A Gaussian Term Structure Model of Credit Spreads and Valuation of Credit Spread Options

by Masaaki KIJIMA*

Abstract:

This paper proposes a simple arbitrage-free model for the term structure of credit spreads and their evolution in time where the likelihood of default is interpreted by a stochastic intensity process. The extended Vasicek model of Hull and White [1990] is used not only for the default-free spot rates but also for the intensity process of default. Assuming that the recovery rate is constant, our model is shown to be very tractable analytically and consistent with the current term structures of default-free interest rates and credit spreads simultaneously. The model will prove useful in practice for pricing and hedging credit derivatives such as credit spread options.

I Introduction

This paper proposes a simple arbitrage-free model for the term structure of credit spreads and their evolution in time where the likelihood of default is interpreted by a stochastic intensity process. The distinguishing feature of the model is that it is consistent with the current term structures of default-free interest rates and credit spreads simultaneously. Also, the intensity process under the risk-neutral probability measure is given in terms of the instantaneous credit spread adjusted by the fractional loss. The model can be used for valuing corporate debt and for pricing and hedging credit derivatives such as credit spread options.

Pricing of corporate debt subject to credit risk has been extensively studied in the finance literature. We refer to Duffie and Singleton [1999] for the survey of such pricing models. Among them, Jarrow and Turnbull [1995] assumed that the payoffs upon default are expressed as an exogenous fraction of the claim and they showed that, under some regularity conditions, the price is given by the expected, discounted payoffs under the risk-neutral probability measure. Duffie and Singleton [1999] proposed another model in which the payoffs are discounted by an interest rate that is adjusted so as to reflect the effect of default risk. These models are classified into the reduced-form approach.[1] Recently, Madan and Unal [2000] developed a two-factor stochastic

---

* Professor, Graduate School of Economics, Kyoto University.

1) The key to value credit risk is how to construct the default model. In the finance literature, there are two major approaches, the *structural model* and the *reduced-form model*. While the former explicitly defines the default event in terms of the firm value and its capital structure to evaluate the probability of default and the payments, the latter leaves the default event undefined and assumes that the default intensity at any time prior to maturity is given exogenously. The primary advantage of the structural approach is its economical intuition while the reduced-form approach is analytically more tractable and easy to implement.

intensity model that incorporates the attractive feature of the structural model.[2)]

For the pricing of corporate debt, however, we also need to model the term structure of default-free interest rates and their evolution in time, since the future cashflow of the risky debt must be discounted in terms of the default-free discount factor. Several models of the default-free term structure have also been proposed in the finance literature and they now appear in standard textbooks of interest rates such as Rebonato [1996]. Among them, the extended Vasicek model proposed by Hull and White [1990],

$$dr(t) = (\phi_0(t) - a_0 r(t))dt + \sigma_0 dz_0(t), \quad 0 \leq t \leq T, \tag{1}$$

is known to be very tractable analytically and useful in practice. Here, $r(t)$ denotes the default-free spot rate, $\phi_0(t)$ is some deterministic function of time t, a_0 and σ_0 are positive constants, $z_0(t)$ denotes the standard Brownian motion under the risk-neutral probability measure P, and T is some future time. The model is Gaussian and consistent with the current term structure of default-free interest rates by choosing

$$\phi_0(t) = a_0 f(0,t) + \frac{\partial}{\partial t} f(0,t) + \frac{\sigma_0^2}{2a_0}(1 - e^{-2a_0 t}), \quad t \geq 0, \tag{2}$$

where $f(t,T)$ denotes the forward rate of the default-free discount bonds. Also, as Inui and Kijima [1998] pointed out, it is consistent with the Heath, Jarrow and Morton framework [1992][3)], so that the model incorporates all current information in the yield curve, and relies on markets being dynamically complete.

In this paper, we take an arbitrage-free setting in which all securities are priced in terms of the risk-neutral probability measure P. Under this probability measure, we let $h(t)$ denote the intensity process of default, and assume that $h(t)$ evolves according to the stochastic differential equation (hereafter SDE),

$$dh(t) = \big(\phi_1(t) - a_1 h(t)\big)dt + \sigma_1 dz_1(t), \quad 0 \leq t \leq T; \tag{3}$$

cf. (1). The SDE (3) describes the "mean reversion" of default rates, since empirical researches suggest such behaviors in company credit outlooks (see, e.g., Fons [1994]). Note that the mean-reverting level $\phi_1(t)$ in (3) cannot be estimated without the knowledge of the *unobservable* market price of risk.[4)] However, as we shall prove in Proposition 1 later, $\phi_1(t)$ can be determined

---

2) A similar two-factor model has been developed by Jarrow and Turnbull [2000] in a different content.

3) Heath, Jarrow and Morton [1992] proved that the forward rate $f(t,T)$ must satisfy the SDE

$$df(t,T) = \left(\sigma_0^f(t,T)\int_t^T \sigma_0^f(t,v)dv\right)dt + \sigma_0^f(t,T)dz_0(t), \quad 0 \leq t \leq T,$$

under the risk-neutral probability measure P. The extended Vasicek model (1) assumes

$$\sigma_0^f(t,T) = \sigma_0 e^{-a_0(T-t)}, \quad 0 \leq t \leq T.$$

An efficient numerical procedure for this model has been developed in Hull and White [1994a] and Kijima and Nagayama [1994].

4) Let $\lambda(t)$ denote the market price of risk, and let $\phi_1^*(t)$ be the mean-reverting level of the default rates under the *physical* probability measure. The mean-reverting level $\phi_1(t)$ under the risk-neutral probability measure P is given by $\phi_1(t) = \phi_1^*(t) - \lambda(t)\sigma_1(t)$.

so that the model is consistent with the current term structure of credit spreads. In other words, our model is flexible enough to describe any term structure of credit spreads observed in the market.

