaginary over a hyperimaginary and two ultraimaginaries being independent over a hyperimaginary.

Definition 1.1 (Ultraimaginaries and an Almost hyperimaginaries). Let $(I, \leq)$ be a directed partial order, and $X$ a sort.

1. An equivalence relation on $X$ is invariant if it is automorphism-invariant.
2. A graded equivalence relation (g.e.r) $R$ on $X$ is the direct limit of reflexive symmetric type-definable relations $(R_i : i \in I)$ on $X$, such that:
   (a) If $i \leq j$ then $R_j$ is coarser than $R_i$.
   (b) For every $i, j$ there is $k$ (which can be taken to be $\geq i, j$) such that $xR_i yR_j z \Rightarrow xR_k z$.

   We then note $R = R_I = \bigvee_{i \in I} R_i$, which is an invariant equivalence relation, and say that the $R_i$ give a grading of $R$. If we want to emphasize $I$, we say $I$-graded and $I$-grading.

3. The class of a modulo $R$ is noted $a_R$. Even when $R$ is just a reflexive symmetric relation we note $a_R = \{ x : xRa \}$ and call this the $R$-class of $a$. For a set $A$ we may also note $A_R = \bigcup_{a \in A} a_R$. We also write $x \in_i A$ instead of $x \in A_{R_i}$, and $\pi(x_{R_i})$ for $\exists y[xR_i y \land \pi(y)]$, where $\pi$ is a partial type. If there are too many indices, we may occasionally use $a/R$ instead.

4. An invariant equivalence relation $R$ is almost type-definable if there is a type-definable symmetric and reflexive relation $R'$ finer than $R$ such that any $R$-class can be covered by boundedly many $R'$-classes. If in addition $R$ is graded and $R'$ is finer than some $R_i$, then we say that it is gradedly almost type-definable (above $i$).

5. A class modulo a (graded) invariant equivalence relation is called a (graded) ultraimaginary. A class modulo a (gradedly) almost type-definable equivalence relation is called a (graded) almost hyperimaginary.
There is an ultraimaginary which is not a almost hyperimaginary (i.e. there is an invariant equivalence relation which is not almost type-definable).

**Example 1.2.** Let $E_i (i \in \omega)$ be equivalence relations such that $E_i$ is a refinement of $E_{i+1}$ with infinitely many $E_{i+1}$-classes and every $E_0$ class has infinite elements. Put a language $L = \{ E_i (i \in \omega) \}$ and consider the above structure. Then an equivalence relation $E = \bigcup_{i \in \omega} E_i$ is automorphism invariant but not almost type-definable.

The following lemma shows that we can define a type and a Lascar strong type of ultraimaginary over a hyperimaginary.

**Lemma 1.3.** For two ultraimaginaries $a_R$ and $b_R$ and a hyperimaginary $c$, the following are equivalent:
1. There are $a' \in a_R$ and $b' \in b_R$, such that $a' \equiv_c b'$ in the usual sense.
2. There is an automorphism fixing $c$ sending $a_R$ to $b_R$.
3. For every $a' \in a_R$ there is $b' \in b_R$ such that $a' \equiv_c b'$.

And the following are also equivalent:
1. There are $a' \in a_R$ and $b' \in b_R$, such that $a' \equiv_{c}^{Ls} b'$.
2. $a_R$ and $b_R$ are equivalent modulo any bounded $c$-invariant equivalence relation.
3. For every $a' \in a_R$ there is $b' \in b_R$ such that $a' \equiv_{c}^{Ls} b'$.

So we define types and Lascar strong types and independence relation for ultraimaginaries.

**Definition 1.4.** 1. Two ultraimaginaries $a_R$ and $b_R$ have the same type over a hyperimaginary $c$, denoted $a_R \equiv_c b_R$, if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv_c b'$ in the usual sense.
2. Two ultraimaginaries $a_R$ and $b_R$ have the same Lascar strong type over a hyperimaginary $c$, denoted $a_R \equiv_{c}^{Ls} b_R$, if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv_{c}^{Ls} b'$ in the usual sense.

**Definition 1.5.** We say that $a_R \downarrow_c b_R$ if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \downarrow_c b'$.

When we define types and independence as above, we have some desired properties as follows:
1. Except finite character, ordinary properties of independence (symmetry, transitivity and etc.) hold for ultraimaginaries.
2. Two almost hyperimaginaries being independent over a hyperimaginary is type-definable.

2. Polygroups

In a stable theory, we construct an interdefinable group configuration from a group configuration. And we have a group and a group operation from germs and a function on germs in an interdefinable group configuration. But, in a simple theory, we only have an interbounded group configuration and have only a polygroup.

**Definition 2.1.** A polygroup is axiomatized in the language \{, $^{-1}$\} by the following axioms:
1. $t \in (x \cdot y) \cdot z \iff t \in x \cdot (y \cdot z)$,
2. $t \in x \cdot y^{-1} \iff x \in t \cdot y$, 

3. \( t \in x^{-1} \cdot y \leftrightarrow y \in x \cdot t \),
4. \( t \in x \cdot e \leftrightarrow t \in x \leftrightarrow t \in e \cdot x \).

