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Non-Colliding System of Brownian Particles as Pfaffian Process

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In the paper [7] we studied the temporally inhomogeneous system of non-colliding Brownian motions and proved that multi-time correlation functions are generally given by the quaternion determinants in the sense of Dyson and Mehta. In this report we give another proof of the equivalent statement using Fredholm determinant and Fredholm pfaffian, and claim that the present system is a typical example of pfaffian processes.

1 Non-Colliding Brownian Motions

By virtue of the Karlin-McGregor formula [5, 6], the transition density of the absorbing Brownian motion in a Weyl chamber

$$\mathbb{R}^N_< = \Big\{ \mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N ; x_1 < x_2 < \cdots < x_N \Big\},$$

is given by

$$f_N(t; \mathbf{x}, \mathbf{y}) = \det_{1 \le i,j \le N} \Big[p_t(x_i, y_j) \Big], \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^N_<, \quad t \in [0, \infty),$$

where p_t is the heat-kernel given by

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}.$$

The integral

$$\mathcal{N}_N(t; \mathbf{x}) = \int_{\mathbb{R}^N_<} f_N(t; \mathbf{x}, \mathbf{y}) d\mathbf{y},$$

where $d\mathbf{y} = \prod_{i=1}^N dy_i$, gives the probability that a Brownian motion starting from $\mathbf{x} \in \mathbb{R}^N_<$ does not hit the boundary of $\mathbb{R}^N_<$ up to time $t > 0$.

For a given $T > 0$, we define

$$g_{N,T}(s, \mathbf{x}; t, \mathbf{y}) = \frac{f_N(t-s; \mathbf{x}, \mathbf{y}) \mathcal{N}_N(T-t; \mathbf{y})}{\mathcal{N}_N(T-s; \mathbf{x})} \tag{1.1}$$

for $0 \le s \le t \le T, \mathbf{x}, \mathbf{y} \in \mathbb{R}^N_<$. It can be regarded as the transition probability density from the state $\mathbf{x} \in \mathbb{R}^N_<$ at time s to the state $\mathbf{y} \in \mathbb{R}^N_<$ at time t. In [8, 9] it was shown that as $|\mathbf{x}| \to 0$, $g^T_N(0, \mathbf{x}; t, \mathbf{y})$ converges to

$$g_{N,T}(0, \mathbf{0}; t, \mathbf{y}) \equiv C(N, T, t) h_N(\mathbf{y}) \prod_{i=1}^N p_t(0, y_i) \mathcal{N}_N(T-t, \mathbf{y}), \tag{1.2}$$

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where $C(N,T,t) = \pi^{N/2} \left(\prod_{j=1}^{N} \Gamma(j/2)\right)^{-1} T^{N(N-1)/4} t^{-N(N-1)/2}$, and

$$h_N(\mathbf{x}) = \prod_{1 \le i < j \le N} (x_j - x_i) = \det_{1 \le i,j \le N} \left[x_j^{i-1} \right].$$

The N particle system of non-colliding Brownian motions $\mathbf{X}(t)$ all started from the origin at time 0, *i.e.* $\mathbf{X}(0) = \mathbf{0} = (0, 0, \cdots, 0)$, and conditioned not to collide with each other in a time interval $(0, T]$ is defined by the process associated with the transition probability density $g_{N,T}$ given by (1.1) and (1.2). This process is temporally inhomogeneous and it was obtained as a diffusion scaling limit of the vicious walker model in [9]. Figure 1 illustrates the process $\mathbf{X}(t)$.

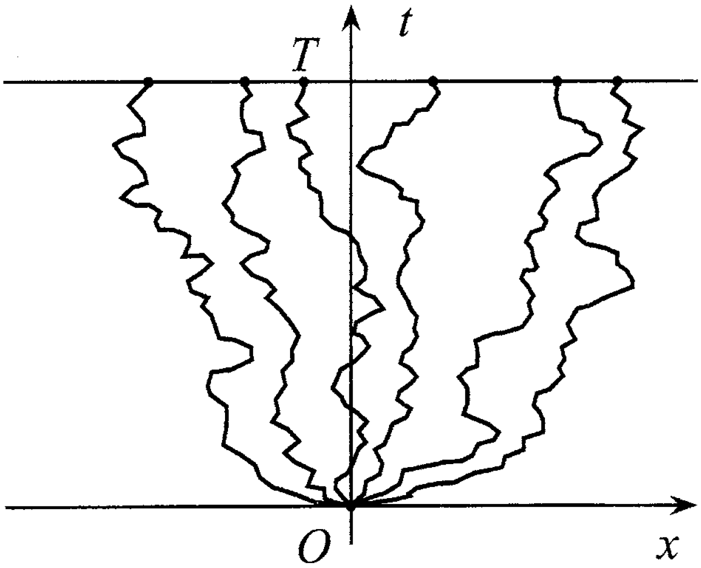

Figure 1: Process $\mathbf{X}(t), t \in [0, T]$, with $\mathbf{X}(0) = \mathbf{0}$.