Let $\Delta(t,T)$ be the time t credit spread of the defaultable discount bond with maturity T. In practice, the relationship between credit spreads and default-free interest rates is important (see, e.g., Duffee [1998] and Madan and Unal [2000]). For such, the proposed model provides the relationship,

$$\lim_{T \to t} \Delta(t,T) = (1-\delta)h(t), \tag{4}$$

provided that default has not occurred until time t. Here δ denotes the constant recovery rate. Hence, assuming that the recovery rate is known exogenously, we can construct the model (3) for the intensity process $h(t)$ from the market data only. Moreover, our model allows any possibility by assuming a desired correlation between $h(t)$ and the default-free spot rate $r(t)$ [5].

This paper is organized as follows. In the next section, we formally define the default model in terms of a stochastic intensity process, and provide a pricing formula of defaultable discount bonds. In Section 3, we propose our Gaussian term structure model of credit spreads with constant recovery of treasury and show that the model has the above desired properties. As an application, Section 4 presents the pricing of credit spread options, while some numerical example is performed in Section 5 to investigate the impact of correlation between default-free spot rates and credit spreads. The model will prove useful in practice for pricing and hedging such credit derivatives. Section 6 concludes this paper. Proofs are given in Appendix.

Throughout the paper, we fix the probability space $(\Omega, \mathcal{F}, P)$ and denote the expectation operator by E. The probability measure P is the *risk-neutral measure* and we assume that such a P exists and is unique, since we are interested in pricing of financial instruments. The canonical filtration generated by the underlying stochastic structure is denoted by $\{\mathcal{F}_t\}$, which defines the information available at each time. The conditional probability measure given $\mathcal{F}_t$ is denoted by P_t and the associated conditional expectation operator is E_t.

II The Framework

In the reduced-form approach, researchers focus on the formulation of default intensity processes. Such a default model for the single asset case was first developed in the work of Madan and Unal [1998].[6] In this section, we formally define our default model and provide pricing formulas of both default-free and defaultable discount bonds.

Let τ be a default time of a corporate discount bond under consideration. The stochastic process $h(t)$ is called an *intensity process* for τ if

5) The empirical evidence on the relationship between credit spreads and interest rates is mixed; some authors report a negative relationship and the other a positive relationship. See Madan and Unal [2000] for details.

6) Recently, the default model was extended to the multivariate asset case by Duffie [1998], Kijima [2000] and Kijima and Muromachi [2000, 2001].

$$P_t\{\tau \leq t+\Delta t \mid \tau > t\} = h(t)\Delta t \tag{5}$$

for sufficiently small $\Delta t > 0$. That is, the intensity process $h(t)$ is the conditional rate of default just after time t given all the information available up to that time.[7] In this respect, we may call $h(t)$ a *default process* in the following. The *cumulative* default process over the time interval $[t,T]$ is denoted by

$$H(t,T) = \int_t^T h(s)ds, \quad 0 \leq t \leq T. \tag{6}$$

Since $h(t) \geq 0$ by definition, the cumulative default rate $H(t,T)$ is non-decreasing in $T \geq t$.

Suppose that the default process $h(t)$ is bounded and that, for each fixed T,

$$Y(t) = E_T\left[e^{-H(t,T)}\right], \quad 0 \leq t \leq T.$$

has no jumps almost surely. Then, it is known (see Proposition 1 in Duffie [1998]) that $Y(t)$ defines the conditional survival probability of τ,[8] i.e.

$$P_t\{\tau > T\} = E_t\left[e^{-H(t,T)}\right] \quad \text{on } \{\tau > t\}, \quad 0 \leq t \leq T. \tag{7}$$

In what follows, we shall often use the formula,

$$P_t\{\tau > T\} = E_t\left[E_T\left[e^{-H(t,T)}\right]\right] \quad \text{on } \{\tau > t\}, \quad 0 \leq t \leq T. \tag{8}$$

which is a simple consequence of the *chain rule* of conditional expectations. Also, from (5) and (7), we have

$$P_u\{u < \tau \leq u+du\} = h(u)e^{-H(t,u)}du, \quad t \leq u \leq T, \tag{9}$$

for sufficiently small $du > 0$. Equation (9) provides the *density function* of τ given $\tau > t$.

Now, it is well known that the time t price, $p(t,T)$, of the default-free discount bond with maturity T is given by

$$p(t,T) = E_t\left[e^{-R(t,T)}\right], \quad t \leq T; \quad R(t,T) = \int_t^T r(s)ds, \tag{10}$$

where $r(t)$ denotes the default-free spot rate.

Next, consider a corporate discount bond with face value 1 and maturity T. Pricing of the defaultable discount bonds depends on the definition of *recovery rate*. Let $\delta(t)$ be the recovery rate at time t given the information $\mathcal{F}_t$. That is, if default were to occur at time t, then

7) A formal definition of the intensity process and the associated default time, we refer to Grandell [1976].

8) For example, if the default process is constant, $h(t) = \lambda$ say, then it defines an exponential distribution with mean $1/\lambda$ via $P_t\{\tau > T\} = e^{-\lambda(T-t)}$, $t \leq T$. In this case, default occurs according to a Poisson process with intensity λ.

claimholders would receive $\delta(t)$ at that time. If no default occurs before the maturity, they will receive the face value at maturity T. Then, under technical conditions, the time t price, $v(t,T)$, of the defaultable discount bond with maturity T is given by

$$v(t,T) = E_t\left[e^{-R(t,\tau)}\delta(\tau)1_{\{\tau \leqq T\}} + e^{-R(t,T)}1_{\{\tau > T\}}\right], \quad t \leq T. \tag{11}$$

This is so under the "risk-neutral valuation" framework, since the payoff of this security is either $\delta(\tau)$ at default epoch τ if $\tau \leq T$ or the face value 1 at maturity T if $\tau > T$.