Since a group operation and an inverse function are multifunctions in a polygroup, \( x \cdot y \) and \( x^{-1} \) are not elements but sets. So \( (x \cdot y) \cdot z \) represents \( \cup \{u \cdot z : u \in x \cdot y\} \). Now we give some examples of a polygroup.

**Example 2.2** (Double coset algebras). Let \( G \) and \( H \) be groups such that \( H \) is a subgroup of \( G \). Put \( M = \{HgH : g \in G\} \) and define a group operation and inverse on \( M \) as follows:
1. \((Hg_1H) \cdot (Hg_2H) = \{Hg_1hg_2H : h \in H\} ;
2. \((HgH)^{-1} = Hg^{-1}H ;
3. \text{identity is the } H(=HeH).

Then a double coset algebra \( \langle M, \cdot, ^{-1}, H \rangle \) is clearly a polygroup and denoted by \( G//H \).

**Example 2.3** (Prenowitz algebras). Let \( \tilde{P} \) be a set of points, \( L \) a set of lines and \( I \subseteq P \times L \) a incidence relation. An incidence system \( (P, L, I) \) is projective geometry if it satisfy the following axioms:
1. any line contains at least three points;
2. two distinct points \( a, b \) are contained in a unique line denoted by \( L(a, b) \);
3. if \( a, b, c, d \) are distinct points and \( L(a, b) \) intersect \( L(b, d) \),
   then \( L(a, c) \) must intersect \( L(c, d) \) (Pasch axiom).

Choose \( e \notin P \) and put \( P' = P \cup \{e\} \). We define \( \circ \) (group operation), \( ^{-1} \) (inverse) and \( e \) (identity) as follows:
1. \( a^{-1} = a \) and \( e \circ a = a = a \circ e \) for all \( a \in P' \);
2. \( a \circ b = L(a, b) \setminus \{a, b\} \) for all \( a \neq b \in P \);
3. \( a \circ a = \{a, e\} \) for all \( a \in P' \).

By Pasche axiom a group operation \( \circ \) is associative, so \( (P', \circ, ^{-1}, e) \) is a polygroup and we called it a Prenowitz algebra.

### 3. Core equivalence relation

In a polygroup, inverse and identity are not unique. To have a unique inverse and a unique identity, we construct a group modulo an equivalence relation. This equivalence relation is called core equivalence. Our goal is to construct a group which has some kind of definability, so we will show that core equivalence is (only) gradedly almost type-definable.

**Definition 3.1.** Let \( P = P_0/R_f \) be a gradedly almost hyperdefinable polygroup.

1. For \( a, b \in P_0 \) and \( i \in I \), we say that \( a \sim_{i_1} b \) if there is a generic \( g \downarrow ab \) such that \( a, b \in_{i} g \cdot h \) for some \( h \) (which must also be generic). \( \sim_{in} \) is the \( n \)-closure of \( \sim_{i_1} \), and \( \bigvee_{in} \sim_{in} \). We shall show that \( \sim \) is an \((I \times \omega)\)-gradedly almost type-definable equivalence relation, which we call the core equivalence.

2. We define the core \( N \) of \( P \) as follows: \( N_{i_1} \subseteq P_0 \) is the set of all \( a \) such that \( a \in_{i} g \cdot g^{-1} \) for some generic \( g \downarrow a \), and \( N_{in} = N_{i_1}^{n} \). One verifies that \( \bigcup_{i} N_{in} \) is a union of \( R \)-classes closed under inverse for all \( n < \omega \), so we can put \( N_n = (\bigcup_{i} N_{in})/R = N_{i_1}^{n} \), and \( N = \bigcup_{n} N_n \leq P \), the sub-polygroup generated by \( N_{i_1} \).

3. \( P \) is coreless if the core equivalence is the same as \( R \), that is for every \((i, n) \in I \times \omega \) there is \( j \in I \) such that \( R_j \) is coarser than \( \sim_{in} \).
By using a stratified rank and boundedness of the product, we show that the core equivalence is almost type-definable. And by the definition of core equivalence, inverse is unique modulo core equivalence.

Lemma 3.2. Let $P = P_0/R$ be an $I$-gradedly almost hyperdefinable polygroup.

1. $\sim$ is an $(I \times \omega)$-gradedly almost type-definable equivalence relation on $P$ coarser than $R$, and every $\sim$-class contains boundedly many $R$-classes (that is, if $a_R \sim b_R$ then $a_R$ and $b_R$ are interbounded as almost hyperimaginaries).
2. $P/\sim$ is coreless. Any almost hyperdefinable group is coreless.
3. If $P$ is coreless, then inverses are unique, and a unique identity exists. This is to say that there are $i \in I$ and $e \in P_0$ such that $(a^{-1})^{-1}, e \cdot a, a \cdot e \subseteq a_{R_i}$ for every $a \in P_0$.

4. BLowing up GENERIC CHUNKS

Our group operation is still a multifuncion. By a blow-up construction, we construct a new group and a new group operation from a coreless polygroup. Then the group operation is a function. Exactly speaking, by a blow-up construction, we construct a gradedly almost hyperdefinable generic group chunk from a coreless gradedly almost hyperdefinable generic group polygroup chunk.