Let $\mathfrak{X}$ be the space of countable subset ξ of $\mathbb{R}$ satisfying $\sharp(\xi \cap K) < \infty$ for any compact subset K. For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \bigcup_{\ell=1}^{\infty} \mathbb{R}^{\ell}$, we denote $\{x_i\}_{i=1}^{n} \in \mathfrak{X}$ simply by $\{\mathbf{x}\}$. The diffusion process $\{\mathbf{X}(t)\}, t \in [0, T]$ on the set $\mathfrak{X}$ is defined by the transition probability density $\mathfrak{g}_{N,T}(s, \{\mathbf{x}\}; t, \{\mathbf{y}\}), 0 \le s \le t \le T$:

$$\mathfrak{g}_{N,T}(s, \{\mathbf{x}\}; t, \{\mathbf{y}\}) = \begin{cases} g_{N,T}(s, \mathbf{x}; t, \mathbf{y}), & \text{if } s > 0, \ \sharp\{\mathbf{x}\} = \sharp\{\mathbf{y}\} = N, \\ g_{N,T}(0, \mathbf{0}; t, \mathbf{y}), & \text{if } s = 0, \ \{\mathbf{x}\} = \{0\}, \ \sharp\{\mathbf{y}\} = N, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the elements of $\mathbb{R}^N_<$. For the given time interval $[0, T]$, we consider the M intermediate times $t_0 = 0 < t_1 < t_2 < \cdots < t_M < t_{M+1} = T$. The multi-time transition density of the process $\{\mathbf{X}(t)\}$ is given by

$$\mathfrak{g}_{N,T}(t_0, \{\mathbf{x}^{(0)}\}; t_1, \{\mathbf{x}^{(1)}\}; \ldots; t_{M+1}, \{\mathbf{x}^{(M+1)}\}) = \prod_{\mu=0}^{M} \mathfrak{g}_{N,T}(t_\mu, \{\mathbf{x}^{(\mu)}\}; t_{\mu+1}, \{\mathbf{x}^{(\mu+1)}\}).$$

Here we set $t_0 = 0$, $t_{M+1} = T$ and $\{\mathbf{x}^{(0)}\} \equiv \{0\}$. From (1.1) and (1.2) we have

$$
\begin{aligned}
(1.3) \qquad & \mathfrak{g}_{N,T}(0,\{0\};t_1,\{\mathbf{x}^{(1)}\};\ldots;t_{M+1},\{\mathbf{x}^{(M+1)}\}) \\
&= C(N,T,t_1)h_N\left(\mathbf{x}^{(1)}\right)\operatorname{sgn}\left(h_N\left(\mathbf{x}^{(M+1)}\right)\right) \\
&\times \prod_{i=1}^{N} p_{t_1}\left(0,x_i^{(1)}\right)\prod_{\mu=1}^{M}\det_{1\le i,j\le N}\left[p_{t_{\mu+1}-t_\mu}\left(x_i^{(\mu)},x_j^{(\mu+1)}\right)\right].
\end{aligned}
$$

For $\mathbf{x}^{(\mu)} \in \mathbb{R}^N_<$, $1 \le \mu \le M+1$, and $N' = 1,2,\ldots,N$, we write $\mathbf{x}^{(\mu)}_{N'} = \left(x_1^{(\mu)}, x_2^{(\mu)},\ldots,x_{N'}^{(\mu)}\right)$. For a sequence $\{N_\mu\}_{\mu=1}^{M+1}$ of positive integers less than or equal to N, we define the multi-time correlation function by

$$
\begin{aligned}
(1.4) \qquad & \rho_{N,T}\left(t_1,\{\mathbf{x}^{(1)}_{N_1}\};t_2,\{\mathbf{x}^{(2)}_{N_2}\};\ldots;t_{M+1},\{\mathbf{x}^{(M+1)}_{N_{M+1}}\}\right) \\
&= \int_{\prod_{\mu=1}^{M+1}\mathbb{R}^{N-N_\mu}} \prod_{\mu=1}^{M+1}\frac{1}{(N-N_\mu)!}\prod_{i=N_\mu+1}^{N} dx_i^{(\mu)} \\
&\qquad \mathfrak{g}_{N,T}\left(0,\{0\};t_1,\{\mathbf{x}^{(1)}_N\};t_2,\{\mathbf{x}^{(2)}_N\};\ldots;t_{M+1},\{\mathbf{x}^{(M+1)}_N\}\right).
\end{aligned}
$$