Using the density function (9) and the chain rule (8), the price (11) can be rewritten as

$$\begin{aligned} v(t,T) &= E_t\left[\int_t^T e^{-R(t,u)}\delta(u)P_u\{u < \tau \leq u+du\} + e^{-R(t,T)}E_T\left[1_{\{\tau > T\}}\right]\right] \\ &= E_t\left[\int_t^T \delta(u)h(u)e^{-\xi(t,u)}du + e^{-\xi(t,T)}\right], \end{aligned} \tag{12}$$

where we define

$$\xi(t,T) = R(t,T) + H(t,T) = \int_t^T\{r(u)+h(u)\}\,du, \quad t \leq T. \tag{13}$$

Note that we have used the identity $E_T[1_{\{\tau > T\}}] = e^{-H(t,T)}$ to obtain (12).

Let $L(t)$ be the *fractional loss* at time t. Then, according to Duffie and Singleton [1999], typical formulations for recovery are the following:

1. RMV (recovery of market value): $\delta(u) = \big(1-L(u)\big)v(u-,T)$
2. RT (recovery of treasury): $\delta(u) = \big(1-L(u)\big)p(u,T)$
3. RFV (recovery of face value): $\delta(u) = \big(1-L(u)\big)$

The RMV assumes that claimholders will loose a fraction $L(t)$ of the market value $v(t-,T)$ just prior to default epoch if it were to occur at time t, while the RFV assumption means they will loose a fraction $L(t)$ of the face value 1. The RMV assumption was used in Duffie and Singleton [1999] whereas the RFV was used by, e.g., Duffee [1998]. On the other hand, the RT assumption implies that claimholders will receive the recovery $\delta(t)$ of the face value at maturity T for sure if default were to occur at time t before the maturity. This formulation was used by Jarrow and Turnbull [1995] and Kijima and Muromachi [2000, 2001]. In this paper, we take the RT assumption for recovery formulation.[9)]

In the RT formulation, the time t price of the defaultable discount bond with maturity T is given, from (12), by

---

9) The pricing formulas under the other recovery formulations and their comparison are available from the author upon request. In particular, if all the parameters except the recovery formulation are the same, then we have
$v_{\mathrm{RMV}}(t,T) \leq v_{\mathrm{RT}}(t,T) \leq v_{\mathrm{RFV}}(t,T)$
where $v_a(t,T)$ denotes the price of defaultable discount bond with recovery formulation a, a = RMV, RT or RFV. This is so, since the recovery values are so ordered, and the pricing functional is a monotone operator.

$$\begin{aligned} v(t,T) &= E_t\left[e^{-R(t,T)}\int_t^T (1-L(u))h(u)e^{-H(t,u)}du + e^{-\xi(t,T)}\right] \\ &= E_t\left[e^{-R(t,T)}(1-L(\tau))1_{\{\tau \leqq T\}} + e^{-\xi(t,T)}\right] \quad \text{on } \{\tau > t\}, \end{aligned}$$

In particular, if the fractional loss $L(t)$ is constant, ℓ say, then, since

$$E_t\left[e^{-R(t,T)}1_{\{\tau \leqq T\}}\right] = p(t,T) - E_t\left[e^{-R(t,T)}E_T\left[1_{\{\tau > T\}}\right]\right] = p(t,T) - E_t\left[e^{-\xi(t,T)}\right],$$

we obtain

$$v(t,T) = (1-\ell)p(t,T) + \ell E_t\left[e^{-\xi(t,T)}\right] \quad \text{on } \{\tau > t\}. \tag{14}$$

Note that we do not assume that $r(t)$ and $h(t)$ are mutually independent. If they are independent, then we obtain from (14) that

$$v(t,T) = p(t,T)[(1-\ell) + \ell P_t\{\tau > T\}] \quad \text{on } \{\tau > t\},$$

as in Jarrow and Turnbull [1995].[10)]

Ⅲ A Gaussian Model with Constant RT Recovery

In this section, we propose a Gaussian term structure model of credit spreads under the constant RT recovery formulation. That is, we will assume the pricing formula (14) in the rest of this paper. Recall that, in this setting, the assumption of constant recovery, i.e. $L(t) = \ell$, simplifies the model substantially.[11)]

Suppose that the default-free spot rate $r(t)$ follows the extended Vasicek model (1) and the default process $h(t)$ follows the SDE (3) with correlation structure[12)]

$$dz_0(t)dz_1(t) = \rho dt$$

under the risk-neutral probability measure P. Since there is a strong evidence that default intensities of corporate bonds vary with the business cycle, it is important to introduce the correlation effect. Note that these processes become negative with positive probability, although the probability may be negligible.

The advantage of the Gaussian model, however, comes from the fact that the linear SDE (3) can be solved as

10) Lando [1994] relaxed the Jarrow and Turnbull model [1995] by allowing $h(t)$ need not be independent of $r(t)$, but at the cost of added computational complexity.

11) As in Madan and Unal [1998], we can assume that the fractional loss $L(t)$ is a random variable while maintaining the assumption that it is independent of time t and the other stochastic structures.

12) The parameters a_i, σ_i, and ρ can be deterministic functions of time t. The analysis even for such a case is identical to the one given in this paper at the expense of notational complexity.

$$h(t) = h(0)e^{-a_1 t} + \int_0^t \phi_1(s)e^{-a_1(t-s)}ds + \sigma_1 \int_0^t e^{-a_1(t-s)}dz_1(s), \quad 0 \le t \le T. \tag{15}$$