Lemma 4.1. For every $i \in I$ there is $j \in I$ such that whenever $a_1, a_2, b_1, b_2, d_1 \in S_0$, the triplet $\{a_1 R, b_1 R, b_2 R\}$ is independent, and $d_1 \in (a_1^{-1} \cdot b_1)_{R_i} \cap (a_2 \cdot b_2^{-1})_{R_i}$, then there is $f \in a_1 \cdot a_2 \cap (b_1 \cdot b_2)_{R_j}$.

Moreover, if we have also $c_1, c_2, d_2 \in S_0$ such that $c_1 R \downarrow c_2 R = a_1 R \downarrow b_1 R, b_2 R c_1 c_2 R$ and $d_2 \in (a_1^{-1} \cdot c_1)_{R_i} \cap (a_2 \cdot c_2^{-1})_{R_i}$, and we take $f' \in a_1 \cdot a_2 \cap (c_1 \cdot c_2)_{R_j}$, then $f \in f'$ for some $i \in I$ dependent only on $i$. In particular, $f$ is unique up to $R_i$.

By the previous lemma, if we choose copies of $a_i (i = 1, 2)$, say $b_i$ and $c_i$, and define $(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = f$ as in the previous lemma, the product is unique modulo $R_i$. So we define a triplet $\tilde{a} = (a, a', a'')$ from $a \in S_0$ and the product of two triplets $\tilde{a} \cdot \tilde{b}$. Then we have a group from a polygroup as desired.

Definition 4.2. 1. We fix some $e \in S_0$, and set $S_0' = \{a \in S_0 : aR \downarrow eR\}$.
2. Define $\tilde{S} = \{(a, a', a'') \in S_0' : a' \in e^{-1} \cdot a \text{ and } a'' \in a \cdot e\}$ and $\tilde{S} = \tilde{S}_0/R$. (We follow a tacit understanding that $R$ may also stand for $R \times R \times R$, where this is clear from the context.)
3. A triplet $\tilde{a} = (a, a', a'') \in \tilde{S}_0$ is called a blow-up of $a$. Conversely, we define the blow-up map $\pi : \tilde{S}_0 \to S_0'$ by $\pi(a, a', a'') = a$, where $a$ is sometimes referred to as the axis of $(a, a', a'')$.
4. Given $\tilde{a}_R \downarrow e \tilde{b}_R$, we wish to define $\tilde{a} \cdot \tilde{b}$. First, we know that $e \in (a_1^{-1} \cdot a'')_{R_i} \cap (b \cdot b''^{-1})_{R_j}$ for some $i \in I$. By Lemma 4.1 there is $c \in a \cdot b \cap (a'' = b'')_{R_1}$, for some $2 \in I$. Again by Lemma 4.1 there is $c \in e^{-1} \cdot c \cap (b \cdot b'')_{R_2}$ and $c'' \in c \cdot e \cap (a \cdot b'')_{R_2}$. Set $\tilde{a} \cdot \tilde{b}$ to be the set of all $\bar{e} = (c, c', c'')$ obtained in this manner.
5. Recall that the inverse is a gradedly definable map, so it is only defined up to some $R_i$. Thus, for $\tilde{a} = (a, a', a'') \in \tilde{S}_0$, we can define its inverse as:

$$\tilde{a}^{-1} = \{ (b, b', b'') \in \tilde{S}_0 : b \in a^{-1}, b' \in a''^{-1}, b'' \in a'^{-1}\}$$

for $j \in I$ big enough to make sure that $\tilde{a}^{-1}$ cannot be empty; re-arranging previous choices we may assume that $j \leq 0$.  

Theorem 4.3. Let $S = S_0/R$ be a coreless gradedly almost hyperdefinable (over $\emptyset$) generic polygroup chunk, and $e \in S_0$. Let $\tilde{S}_0$ be as above. Then $\tilde{S} = \tilde{S}_0/R$ is a gradedly almost hyperdefinable generic group chunk over $e$.

5. Constructing an almost hyperdefinable group

Strictly speaking, we have a gradedly almost hyperdefinable generic group chunk from a group configuration in a simple theory. So we need the Weil-Hrushovski group chunk theorem for almost hyperdefinable group chunk to get a group.

Theorem 5.1. Let $<S_0/R, \cdot, -1>$ be an $I$-gradedly almost hyperdefinable group chunk. Then there is an I-g.e.r. $R'$ on $S_0^2$, such that $G = S_0^2/R'$ is a gradedly almost hyperdefinable group. Moreover, there is a gradedly type-definable map $\sigma : S \rightarrow G$ whose image generates $S$, and the couple $(G, \sigma)$ is gradedly unique as such, up to a unique graded isomorphism (i.e., for every other couple $(G', \sigma')$, there is a unique isomorphism, up to graded equality of maps, rendering $\sigma$ and $\sigma'$ gradedly qual).

References

[5] Itay Ben-Yaacov, Ivan Tomasic and Frank Wagner, Constructing an almost hyperdefinable group, preprint