Let $C_0(\mathbb{R})$ be the set of all continuous real functions with compact supports. For $\mathbf{f} = (f_1, f_2,\cdots,f_{M+1}) \in C_0(\mathbb{R})^{M+1}$, and $\boldsymbol{\theta} = (\theta_1,\theta_2,\cdots,\theta_{M+1}) \in \mathbb{R}^{M+1}$, the multi-time characteristic function is defined for the process $\mathbf{X}(t) = (X_1(t), X_2(t),\cdots,X_N(t))$ as

$$
\Psi_{N,T}(\mathbf{f};\boldsymbol{\theta}) = \mathbf{E}_{N,T}\left[\exp\left\{\sqrt{-1}\sum_{\mu=1}^{M+1}\theta_\mu\sum_{i_\mu=1}^{N}f_\mu(X_{i_\mu}(t_\mu))\right\}\right],
$$

where $\mathbf{E}_{N,T}[\,\cdot\,]$ denotes the expectation determined by $\mathfrak{g}_{N,T}$. Let

$$
\chi_\mu(x) = e^{\sqrt{-1}\theta_\mu f_\mu(x)} - 1, \quad 1 \le \mu \le M+1,
$$

then by the definition of multi-time correlation function (1.4), we have

$$
\begin{aligned}
\Psi_{N,T}(\mathbf{f};\boldsymbol{\theta}) &= \sum_{N_1=0}^{N}\sum_{N_2=0}^{N}\cdots\sum_{N_{M+1}=0}^{N}\int_{\mathbb{R}^{N_1}}d\mathbf{x}^{(1)}_{N_1}\int_{\mathbb{R}^{N_2}}d\mathbf{x}^{(2)}_{N_2}\cdots\int_{\mathbb{R}^{N_{M+1}}}d\mathbf{x}^{(M+1)}_{N_{M+1}} \\
(1.5) \qquad & \prod_{\mu=1}^{M+1}\prod_{i_\mu=1}^{N_\mu}\chi_\mu(x^{(\mu)}_{i_\mu})\rho_{N,T}\left(t_1,\{\mathbf{x}^{(1)}_{N_1}\};t_2,\{\mathbf{x}^{(2)}_{N_2}\};\cdots;t_{M+1},\{\mathbf{x}^{(M+1)}_{N_{M+1}}\}\right).
\end{aligned}
$$

2 Fredholm Pfaffian Representation of Characteristic Function

Let

$$
\begin{aligned}
Z_{N,T}[\chi] &= \left(\frac{1}{N!}\right)^{M+1}\int_{\mathbb{R}^{N(M+1)}}\prod_{\mu=1}^{M+1}d\mathbf{x}^{(\mu)}\det_{1\le i,j\le N}\left[M_{i-1}(x_j^{(1)})p_{t_1}(0,x_j^{(1)})(1+\chi_1(x_j^{(1)}))\right] \\
&\times\prod_{\mu=1}^{M}\det_{1\le i,j\le N}\left[p_{t_{\mu+1}-t_\mu}(x_i^{(\mu)},x_j^{(\mu+1)})(1+\chi_{\mu+1}(x_j^{(\mu+1)}))\right]\operatorname{sgn}\left(h_N(\mathbf{x}^{(M+1)}_N)\right).
\end{aligned}
$$

Here $M_i(x)$ is an arbitrary polynomial of x with degree i in the form $M_i(x) = b_i x^i + \cdots$ with a constant b_i for $i = 0, 1, 2, \cdots$, and thus $h_N(\mathbf{x}) = \det_{1\le i,j\le N}\left[M_{i-1}(x_j)\right] / \prod_{k=1}^{N} b_{k-1}$. Then (1.3) gives

$$\Psi_{N,T}(\mathbf{f};\boldsymbol{\theta}) = \frac{Z_{N,T}[\chi]}{Z_{N,T}[0]}, \tag{2.1}$$

where $Z_{N,T}[0]$ is obtained from $Z_{N,T}[\chi]$ by setting $\chi_\mu(x) = 0$ for all μ.

By the well-known formula

$$\int_{\mathbb{R}^N_<} d\mathbf{x} \det_{1\le i,j\le N}\left[\phi_i(x_j)\right] \det_{1\le i,j\le N}\left[\bar{\phi}_i(x_j)\right] = \det_{1\le i,j\le N}\left[\int_{\mathbb{R}} dx\, \phi_i(x)\bar{\phi}_j(x)\right]$$

for square integrable continuous functions $\phi_i, \bar{\phi}_i, 1 \le i \le N$, we have

$$\begin{aligned} Z_{N,T}[\chi] \;=\; & \int_{\mathbb{R}^N_<} d\mathbf{y} \det_{1\le i,j\le N}\Bigg[\int_{\mathbb{R}^{M+1}} \prod_{\mu=1}^{M+1} dx^{(\mu)} \left\{ M_{i-1}(x^{(1)}) p_{t_1}(0, x^{(1)})(1+\chi_1(x^{(1)}))\right\} \\ & \times \prod_{\mu=1}^{M} \left\{ p_{t_{\mu+1}-t_\mu}(x^{(\mu)}, x^{(\mu+1)})(1+\chi_{\mu+1}(x^{(\mu+1)}))\right\} p_{T-t_{M+1}}(x^{(M+1)}, y_j)\Bigg], \end{aligned}$$

where $p_{T-t_{M+1}}(x,y) = p_0(x,y) = \delta(x-y)$.