The solution $h(t)$ in (15) is a Gauss-Markov process, and since

$$h(t+v) = h(t)e^{-a_1 v} + \int_t^{t+v} \phi_1(s)e^{-a_1(t+v-s)}ds + \sigma_1 \int_t^{t+v} e^{-a_1(t+v-s)}dz_1(s), \qquad v \ge 0, \tag{16}$$

we obtain, after interchange of the order of integration,

$$H(t,T) = h(t)B_1(t,T) + \hat{A}_1(t,T) + \sigma_1 \int_t^T B_1(s,T)dz_1(s), \qquad t \le T, \tag{17}$$

where

$$B_1(t,T) = \frac{1-e^{-a_1(T-t)}}{a_1}$$

and

$$\hat{A}_1(t,T) = \int_t^T \phi_1(s)\frac{1-e^{-a_1(T-s)}}{a_1}ds.$$

It should be noted that the cumulative default process $H(t,T)$ is normally distributed with mean

$$\mu_1(t,T) \equiv E_t[H(t,T)] = h(t)\frac{1-e^{-a_1(T-t)}}{a_1} + \int_t^T \phi_1(s)\frac{1-e^{-a_1(T-s)}}{a_1}ds,$$

and variance

$$S_1^2(t,T) \equiv V_t[H(t,T)] = \frac{\sigma_1^2}{a_1^2}\left[(T-t) - 2\frac{1-e^{-a_1(T-t)}}{a_1} + \frac{1-e^{-2a_1(T-t)}}{2a_1}\right].$$

Similar results hold for the default-free spot rate $r(t)$ with the subscript 1 being replaced by 0. We denote the mean and variance of $R(t,T)$ by $\mu_0(t,T)$ and $S_0^2(t,T)$, respectively. Also, the covariance between $R(t,T)$ and $H(t,T)$ is given by

$$\begin{aligned} C(t,T) &\equiv C_t[R(t,T),H(t,T)] \\ &= \rho\frac{\sigma_0\sigma_1}{a_0 a_1}\left[(T-t) - \frac{1-e^{-a_0(T-t)}}{a_0} - \frac{1-e^{-a_1(T-t)}}{a_1} + \frac{1-e^{-(a_0+a_1)(T-t)}}{(a_0+a_1)}\right]. \end{aligned}$$

From (10), the time t price of the default-free discount bond with maturity T is given by

$$p(t,T) = \exp\left\{-\mu_0(t,T) + \frac{1}{2}S_0^2(t,T)\right\} = A_0(t,T)e^{-B_0(t,T)r(t)}, \tag{18}$$

where

$$A_0(t,T) = \exp\left\{\frac{1}{2}S_0^2(t,T) - \int_t^T \phi_0(s)B_0(s,T)ds\right\}.$$

This is so, because if random variable X is normally distributed with mean μ and variance σ^2, then its *moment generating function* is given by

$$E\left[e^{\theta X}\right] = \exp\left\{\theta\mu + \frac{\theta^2\sigma^2}{2}\right\}. \tag{19}$$

Hence, we have recovered the result obtained in Hull and White [1990].

Let $0 < \ell < 1$. In the pricing formula (14), since the random variable $\xi(t,T) = R(t,T) + H(t,T)$ is normally distributed with mean $\mu_0(t,T)+\mu_1(t,T)$ and variance $S_0^2(t,T) + S_1^2(t,T) + 2C(t,T)$, it follows from (19) that

$$E_t\left[e^{-\xi(t,T)}\right] = p(t,T)\mathrm{epx}\left\{-\mu_1(t,T) + \frac{1}{2}S_1^2(t,T) + C(t,T)\right\}. \tag{20}$$

Hence, in this Gaussian setting, the time t price of the defaultable discount bond with maturity T is given by

$$v(t,T) = p(t,T)\left[1-\ell+\ell A_1(t,T)e^{-B_1(t,T)h(t)}\right] \quad \text{on}\,\{\tau > t\}, \tag{21}$$

where

$$A_1(t,T) = \exp\left\{\frac{1}{2}S_1^2(t,T) - \int_t^T \phi_1(s)B_1(s,T)ds + C(t,T)\right\}.$$

The correlation effect appears in the pricing formula (21) through the term $C(t,T)$.

The significant property of our model is that we can determine the function $\phi_1(t)$ so that the model is consistent with the current term structure of the defaultable discount bonds. To this end, we define, from (21),

$$\gamma(t,T) = \frac{1}{\ell}\left[\frac{v(t,T)}{p(t,T)} - (1-\ell)\right] = A_1(t,T)e^{-B_1(t,T)h(t)}, \quad t \le T, \tag{22}$$

and

$$g(t,T) = -\frac{\partial}{\partial T}\log\gamma(t,T). \tag{23}$$

The function $g(t,T)$ is the forward spread between default-free discount bonds and defaultable discount bonds adjusted by the recovery rate $\delta = 1-\ell$. In particular, if $\delta = 0$, *i.e.* the case of no recovery, we have

$$g(t,T) = -\frac{\partial}{\partial T}\log\frac{v(t,T)}{p(t,T)} = f_1(t,T)-f(t,T), \quad t \le T,$$

where $f_1(t,T)$ is the forward rate of the defaultable discount bond with maturity T. The next result should be compared with the result (2).

Proposition 1 *Under the assumptions stated above, if*

$$\phi_1(t) = a_1 g(0,t)+\frac{\partial}{\partial t}g(0,t)+\frac{\sigma_1^2}{2a_1}(1-e^{-2a_1 t})+\rho\sigma_0\sigma_1\left(\frac{1-e^{-a_0 t}}{a_0}+\frac{e^{-a_0 t}-e^{-(a_0+a_1)t}}{a_1}\right),$$

then the model is consistent with the current term structure of defaultable discount bonds.

Let $\Delta(t,T)$ denote the time t credit spread for the discount bond with maturity T. Formally, we define

$$\Delta(t,T) = -\frac{1}{T-1}\log\frac{v(t,T)}{p(t,T)}. \tag{24}$$

It follows from (21) that

$$\Delta(t,T) = -\frac{1}{T-1}\log\left[1-\ell+\ell A_1(t,T)e^{-B_1(t,T)h(t)}\right] \quad \text{on}\{\tau > \mathrm{t}\}. \tag{25}$$

By the L'Hospital rule applying to (25), we then obtain the result (4), i.e.

$$\lim_{T\to t}\Delta(t,T) = \ell h(t) \quad \text{on}\{\tau > \mathrm{t}\}.$$

Hence, in this model, the default process $h(t)$ is interpreted as the *instantaneous* credit spread at time t adjusted by the fractional loss ℓ, provided that the default has not yet occurred. It follows that the spot rate of the defaultable discount bonds, $r_1(t)$ say, is given by

$$r_1(t) = r(t)+\ell h(t), \quad r_1(t) \equiv -\lim_{T\to t}\frac{\log v(t,T)}{T-t}, \tag{26}$$

which is the same as the default-adjusted spot rate of Duffie and Singleton [1999].

