DIMENSIONAL ANALYSIS OF CONSTANT IN EQUATION OF VISCOSITY FOR ROLLING BALL METHOD

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In the present paper, it was found that the equationot H. W. Lewis Eq. (5) deviates from experimental results in the range below $d/D = 0.95$, and that the equation proposed satisfactorily agrees with experimental results with correction factor.

Introduction

For many years the system of an inclined tube and a rolling ball, the so-called rolling ball viscometer, has been used as the valuable viscometer because it can be applied to a fluid which has the viscosity of $10^{-4} \sim 10^4$ poise[1] in a wide region of temperature and pressure. It is useful especially at high temperatures and pressures.

There are few papers that have theoretically analysed the equation of viscosity for rolling ball method. R. M. Hubbard and G. G. Brown[2] obtained a formula by the dimensional analysis. The formula given by them is

$$\mu = \frac{5\pi}{42} \cdot g \cdot \sin\theta \cdot \frac{\rho_\bullet - \rho}{V} \cdot K \cdot d \cdot (D + d) \qquad (1)$$

where,

μ : viscosity of fluid (poise)

g : acceleration of gravity (980 cm/sec²)

θ : angle of inclination of tube to the horizontal

ρ : density of fluid (g/cm³)

$\rho_\bullet$: density of ball (g/cm³)

V : terminal rolling velocity of ball (cm/sec)

d : diameter of ball (cm)

D : diameter of tube (cm)

In their analysis, the variables used are:

Variable	Symbol	Dimension
Driving force	R	MLT^{-2}
Equivalent diameter	h	L
Density of fluid	ρ	ML^{-3}

1) W. H. Markwood, *A. S. T. M. proc.*, **51**, 457 (1951)

2) R. M. Hubbard, G. G. Brown, *Ind. Eng. Chem., Anal. Ed.*, **15**, 212 (1943)

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| Viscosity of fluid | μ | $ML^{-1}T^{-1}$ |
| Velocity of fluid | u | LT^{-1} |

where,

R : driving force on a ball or the resistance of fluid to the motion of a ball (g·cm/sec²), and is written as

$$R=\frac{5}{7}\cdot g\cdot\sin\theta\cdot\frac{\pi d^{3}}{6}\cdot(\rho_{s}-\rho) \tag{2}$$

h : equivalent diameter of the annular space between a ball and a tube (equal to four times the value which is defined as the cross-sectional area of the channel divided by the wetted perimeter hydrodynamically) (cm), and is written as

$$h=4\cdot\frac{\frac{\pi}{4}(D^{2}-d^{2})}{\pi(D+d)}=D-d \tag{3}$$

u : average fluid velocity through the annular space between a ball and a tube (cm/sec), and is written as

$$u=V\cdot\frac{d^{2}}{D^{2}-d^{2}} \tag{4}$$

Then, two dimensionless products or groups must be found.

The Reynolds number ($hu\rho/\mu$) and the resistance factor ($R/h^{2}\rho u^{2}$) as two dimensionless groups were assumed to derive the above Eq.(1). Equation (1) contains K which is a function of the ratio d/D. The constant, K, must be calculated for a set of a tube and a ball by the measurement using viscosity-known fluid.

Therefore the Eq.(1) is unsatisfactory for the practical use, because the constant in their equation varies according the diameters of the tube and the ball.

Afterwards H. W. Lewis[3] derived the constant, K, as a function of diameters of the tube D, and the ball, d, from the hydrodynamical analysis, as follows:

$$K=0.0891\left(\frac{D-d}{D}\right)^{\frac{1}{2}} \tag{5}$$

In the present paper, a new, more satisfactory formula is derived by the dimensional analysis.

Dimensional Analysis

As the constant K in Eq.(1) is a function of the ratio d/D, K should be related to the crescent shape between a tube and a ball. It is necessary to determine a new dimensionless group relating to this crescent shape and containing a new variable other than R, h, ρ, μ and u. Although the ratio d/D is a simple quantity representing the crescent shape, it is unsuitable because of its being dimensionless. Therefore a new quantity, ε, is defined as a function of the circumference, L, and the cross-sectional area, S, of the crescent as follows:

3) H. W. Lewis, *Anal. Chem.*, **25**, 507 (1953).