For simplicity of expressions, we will assume that the number of particles N is even from now on. We use the formula [2]

$$\int_{\mathbb{R}^N_<} d\mathbf{y} \det_{1\le i,j\le N}\left[\phi_i(y_j)\right] = \mathrm{Pf}_{1\le i,j\le N}\left[\int_{\mathbb{R}} dy \int_{\mathbb{R}} d\tilde{y}\, \mathrm{sgn}(\tilde{y}-y)\phi_i(y)\phi_j(\tilde{y})\right]$$

for integrable continuous functions $\phi_i, 1 \le i \le N$. Here, for an integer n and an antisymmetric $2n \times 2n$ matrix $A = (a_{ij})$, the pfaffian is defined as

$$\mathrm{Pf}(A) = \mathrm{Pf}_{1\le i<j\le 2n}\left[\, a_{ij} \,\right] = \frac{1}{n!}\sum_{\sigma} \mathrm{sgn}(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2n-1)\sigma(2n)},$$

where the summation is extended over all permutations σ of $(1, 2, \ldots, 2n)$ with restriction $\sigma(2k-1) < \sigma(2k), k = 1, 2, \ldots, n$. Since $(\mathrm{Pf}(A))^2 = \det A$ for any antisymmetric $2n \times 2n$ matrix A, we have

$$\begin{aligned} \left(Z_{N,T}[\chi]\right)^2 \;=\; & \det_{1\le i,j\le N}\Bigg[\int dy \int d\tilde{y}\, \mathrm{sgn}(\tilde{y}-y) \\ & \times \int_{\mathbb{R}^{M+1}} \prod_{\mu=1}^{M+1} dx^{(\mu)} \left\{ M_{i-1}(x^{(1)}) p_{t_1}(0, x^{(1)})(1+\chi_1(x^{(1)}))\right\} \\ & \times \prod_{\mu=1}^{M} \left\{ p_{t_{\mu+1}-t_\mu}(x^{(\mu)}, x^{(\mu+1)})(1+\chi_{\mu+1}(x^{(\mu+1)}))\right\} p_{T-t_{M+1}}(x^{(M+1)}, y) \\ & \times \int_{\mathbb{R}^{M+1}} \prod_{\mu=1}^{M+1} d\tilde{x}^{(\mu)} \left\{ M_{j-1}(\tilde{x}^{(1)}) p_{t_1}(0, \tilde{x}^{(1)})(1+\chi_1(\tilde{x}^{(1)}))\right\} \end{aligned}$$

$$\times \prod_{\mu=1}^{M} \left\{ p_{t_{\mu+1}-t_\mu}(\tilde{x}^{(\mu)}, \tilde{x}^{(\mu+1)})(1+\chi_{\mu+1}(\tilde{x}^{(\mu+1)})) \right\} p_{T-t_{M+1}}(\tilde{x}^{(M+1)}, \tilde{y}) \Big]$$

$$= \det_{1 \le i,j \le N} \Big[(A_0)_{ij} + (A_1)_{ij} + (A_2)_{ij} + (A_3)_{ij} \Big]$$