IV Credit Spread Options

In this section, we consider a European option written on the credit spread $\Delta(t,T)$ with the maturity s, $t < s < T$. The payoff function at the maturity s of the option is denoted by $G(x)$. Then, the time t price of the option is given by

$$\pi(t) = E_t\left[e^{-R(t,s)}G\big(\Delta(s,T)\big)\right], \quad t < s < T. \tag{27}$$

For example, the price of the corresponding put option with exercise spread K is obtained by calculating

$$\pi(t) = E_t\left[e^{-R(t,s)}\left\{K-\Delta(s,T)\right\}_+\right], \quad t < s < T, \tag{28}$$

where $\{x\} = \max\{0,x\}$.

Valuation of credit spread options is not new. For example, Longstaff and Schwartz [1995] assumed that the process $X(t)$ defined by $e^{X(t)} = \Delta(t,t+T)$ follows the Vasicek model [1977] and derived a closed form solution for credit spread options. However, this approach does not consider the possibility of default, which is somewhat contradictory because credit spreads exist due to the possibility of default. Kijima and Komoribayashi [1998] formulated a Markov chain model to evaluate credit spread options based on Jarrow, Lando and Turnbull model [1997], but they assumed that the default process and the default-free interest rates are mutually independent. See also Das and Tufano [1996]. In this section, we consider credit spread options when the defaultable discount bond price is given by (21). See Duffie and Singleton [1999] for the case of RMV formulation.

For the case of RT formulation with constant recovery, the credit spread $\Delta(t,T)$ is given by (25). Note that the random variables included there are just $h(t)$ and τ. Hence the option price $\pi(t)$ in (27) can be rewritten as

$$\begin{aligned}\pi(t) &= E_t\left[e^{-R(t,s)}G\big(\Delta(s,T)\big)1_{\{\tau > s\}}\right]+E_t\left[e^{-R(t,s)}G\big(\Delta(s,T)\big)1_{\{\tau \le s\}}\right] \\ &= E_t\left[e^{-R(t,s)}F_T\big(h(s)\big)1_{\{\tau > s\}}\right]+G\Big(-\frac{1}{T-s}\log\delta\Big)E_t\left[e^{-R(t,s)}1_{\{\tau \le s\}}\right],\end{aligned} \tag{29}$$

where $\delta = 1-\ell$ is the recovery rate at maturity and, from (25),

$$F_T(x) = G\Big(-\frac{1}{T-s}\log\Big[1-\ell+\ell A_1(s,T)e^{-B_1(s,T)x}\Big]\Big).$$

We will denote the first term of the right hand side in (29) by $\pi_1(t)$ and the second term by $\pi_2(t)$, so that $\pi(t) = \pi_1(t)+\pi_2(t)$.

For $\pi_2(t)$, we obtain, as before,

$$E_t\left[e^{-R(t,s)}1_{\{\tau \le s\}}\right] = p(t,s)-E_t\left[e^{-\xi(t,s)}\right],$$

so that

$$\pi_2(t) = p(t,s)\big(1-\gamma(t,s)\big)G\Big(-\frac{1}{T-s}\log\delta\Big), \quad \delta = 1-\ell, \tag{30}$$

where $\gamma(t,T)$ is defined in (22). For $\pi_1(t)$, on the other hand, we need the following result.

Lemma 1 *Suppose that* (X,Y) *is jointly, normally distributed. Then*

$$E\left[e^{-Y}f(X)\right] = E\left[e^{-Y}\right]E\left[f\left(X-C[X,Y]\right)\right]$$

for any function $f(x)$ *for which the expectation exists.*

Now, for $\pi_1(t)$, we first obtain

$$\pi_1(t) = E_1\left[e^{-\xi(t,s)}F_T\left(h(s)\right)\right]. \tag{31}$$

Next, referring to (16), we obtain
\

$$h(s) = h(t)e^{-a_1(s-t)} + \int_t^s \phi_1(u)e^{-a_1(s-u)}du + \sigma_1 \int_t^s e^{-a_1(s-u)}dz_1(u), \quad t \leq s. \tag{32}$$

Hence $h(s)$ is normally distributed with mean

$$m_1(t,s) \equiv E_t[h(s)] = h(t)e^{-a_1(s-t)} + \int_t^s \phi_1(u)e^{-a_1(s-u)}du, \quad t < s,$$

and variance

$$V_1^2(t,s) \equiv V_t[h(s)] = \frac{\sigma_1^2}{2a_1}\left(1-e^{-2a_1(s-t)}\right), \quad t < s.$$

From (17) and (32), we know that the covariance between $h(s)$ and $\xi(t,s)$ is given by

$$\begin{aligned} C_h(t,s) &\equiv C_t\left[h(s), \xi(t,S)\right] \\ &= \rho\frac{\sigma_0\sigma_1}{a_0}\left[\frac{1-e^{-a_1(s-t)}}{a_1} - \frac{1-e^{-(a_0+a_1)(s-t)}}{a_0+a_1}\right] \\ &\qquad + \frac{\sigma_1^2}{a_1}\left[\frac{1-e^{-a_1(s-t)}}{a_1} - \frac{1-e^{-2a_1(s-t)}}{2a_1}\right]. \end{aligned}$$

The next result can now be proved using Lemma 1.