Dimensional Analysis of Constant in Equation 9

$$\varepsilon = \frac{L^2}{S}. \tag{6}$$

S is written as

$$S = \frac{\pi}{4}(D^2 - d^2) \tag{7}$$

then, from the definition of L,

$$L = 4 \cdot \frac{S}{h} \tag{8}$$

Substituting Eq.(8) into Eq.(6), the following relation is obtained:

$$\varepsilon = 16\frac{S}{h_2} \tag{9}$$

The dimensional analysis is carried out using six variables and three kinds of fundamental units as summarized in the next table.

Variable	Symbol	Dimension
Driving force	R	MLT^{-2}
Equivalent diameter	h	L
Density of fluid	ρ	ML^{-3}
Viscosity of fluid	μ	$ML^{-1}T^{-1}$
Velocity of fluid	u	LT^{-1}
Area of crescent shape	S	L^2

Furthermore, the Reynolds number, $(hu\rho)/\mu$, the resistance factor, $R/(h^2\rho u^2)$ and the new difined shape factor, $16S/h^2$, are used as three dimensionless quantities. A general relation is

$$C\left(\frac{R}{u^2h^2\rho}\right)^a\left(\frac{\mu}{uh\rho}\right)^b\left(\frac{S}{h^2}\right)^f = 1, \tag{10}$$

16 is included in the coefficient C. Indexes, a, b and f, and the coefficient C in Eq.(10) should be determined from the results of measurement.

The equation, which is used by R. M. Hubbard and G. G. Brown for the derivation of Eq.(1), is

$$K\left(\frac{R}{h^2\rho u^2}\right)\left(\frac{\mu}{hu\rho}\right)^{-1} = 1. \tag{11}$$

Comparing Eq.(10) with Eq.(11), the following solutions are obtained.

$$a = 1, \quad b = -1 \quad \text{and} \quad C\left(\frac{S}{h^2}\right)^f = K. \tag{12}$$

These relations are substantially valid in the region of streamline flow.

$$K = C\left(\frac{S}{h^2}\right)^f = C\left(\frac{\pi}{4}\frac{D+d}{D-d}\right)^f \tag{13}$$

1/20 as the intercept and -2 as the slope are derived from the log-log straight line of Eq. (13). Furthermore, substituting 1/20 for c and -2 for f into the Eq.(13), then

$$K = \frac{1}{20}\left(\frac{\pi}{4}\frac{D+d}{D-d}\right)^{-2}. \tag{14}$$

This equation is fairly agreeable with the experimental result of R. M. Hubbard and G. G. Brown in the range below $d/D = 0.95$, and beyond this value, Eq. (14) gives higher one. This

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reason is considered to be ascribed to the difference between the real diameters. D, d and the effective diameters of them. It can be assumed that there is a limit at which a ball can not roll down as the ratio, d/D, approaches to one, because the cohesive resistance of fluid to the surface of a ball and a tube becomes larger than the component force of the gravity for a ball. From this standpoint, a correction is attempted to allow for the increase in the size of the ball and the decrease in the size of the tube. The diameters, d and D, of the Eqs. (3), (4) and (7) are necessary to be correct, but d of Eq. (2) is not necessary to be correct. The correction factor α for D and d is simply assumed and αD and d/α are used instead of D and d respectively. Then Eq. (14) is

$$K = \frac{20}{1}\left(\frac{4}{\pi} \cdot \frac{\alpha^2 D - d}{\alpha^2 D + d}\right)^2, \tag{15}$$

and Eq. (1) becomes

$$\mu = \frac{2}{21\pi} \cdot g \cdot \sin\theta \cdot \frac{\rho_s - \rho}{V} \cdot \frac{\alpha d (\alpha^2 D - d)^2}{\alpha^2 D + d} \tag{16}$$

and it is possible to show that

$$\frac{\alpha^2 D - d}{\alpha^2 D + d} \doteqdot \frac{\alpha^2 D - d}{D + d}, \qquad \frac{\alpha d}{\alpha^2 D + d} \doteqdot \frac{d}{D + d},$$

therefore

$$K = \frac{1}{20}\left(\frac{4}{\pi} \cdot \frac{\alpha^2 D - d}{D + d}\right)^2 \quad \text{and} \quad \mu = \frac{2}{21\pi} \cdot g \cdot \sin\theta \cdot \frac{\rho_s - \rho}{V} \cdot \frac{d(\alpha^2 D - d)^2}{D + d} \tag{17},(18)$$

Experiment and Experimental Results

The tubes used in this experiment should be circular in bore, have equal diameter over the whole length, have a smooth inner surface, and were obtained from among many burettes. The smooth and completely spherical steel-ball needs to correspond to the diameter of the tube, and was obtained from a commercial ball bearing. The viscometer-tube is set in an airconditioned room, the temperature of which is regulated within $\pm 0.3°C$. The inclination of the tube and the distance traversed by the ball were measured with a cathetometer. The experimental data and the calculated values from Eqs. (5) and (17) are given in Table 1. The experimental data of Hubbard and Brown, Hoeppler[4], Benning and Markwood[5], Spée[6] Kiyama and Makita[7], and the author are plotted in Fig. 1 and Eqs. (5) and (17) also are shown graphically in Fig. 1.