with

$$\begin{aligned}
(A_0)_{ij} &= \int dy \int d\tilde{y}\, \mathrm{sgn}(\tilde{y}-y) \int dx M_{i-1}(x) p_{t_1}(0,x) p_{T-t_1}(x,y) \\
&\quad \times \int d\tilde{x} M_{j-1}(\tilde{x}) p_{t_1}(0,\tilde{x}) p_{T-t_1}(\tilde{x},\tilde{y}), \\
(A_1)_{ij} &= \sum_{\ell=1}^{M+1} \sum_{1 \le \mu_1 < \mu_2 < \cdots < \mu_\ell \le M+1} \int dy \int d\tilde{y}\, \mathrm{sgn}(\tilde{y}-y) \\
&\quad \times \int_{\mathbb{R}^\ell} \prod_{k=1}^{\ell} dx^{(\mu_k)} \int dx\, M_{i-1}(x) p_{t_1}(0,x) p_{t_{\mu_1}-t_1}(x, x^{(\mu_1)}) \\
&\quad \times \prod_{k=1}^{\ell-1} \left\{ \chi_k(x^{(\mu_k)}) p_{t_{\mu_{k+1}}-t_{\mu_k}}(x^{(\mu_k)}, x^{(\mu_{k+1})}) \right\} \chi_\ell(x^{(\mu_\ell)}) p_{T-t_{\mu_\ell}}(x^{(\mu_\ell)}, y) \\
&\quad \times \int d\tilde{x} M_{j-1}(\tilde{x}) p_{t_1}(0,\tilde{x}) p_{T-t_1}(\tilde{x},\tilde{y}), \\
(A_2)_{ij} &= \sum_{\ell=1}^{M+1} \sum_{1 \le \mu_1 < \mu_2 < \cdots < \mu_\ell \le M+1} \int dy \int d\tilde{y}\, \mathrm{sgn}(\tilde{y}-y) \\
&\quad \times \int dx M_{i-1}(x) p_{t_1}(0,x) p_{T-t_1}(x,y), \\
&\quad \times \int_{\mathbb{R}^\ell} \prod_{k=1}^{\ell} d\tilde{x}^{(\mu_k)} \int d\tilde{x}\, M_{j-1}(\tilde{x}) p_{t_1}(0,\tilde{x}) p_{t_{\mu_1}-t_1}(\tilde{x}, \tilde{x}^{(\mu_1)}) \\
&\quad \times \prod_{k=1}^{\ell-1} \left\{ \chi_k(\tilde{x}^{(\mu_k)}) p_{t_{\mu_{k+1}}-t_{\mu_k}}(\tilde{x}^{(\mu_k)}, \tilde{x}^{(\mu_{k+1})}) \right\} \chi_\ell(\tilde{x}^{(\mu_\ell)}) p_{T-t_{\mu_\ell}}(\tilde{x}^{(\mu_\ell)}, \tilde{y}), \\
(A_3)_{ij} &= \sum_{\ell=1}^{M+1} \sum_{m=1}^{M+1} \sum_{1 \le \mu_1 < \mu_2 < \cdots < \mu_\ell \le M+1} \sum_{1 \le \nu_1 < \nu_2 < \cdots < \nu_m \le M+1} \int dy \int d\tilde{y}\, \mathrm{sgn}(\tilde{y}-y) \\
&\quad \times \int_{\mathbb{R}^\ell} \prod_{k=1}^{\ell} dx^{(\mu_k)} \int dx\, M_{i-1}(x) p_{t_1}(0,x) p_{t_{\mu_1}-t_1}(x, x^{(\mu_1)}) \\
&\quad \times \prod_{k=1}^{\ell-1} \left\{ \chi_k(x^{(\mu_k)}) p_{t_{\mu_{k+1}}-t_{\mu_k}}(x^{(\mu_k)}, x^{(\mu_{k+1})}) \right\} \chi_\ell(x^{(\mu_\ell)}) p_{T-t_{\mu_\ell}}(x^{(\mu_\ell)}, y) \\
&\quad \times \int_{\mathbb{R}^\ell} \prod_{n=1}^{m} d\tilde{x}^{(\nu_m)} \int d\tilde{x}\, M_{j-1}(\tilde{x}) p_{t_1}(0,\tilde{x}) p_{t_{\nu_1}-t_1}(\tilde{x}, \tilde{x}^{(\nu_1)}) \\
&\quad \times \prod_{n=1}^{m-1} \left\{ \chi_n(\tilde{x}^{(\nu_n)}) p_{t_{\nu_{n+1}}-t_{\nu_n}}(\tilde{x}^{(\nu_n)}, \tilde{x}^{(\nu_{n+1})}) \right\} \chi_m(\tilde{x}^{(\nu_m)}) p_{T-t_{\nu_m}}(\tilde{x}^{(\nu_m)}, \tilde{y}),
\end{aligned}$$

where we have used the Chapman-Kolmogorov equation for the heat-kernel

$$\int dy\, p_{t-s}(x,y)p_{u-t}(y,z) = p_{u-s}(x,z), \quad 0 < s < t < u,\ x,y \in \mathbb{R},$$

and the fact that $p_0(x,y) = \lim_{t\to 0} p_t(x,y) = \delta(x-y)$.

We consider a vector space $\mathcal{V}$ with the orthonormal basis $\left\{|\mu, x\rangle\right\}_{\mu=1,2,\cdots,M+1, x\in\mathbb{R}}$ which satisfy

$$\langle \mu, x|\nu, y\rangle = \delta_{\mu\nu}\delta(x-y),$$

$\mu, \nu = 1, 2, \cdots, M+1, x, y \in \mathbb{R}$. We introduce the operators $\hat{J}, \hat{p}, \hat{p}_+, \hat{p}_-$ and $\hat{\chi}$ acting on $\mathcal{V}$ as follows

$$\begin{aligned}
&\langle \mu, x|\hat{J}|\nu, y\rangle = \mathbf{1}_{(\mu=\nu=M+1)}\mathrm{sgn}(y-x),\\
&\langle \mu, x|\hat{p}|\nu, y\rangle = p_{|t_\nu - t_\mu|}(x,y),\\
&\langle \mu, x|\hat{p}_+|\nu, y\rangle = p_{t_\nu - t_\mu}(x,y)\mathbf{1}_{(\mu<\nu)} = \langle \nu, y|\hat{p}_-|\mu, x\rangle,\\
&\langle \mu, x|\hat{\chi}|\nu, y\rangle = \chi_\mu(x)\delta_{\mu\nu}\delta(x-y),
\end{aligned} \tag{2.2}$$

where $\mathbf{1}_{(\omega)}$ is the indicator function: $\mathbf{1}_{(\omega)} = 1$ if ω is satisfied and $\mathbf{1}_{(\omega)} = 0$ otherwise, and we will use the convention

$$\langle \mu, x|\hat{A}|\nu, y\rangle\langle \nu, y|\hat{B}|\rho, z\rangle = \sum_{\nu=1}^{M+1}\int_{\mathbb{R}} dy\, A(\mu, x; \nu, y)B(\nu, y; \rho, z) = \langle \mu, x|\hat{A}\hat{B}|\rho, z\rangle$$

for operators $\hat{A}, \hat{B}$ with $\langle \mu, x|\hat{A}|\nu, y\rangle = A(\mu, x; \nu, y), \langle \mu, x|\hat{B}|\nu, y\rangle = B(\mu, x; \nu, y)$.