Proposition 2 *Let* $\tilde{h}(s)$ *be normally distributed with mean* $m_1(t,s) - C_h(t,s)$ *and variance* $V_1^2(t,s)$. *Then the option price* $\pi(t)$ *in (27) is given by*

$$\pi(t) = v_0(t,s)\left(\gamma(t,s)E_t\left[F_T\left(\tilde{h}(s)\right)\right] + \left(1-\gamma(t,s)\right)G\left(-\frac{1}{T-s}\log\delta\right)\right), \quad t < s < T.$$

Table 1. Parameter values for the numerical example

a_0	σ_0	a_1	t	T	s	K	$f(0,t)=f_0$
0.2	0.02	0.1	0	5	1	0.1	0.05

$f_1(0,t)=f_1$	0.06, **0.07**, 0.08
σ_1	**0.01**, 0.02
δ	0.3, 0.4, **0.5**, 0.6, 0.7
ρ	From -1 to 1 with step size 0.2

In the table, a_0 (a_1, respectively) represents the restoring force towards the mean-reverting level in the default-free spot rate $r(t)$ (the default process $h(t)$) and $\sigma_0(\sigma_1)$ is its volatility. Also, t is the current time, T the maturity of the underlying discount bonds, and s the maturity of the put option with strike spread K. $f(t,T)$ denotes the forward rate of the default-free discount bonds, whereas $f_1(t,T)$ the forward rate of the defaultable discount bonds. δ is the recovery rate and ρ represents the correlation coefficient between $r(t)$ and $h(t)$. These parameter values are used throughout the numerical example. The bold-face numbers indicate that they are used as the base model.

V A Numerical Example

In this section, we provide a numerical example to investigate the impact of the correlation coefficient ρ for various values of recovery rate $\delta = 1-\ell$ and initial term structure $f_1(0,t)$ on the prices of put option (28). Throughout the example, we use the parameter values listed in Table 1. The current forward rate curve $f(0,t)$ for the default-free discount bonds is assumed to be flat and hence $r(0) = f_0$, say. The time-dependent level $\phi_0(t)$ is obtained from (2) and the time t price, $p(t,T)$, of the default-free discount bond with maturity T is given by (18).

The current forward rate curve $f_1(0,t)$ of the defaultable discount bonds is also assumed to be flat and so, from (26), we have

$$h(0) = \frac{f_1-f_0}{1-\delta}, \quad \delta = 1-\ell. \tag{33}$$

Since then

$$p(0,s) = e^{-f_0 s} \quad \text{and} \quad v(0,s) = e^{-f_1 s}, \quad s \geq 0, \tag{34}$$

we have from (23) that

$$g(0,s) = -\frac{f_1-f_0}{1-\delta} e^{-(f_1-f_0)s}, \quad s \geq 0.$$

The time-dependent level $\phi_1(t)$ is now obtained from Proposition 1, once the parameter values for δ, ρ and f_1 are specified.

The mean and variance of normally distributed random variable $\tilde{h}(s)$ is given in

Proposition 2. The put option price can now be calculated from (34), (22) and numerical integration of

$$E_0\Big[F_T\big(\tilde{h}(s)\big)\Big] = \int_{-\infty}^{\infty} F_T(x)\tilde{n}(x)dx, \tag{35}$$

where $\tilde{n}(x)$ denotes the density function of $\tilde{h}(s)$. Note that, since $G(x) = \{K-x\}_+$, we have

$$G\Big(-\frac{1}{T-s}\log\delta\Big) \iff K+\frac{1}{T-s}\log\delta \leq 0.$$

Integration in (35) causes a bit trouble because it ranges from $-\infty$ to some point x_0, where $F_T(x_0) = 0$. Also, the range $(-\infty, 0)$ means that the default rate $\tilde{h}(s)$ is negative at time s. So we modify the theoretical value (35) in such a way that

$$E_0\Big[F_T\big(\tilde{h}(s)\big)\Big] \approx \frac{1}{P\{\tilde{h}(s) \geq 0\}}\int_{0}^{\infty} F_T(x)\tilde{n}(x)dx \tag{36}$$

The put option values obtained from (36) are compared with those from (35) in Table 2. The option values are surprisingly robust with respect to the correlation coefficient ρ. The differences between the two methods are rather large compared to the sensitivity in ρ. Of course, the difference depends on the variance $V_1^2(0,s)$.

In the following computation, we utilize the approximation (36) for simplicity. Table 3 shows put option prices for various initial term structures when the recovery rate is fixed as $\delta = 0.5$, where $h(0) = 0.02$ from (33), while Table 4 gives put option prices for various recov-

Table 2. Exact vs. approximated values for numerical integration

	$\sigma_1 = 0.01$		$\sigma_1 = 0.02$	
ρ	Exact	Approximation	Exact	Approximation
-1.0	7.286	7.286	7.297	7.269
-0.8	7.282	7.282	7.290	7.262
-0.6	7.278	7.278	7.282	7.254
-0.4	7.274	7.274	7.275	7.247
-0.2	7.271	7.271	7.267	7.240
0.0	7.267	7.267	7.260	7.233
0.2	7.263	7.263	7.252	7.225
0.4	7.259	7.259	7.245	7.218
0.6	7.256	7.256	7.237	7.211
0.8	7.252	7.252	7.230	7.203
1.0	7.248	7.248	7.222	7.196

The prices of put options written on a 5-year credit spread are calculated using the exact form (35) and approximation (36). The listed values are multiplied by 100. The maturity of the options is 1 year. The parameters are set as for the base model in Table 1 so that $h(0) = 0.04$ from (33). The model is chosen to fit the initial term structures of default-free interest rates and credit spreads simultaneously.