4) F., Hoeppler, *Z. tech. Physik*, 14, 165 (1933)

5) A. F., Benning, and W. H., Markwood, *Refrig. Eng.*, 37, 243 (1939)

6) V. Spée, *Trans. Chem. Eng. Congr. World Power Conf.*, 2, 1 (1937), London, Perry Lund, Humphires and Co.

7) R. Kiyama and T. Makita, *This Journal*, 21, 63 (1961). 22, 49 (1952), 24, 74 (1954)

Table 1. Summary of experimental conditions and results

	Benzene	Benzene	Sugar solution[a]	Sugar solution[a]
Viscosity of Fluid[b] (c. p.)	0.63	·0.63	32	32
Tube No.	1	2	3	3
Tube Diameter (D, cm)	0.8916	0.9830	1.0570	1.0570
Ball Diameter (d, cm)	0.8733	0.9535	0.9535	0.8733
Diameter Ratio (d/D)	0.980	0.970	0.902	0.826
Inclination (θ)	2–28	2–27	2–76	2–38
Ball Density (ρ_s, g/cm^3)	7.758	7.758	7.758	7.758
Fluid Density (ρ, g/cm^3)	0.877	0.877	1.275	1.275
Critical Reynolds No.	54.4	37.7	9.0	8.7
Correlation Factor K				
⎧Experimental[c]	7.22×10^{-6}	1.55×10^{-5}	2.18×10^{-4}	7.29×10^{-4}
⎨Calculated from equation (17)	7.10×10^{-6}	1.64×10^{-5}	2.06×10^{-4}	7.18×10^{-4}
⎩Caluculated from equation (5)	5.36×10^{-6}	1.40×10^{-5}	2.67×10^{-4}	1.29×10^{-3}

a): aqueous sucrose solution (57.8 Wt.%)

b): *International Critical Table* 5, 12 (1933)

   (fitted with arthors' experimental result using Ostwald Viscometer)

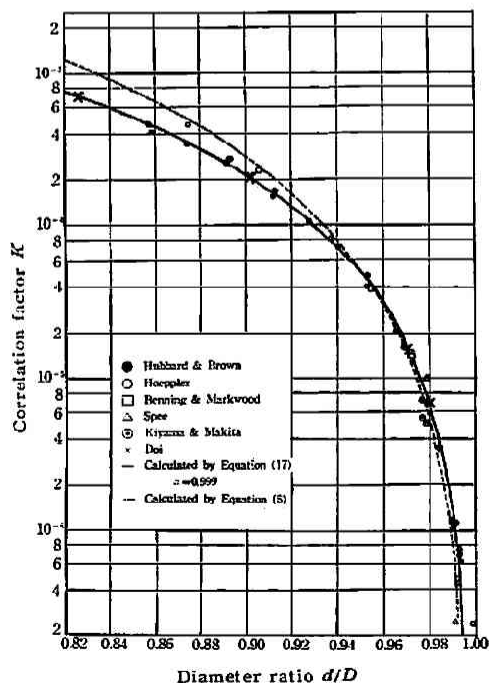c): These values are determined by data within critical Reynolds No.

Fig. 1. Correlation for rolling ball viscometer

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Considerations

In the calculation of Eq. (17), the correction factor, α for the diameters of the tube and the ball is used, and 0.999 is experimentally adopted for α. α may be a factor that varies according to the kind of fluid and surface roughness of a tube and a ball. A correction for the roughness of the inner wall of a capillary tube is seen, for instance, in the explanation[8] for viscous behavior of dilute high polymer solution. The value 0.999, which is common for all liquids, is very satisfactory.

It is technically difficult to make the ratio d/D approach one, because of inaccuracy of the tube and the ball. Therefore, it is not attempted to consider the fit in the range above $d/D= 0.99$.

No consideration is given to the factor α for gas in this work.

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8) W. Knappe and H. Lange, *Kolloid-Z.*, 179, 97 (1961)