Consider another basis $\left\{|i\rangle\, ;\, i = 1, 2, \cdots\right\}$ in $\mathcal{V}$ and we assume that transformation matrix between the two bases is given by

$$\langle i|\mu, x\rangle = \langle \mu, x|i\rangle = \int dy M_{i-1}(y)p_{t_1}(0,y)p_{t_\mu - t_1}(y,x), \tag{2.3}$$

$i = 1, 2, \cdots, \mu = 1, 2, \cdots, M+1, x \in \mathbb{R}$. Then the quantity

$$\begin{aligned}
&\int dy\, M_{i-1}(y)p_{t_1}(0,y)p_{t_\mu - t_1}(y,x)\\
&+\sum_{m=1}^{\mu-1}\sum_{1\le\mu_1<\mu_2<\cdots<\mu_m<\mu}\int_{\mathbb{R}^m}\prod_{j=1}^{m}dx^{(\mu_j)}\int dy\, M_{i-1}(y)p_{t_1}(0,y)p_{t_{\mu_1}-t_1}(y, x^{(\mu_1)})\\
&\qquad\times\prod_{k=1}^{m-1}\left\{\chi_{\mu_k}(x^{(\mu_k)})p_{t_{\mu_{k+1}}-t_{\mu_k}}(x^{(\mu_k)}, x^{(\mu_{k+1})})\right\}\chi_{\mu_m}(x^{(\mu_m)})p_{t_\mu - t_{\mu_m}}(x^{(\mu_m)}, x),
\end{aligned}$$

for $\mu = 1, 2, \cdots, M+1, x \in \mathbb{R}, i = 1, 2, \cdots$, can be written as

$$\begin{aligned}
&\langle i|\mu, x\rangle + \sum_{m\ge1}\langle i|\mu_1, x^{(\mu_1)}\rangle\langle \mu_1, x^{(\mu_1)}|\hat{\chi}\hat{p}_+|\mu_2, x^{(\mu_2)}\rangle\\
&\qquad\cdots\langle \mu_{m-1}, x^{(\mu_{m-1})}|\hat{\chi}\hat{p}_+|\mu_m, x^{(\mu_m)}\rangle\langle \mu_m, x^{(\mu_m)}|\hat{\chi}\hat{p}_+|\mu, x\rangle\\
= &\ \langle i|\mu, x\rangle + \sum_{m\ge1}\langle i|(\hat{\chi}\hat{p}_+)^m|\mu, x\rangle\\
= &\ \langle i|\frac{1}{1-\hat{\chi}\hat{p}_+}|\mu, x\rangle.
\end{aligned}$$

It is also expressed as $\langle\mu, x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle$.

It should be noted that the basis $\{|i\rangle\,;\, i=1,2,\cdots\}$ is in general not orthonormal. Here we introduce an operator $\hat{\delta}$ such that

$$\langle i|\hat{\delta}|j\rangle = \langle j|\hat{\delta}|i\rangle = \delta_{ij}, \quad i,j=1,2,\cdots. \tag{2.4}$$

We will use the following convention

$$\langle i|\hat{A}|j\rangle\langle j|\hat{B}|\mu,x\rangle = \sum_{j=1}^{\infty} A_{ij}B_j^{(\mu)}(x)$$

for $A_{ij} = \langle i|\hat{A}|j\rangle \equiv \langle i|\mu,x\rangle\langle\mu,x|\hat{A}|\nu,y\rangle\langle\nu,y|j\rangle$, $B_j^{(\mu)}(x) = \langle j|\hat{B}|\mu,x\rangle \equiv \langle j|\nu,y\rangle\langle\nu,y|\hat{B}|\mu,x\rangle$, but we will not write it as $\langle i|\hat{A}\hat{B}|\mu,x\rangle$, since $\{|i\rangle\,;\, i=1,2,\cdots\}$ is in general not a complete basis. By this basis, any operator $\hat{A}$ on $\mathcal{V}$ may have a semi-infinite matrix representation $A = \left(\langle i|\hat{A}|j\rangle\right)_{i,j=1,2,\cdots}$. If the matrix A representing an operator $\hat{A}$ is invertible, we define the operator $\hat{A}^{\triangle}$ such that its matrix representation is the inverse of A;

$$\left(\langle i|\hat{A}^{\triangle}|j\rangle\right)_{i,j=1,2,\cdots} = A^{-1}. \tag{2.5}$$

In other words,

$$\langle i|\hat{A}|j\rangle\langle j|\hat{A}^{\triangle}|k\rangle = \langle i|\hat{A}^{\triangle}|j\rangle\langle j|\hat{A}|k\rangle = \langle i|\hat{\delta}|k\rangle, \qquad i,k=1,2,\cdots.$$

For any given operator $\hat{A}$, the equality

$$\langle i|\hat{A}|j\rangle\langle j|\hat{A}^{\triangle}|k\rangle\langle k|\hat{B}|\ell\rangle = \langle i|\hat{B}|\ell\rangle.$$

holds for arbitrary $i,\ell = 1,2,\cdots$ and $\hat{B}$. Then the equality

$$\hat{A}|j\rangle\langle j|\hat{A}^{\triangle}|k\rangle\langle k| = 1 \tag{2.6}$$

should be established for each $\hat{A}$. We will use this equality later.