Table 3. Put option prices with respect to the initial forward rate

ρ	$f_1 = 6.0$ %	$f_1 = 7.0$ %	$f_1 = 8.0$ %
-1.0	8.399	7.286	6.191
-0.8	8.395	7.282	6.187
-0.6	8.390	7.278	6.184
-0.4	8.386	7.274	6.180
-0.2	8.382	7.271	6.177
0.0	8.378	7.267	6.173
0.2	8.374	7.263	6.170
0.4	8.370	7.259	6.166
0.6	8.366	7.256	6.163
0.8	8.362	7.252	6.159
1.0	8.358	7.248	6.156
$h(0)$	2.0%	4.0%	6.0%

The prices of put options written on a 5-year credit spread are calculated using (36). The listed values are multiplied by 100. The maturity of the options is 1 year. The recovery rate is fixed as $\delta = 0.5$ while other parameters are set as in Table 1. The model is chosen to fit the initial term structures of default-free interest rates and credit spreads simultaneously.

Table 4. Put option prices with respect to the recovery rate

ρ	$\delta = 0.3$	$\delta = 0.4$	$\delta = 0.5$	$\delta = 0.6$	$\delta = 0.7$	$\delta = 0.8$
-1.0	7.401	7.352	7.286	7.187	7.093	7.120
-0.8	7.395	7.348	7.282	7.184	7.091	7.119
-0.6	7.389	7.343	7.278	7.181	7.089	7.118
-0.4	7.384	7.338	7.274	7.179	7.087	7.117
-0.2	7.378	7.334	7.271	7.176	7.085	7.116
0.0	7.373	7.329	7.267	7.173	7.083	7.115
0.2	7.367	7.324	7.263	7.170	7.081	7.114
0.4	7.361	7.320	7.259	7.167	7.079	7.113
0.6	7.356	7.315	7.256	7.164	7.078	7.112
0.8	7.350	7.310	7.252	7.162	7.076	7.111
1.0	7.344	7.305	7.248	7.159	7.074	7.110
$h(0)$	2.86%	3.33%	4.00	5.00	6.67%	10.00%

Prices of put options written on a 5-year credit spread are calculated using (36). The listed values are multiplied by 100. The maturity of the options is 1 year. The current forward rate curve of the defaultable discount bonds is flat and fixed as $f_1(0,t) = 0.07$ while other parameters are set as in Table 1. The initial values for the default process $h(t)$ are calculated from (33). The model is chosen to fit the initial term structures of default-free interest rates and credit spreads simultaneously.

ery rates when the current term structure is $f_1(0,t) = 0.08$. These tables show that, although the put option price is decreasing in ρ, the price is robust with respect to the correlation coefficient ρ. This seems against our intuition because the correlation coefficient is one of the most sensitive parameters for the portfolio management (e.g. for Value at Risk). The decreasing property is also against our intuition since the credit spread is decreasing in ρ provided that $\phi_1(t)$ is held fixed. Note, however, that the correlation coefficient ρ affects $\phi_1(t)$ as well as the mean of $\tilde{h}(s)$ in a complex way. Hence the effect from ρ seems indeterminate. Through our numerical experiments, we have encountered only such a case that the put option prices decrease with respect to ρ.

According to Table 4, we also conclude that the put option prices are robust with respect to the recovery rate $\delta = 1 - \ell$. The robustness with respect to the recovery rate and the correlation coefficient seems due to the fact that our model is consistent with the current term structure. Indeed, when we change the current term structure as in Table 3, option prices change rather drastically. Hence, we conclude that, in our Gaussian term structure model, credit spread options are sensitive to the initial term structure of credit spreads, and insensitive to the correlation between default-free interest rates and credit spreads. The recovery rate is not a major factor either. Note, however, that the robustness property is important in practice, because the recovery rate as well as the correlation coefficient is usually very difficult to estimate.

VI Concluding Remarks

This paper provides a simple arbitrage-free model for the term structure of credit spreads and their evolution in time. The likelihood of default of corporate debt is described by the extended Vasicek model of Hull and White [1990] and the pricing of defaultable discount bonds is derived under the RT framework of constant recovery. Our model is very tractable analytically and consistent with the current term structures not only of default-free interest rates but also of credit spreads. As an application, we obtain a valuation formula for European options written on the credit spread. Through numerical experiments, we observed that, in our Gaussian term structure model, credit spread options are sensitive to the initial term structure of credit spreads, and insensitive to the correlation between default-free interest rates and credit spreads.

The valuation of American derivatives is also straightforward within our framework. We can develop a two-dimensional discrete model (e.g. trinomial tree model) with the state space consisting of states for $\left(r(t), h(t)\right)$ and an absorbing state that represents default of a corporate debt. Except the appended absorbing state, the construction of the discrete model is the same as usual and the valuation of American derivatives is performed backwardly. If we use a two-dimensional trinomial tree that incorporates the correlation structure suitably, the approximation technique developed by Hull and White [1994b] may be useful. We believe that our model will prove useful in practice for pricing and hedging credit derivatives such as credit spread options.

A Proof of Proposition 1

From (20) and (23), we obtain

$$g(0,t) = \mu_1'(0,t) - \frac{1}{2}\big(S_1^2(0,T)\big)' - C'(0,t),$$

where the prime $'$ denotes the derivative with respect to t. Using the identity

$$\frac{\partial}{\partial T}B_i(t,T) = -\frac{\partial}{\partial t}B_i(t,T) = e^{-a_i(T-t)}, \quad i = 0,1,$$

it follows that

$$g(0,t) = e^{-a_1 t}h(0) + \int_0^t \phi_1(u)e^{-a_1(t-u)}du - \frac{\sigma_1^2}{2}B_1^2(0,t) - \rho\sigma_0\sigma_1 B_0(0,t)B_1(0,t),$$

so that

$$g'(0,t) = -a_1 e^{-a_1 t}h(0) + \phi_1(t) - a_1\int_0^t \phi_1(u)e^{-a_1(t-u)}du - \sigma_1^2 B_1^2(0,t)e^{-a_1 t}$$
$$-\rho\sigma_0\sigma_1\big(e^{-a_0 t}B_1(0,t) + e^{-a_1 t}B_0(0,t)\big).$$

The result follows by calculating $a_1 g(0,t) + g'(0,t)$.

B Proof of Lemma 1

Let $\zeta(x,y)$ be the joint density function of (X,Y) and define

$$\zeta_X(x) = \int_{-\infty}^{\infty} e^{-y}\zeta(x,y)dy, \qquad -\infty < x < \infty.$$

Then,

$$E\Big[e^{-Y}f(X)\Big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-y}f(x)\zeta(x,y)dxdy = \int_{-\infty}^{\infty} f(x)\zeta_X(x)dx.$$

Denoting the moment generating function of (X,Y) by $\eta(s,t) = E_t\Big[e^{sX+tY}\Big]$, we obtain

$$\eta(s,-1) = \int_{-\infty}^{\infty} e^{sx}\zeta_X(x)dx.$$

Since $sX - Y$ is normally distributed, it follows from (19) that

$$\eta(s,-1) \quad = \exp\Big\{sE[X] + \frac{s^2}{2}V[X] - E[Y] + \frac{1}{2}V[Y] - sC[X,Y]\Big\}$$

$$= E\left[e^{-Y}\right]\exp\left\{s\left(E[X]-C[X,Y]\right)+\frac{s^2}{2}V[X]\right\}$$
$$= E\left[e^{-Y}\right]E\left[e^{s(X-C[X,Y])}\right].$$

Recall that the moment generating function determines the distribution uniquely, if it exists. Hence, we conclude that $\zeta_X(x)/E[e^{-Y}]$ is a density function of $\left(X-C[X,Y]\right)$. It follows that

$$E\left[e^{-Y}f(X)\right] = E\left[e^{-Y}\right]\int_{-\infty}^{\infty} f(x)\frac{\zeta_X(x)}{E[e^{-Y}]}dx = E\left[e^{-Y}\right]E\left[f\left(X-C[X,Y]\right)\right],$$

completing the proof.

Acknowledgement. The author is grateful to Dr. Yukio Muromachi of NLI Research Institute for helpful discussions and numerical assistant.

References

Das, S.R., and P. Tufano. (1996). "Pricing Credit-Sensitive Debt when Interest Rates, Credit Ratings and Credit Spreads are Stochastic." *Journal of Financial Engineering*, **5**, 161-198.

Duffee, G. (1998). "The Relation between Treasury Yields and Corporate Bond Yield Spreads." *Journal of Finance*, **53**, 2225-2242.

Duffie, D. (1998). "First-to-Default Valuation." Working paper, Stanford University.

Duffie, D., and K. Singleton. (1999). "Modeling Term Structures of Defaultable Bonds." *Review of Financial Studies*, **12**, 687-720.

Fons, J.S. (1994). "Using Default Rates to Model the Term Structure of Credit Risk." *Financial Analysts Journal*, September-October, 25-32.

Grandell, J. (1976). *Double Stochastic Poisson Processes*, Lecture Notes in Mathematics, **529**, New York, NY: Springer.

Heath, D., R. Jarrow, and A. Morton. (1992). "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation." *Econometrica*, **60**, 77-105.

Hull, J., and A. White. (1990). "Pricing Interest-Rate-Derivative Securities." *Review of Financial Studies*, **3**, 573-592.

Hull, J., and A. White. (1994a). "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models." *Journal of Derivatives*, **2**, 7-16.

Hull, J., and A. White. (1994b). "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models." *Journal of Derivatives*, **2**, 37-48.

Inui, K., and M. Kijima. (1998). "A Markovian Framework in Multi-Factor Heath-Jarrow-Morton Models." *Journal of Financial and Quantitative Analysis*, **33**, 423-440.

Jarrow, R.A., D. Lando, and S.M. Turnbull. (1997). "A Markov Model for the Term Structure of Credit Risk Spread." *Review of Financial Studies*, **10**, 481-523.

Jarrow, R.A., and S.M. Turnbull. (1995). "Pricing Options on Financial Securities subject to Default Risk." *Journal of Finance*, **50**, 53-86.

Jarrow, R.A. and S.M. Turnbull. (2000). "The Intersection of Market and Credit Risk." *Journal of Banking and Finance*, **24**, 271-299.

Kijima, M. (2000). "Valuation of a Credit Swap of the Basket Type." *Review of Derivatives Research*, **4**, 79-95.

Kijima, M., and K. Komoribayashi. (1998). "A Markov Chain Model for Valuing Credit Risk Derivatives." *Journal of Derivatives*, **6** (Fall), 97-108.

Kijima, M., and Y. Muromachi. (2000). "Credit Events and the Valuation of Credit Derivatives of Basket Type." *Review of Derivatives Research*, **4**, 79-95.

Kijima, M. and Muromachi, Y. (2000), "Evaluation of Credit Risk of a Portfolio with Stochastic Interest Rate and Default Processes" *Journal of Risk*, **3**, 5-36.

Kijima, M., and I. Nagayama. (1994). "Efficient Numerical Procedures for the Hull-White Extended Vasicek Model." *Journal of Financial Engineering*, **3**, 275-292.

Lando, D. (1994). "Three Essays on Contingent Claims Pricing." PhD. Thesis, Department of Statistics, Cornell University.

Longstaff, F., and E. Schwartz. (1995). "Valuing Credit Derivatives." *Journal of Fixed Income*, **5**, 6-12.

Madan, D., and H. Unal. (1998). "Pricing the Risks of Default." *Review of Derivatives Research*, **2**, 121-160.

Madan, D. and H. Unal. (2000). "A Two-Factor Hazard Rate Model for Pricing Risky Debt and the Term Structure of Credit Spreads." *Journal of Financial and Quantitative Analysis,* **35**, 43-65.

Rebonato, R. (1996). *Interest-rate Option Models — Understanding, Analysing and Using Models for Exotic Interest-rate Options*. New York, NY: Wiley.

Vasicek, O.A. (1977). "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics,* **5**, 177-188.