Let $\mathcal{P}_N$ be a projection operator from the space spanned by $\{|i\rangle\,;\, i=1,2,\cdots\}$ to its N-dimensional subspace spanned by $\{|i\rangle\,;\, i=1,2,\cdots,N\}$, and thus

$$\langle i|\mathcal{P}_N|\mu,x\rangle = \langle\mu,x|\mathcal{P}_N|i\rangle = \begin{cases} \langle i|\mu,x\rangle, & \text{if } 1\le i\le N, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following expressions for $(A_\alpha)_{ij}, \alpha=0,1,2,3, i,j=1,2,\cdots,N$,

$$\begin{aligned} (A_0)_{ij} &= \langle i|\mathcal{P}_N|\mu,x\rangle\langle\mu,x|\hat{J}|\nu,y\rangle\langle\nu,y|\mathcal{P}_N|j\rangle \\ &= \langle i|\mathcal{P}_N\hat{J}\mathcal{P}_N|j\rangle, \end{aligned}$$

$$
\begin{aligned}
(A_1)_{ij} &= \langle i|\mathcal{P}_N \frac{1}{1-\hat{\chi}\hat{p}_+}\hat{\chi}|\mu,x\rangle\langle\mu,x|\hat{p}\hat{J}\mathcal{P}_N|j\rangle \\
&= \langle i|\mathcal{P}_N \hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\mu,x\rangle\langle\mu,x|\hat{p}\hat{J}\mathcal{P}_N|j\rangle \\
&= \langle i|\mathcal{P}_N \hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}\hat{p}\hat{J}\mathcal{P}_N|j\rangle, \\
(A_2)_{ij} &= \langle i|\mathcal{P}_N \hat{J}\hat{p}|\mu,x\rangle\langle\mu,x|\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\mathcal{P}_N|j\rangle, \\
&= \langle i|\mathcal{P}_N \hat{J}\hat{p}\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\mathcal{P}_N|j\rangle, \\
(A_3)_{ij} &= \langle i|\mathcal{P}_N \frac{1}{1-\hat{\chi}\hat{p}_+}\hat{\chi}|\mu,x\rangle\langle\mu,x|\hat{p}\hat{J}\hat{p}|\nu,y\rangle\langle\nu,y|\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\mathcal{P}_N|j\rangle \\
&= \langle i|\mathcal{P}_N \hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\mu,x\rangle\langle\mu,x|\hat{p}\hat{J}\hat{p}|\nu,y\rangle\langle\nu,y|\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\mathcal{P}_N|j\rangle \\
&= \langle i|\mathcal{P}_N \hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}\hat{p}\hat{J}\hat{p}\hat{\chi}\frac{1}{1-\hat{p}_-\hat{\chi}}\mathcal{P}_N|j\rangle.
\end{aligned}
$$

That is, the matrices $A_\alpha = \Big((A_\alpha)_{ij}\Big)_{i,j=1,2,\cdots,N}$, $\alpha = 0,1,2,3$, can be regarded as the matrix representations in the basis $\Big\{|i\rangle\,;\, i=1,2,\cdots\Big\}$ of the operators.

Since $\Big(Z_{N,T}[0]\Big)^2 = \det_{1\le i,j\le N}\Big[(A_0)_{ij}\Big]$, (2.1) gives

$$
\Big\{\Psi_{N,T}(\mathbf{f};\boldsymbol{\theta})\Big\}^2 = \det_{1\le i,j\le N}\Big[\delta_{ij} + (A_0^{-1}A_1)_{ij} + (A_0^{-1}A_2)_{ij} + (A_0^{-1}A_3)_{ij}\Big]. \tag{2.7}
$$

By our notation (2.5),

$$
(A_0^{-1})_{ij} = \langle i|\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N|j\rangle,
$$

since it satisfies the relation

$$
\begin{aligned}
&\langle i|\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N|j\rangle\langle j|\mathcal{P}_N\hat{J}\mathcal{P}_N|k\rangle \\
&= \mathbf{1}_{(1\le i,k\le N)}\sum_{j=1}^{N}(A_0^{-1})_{ij}(A_0)_{jk} = \mathbf{1}_{(1\le i,k\le N)}\delta_{ik} \\
&= \langle i|\mathcal{P}_N\hat{\delta}\mathcal{P}_N|j\rangle.
\end{aligned}
$$

We will use the abbreviation

$$
\hat{A}_N = \mathcal{P}_N\hat{A}\mathcal{P}_N
$$

for an operator $\hat{A}$. Then it is easy to see that (2.7) is written in the form

$$
\begin{aligned}
\Big\{\Psi_{N,T}(\mathbf{f};\boldsymbol{\theta})\Big\}^2 &= \det_{1\le i,j\le N}\Big[\delta_{ij} + \langle i|\mathbf{B}|\mu,x\rangle\langle\mu,x|\mathbf{C}|j\rangle\Big] \\
&= \det_{1\le i,j\le N}\langle i|\Big[\hat{\delta}_N + \mathbf{BC}\Big]|j\rangle,
\end{aligned} \tag{2.8}
$$

where $\langle i|\mathbf{B}|\mu,x\rangle$ and $\langle \mu,x|\mathbf{C}|j\rangle$ are the two-dimensional row and column vectors, respectively, given by

$$\langle i|\mathbf{B}|\mu,x\rangle = \begin{pmatrix} \langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|k\rangle\langle k|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\mu,x\rangle & \begin{array}{l} -\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|k\rangle\langle k|\hat{J}\hat{p}\hat{\chi}|\mu,x\rangle \\ -\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|k\rangle \\ \quad\times\langle k|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\nu,y\rangle\langle\nu,y|\hat{p}\hat{J}\hat{p}\hat{\chi}|\mu,x\rangle \end{array} \end{pmatrix},$$

$$\langle \mu,x|\mathbf{C}|j\rangle = \begin{pmatrix} \langle\mu,x|\hat{p}\hat{J}\mathcal{P}_N|j\rangle \\ -\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}\mathcal{P}_N|j\rangle \end{pmatrix}.$$

The determinant (2.8) is written using the Fredholm determinant,

$$
\begin{aligned}
&\mathrm{Det}\langle\mu,x|\left[I_2+\mathbf{CB}\right]|\nu,y\rangle \\
&= \mathrm{Det}\begin{bmatrix}
\begin{array}{l}\langle\mu,x|\nu,y\rangle \\ +\langle\mu,x|\hat{p}\hat{J}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\nu,y\rangle\end{array} &
\begin{array}{l}-\langle\mu,x|\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle \\ -\langle\mu,x|\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\rho,z\rangle \\ \quad\times\langle\rho,z|\hat{p}\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle\end{array} \\
\begin{array}{l}-\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\nu,y\rangle\end{array} &
\begin{array}{l}\langle\mu,x|\nu,y\rangle \\ +\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle \\ +\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\rho,z\rangle \\ \quad\times\langle\rho,z|\hat{p}\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle\end{array}
\end{bmatrix} \\
&= \mathrm{Det}\left(\begin{bmatrix}
\begin{array}{l}\langle\mu,x|\rho,z\rangle \\ +\langle\mu,x|\hat{p}\hat{J}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\rho,z\rangle\end{array} &
\begin{array}{l}-\langle\mu,x|\hat{p}\hat{J}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}|\rho,z\rangle \\ +\langle\mu,x|\hat{p}\hat{J}\hat{p}\hat{\chi}|\rho,z\rangle\end{array} \\
\begin{array}{l}-\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\rho,z\rangle\end{array} &
\begin{array}{l}\langle\mu,x|\rho,z\rangle \\ +\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}|\rho,z\rangle\end{array}
\end{bmatrix}\right. \\
&\qquad\qquad\left.\times\begin{bmatrix}\langle\rho,z|\nu,y\rangle & -\langle\rho,z|\hat{p}\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle \\ 0 & \langle\rho,z|\nu,y\rangle\end{bmatrix}\right) \\
&= \mathrm{Det}\begin{bmatrix}
\begin{array}{l}\langle\mu,x|\nu,y\rangle \\ +\langle\mu,x|\hat{p}\hat{J}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\nu,y\rangle\end{array} &
\begin{array}{l}-\langle\mu,x|\hat{p}\hat{J}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle \\ +\langle\mu,x|\hat{p}\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle\end{array} \\
\begin{array}{l}-\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi}\frac{1}{1-\hat{p}_+\hat{\chi}}|\nu,y\rangle\end{array} &
\begin{array}{l}\langle\mu,x|\nu,y\rangle \\ +\langle\mu,x|\frac{1}{1-\hat{p}_-\hat{\chi}}|i\rangle \\ \quad\times\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}|\nu,y\rangle\end{array}
\end{bmatrix},
\end{aligned}
$$

where I_2 denotes the unit matrix with size 2. It is further rewritten as

$$
\begin{aligned}
&\mathrm{Det}\langle\mu,x|\left(I_2+\begin{bmatrix}1 & 0\\ 0 & \frac{1}{1-\hat{p}_-\hat{\chi}}\end{bmatrix}\right.\\
&\qquad\times\begin{bmatrix}\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi} & \left(-\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}+\hat{p}\hat{J}\hat{p}\right)\hat{\chi}\\ -|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi} & |i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}\end{bmatrix}\\
&\qquad\qquad\left.\times\begin{bmatrix}\frac{1}{1-\hat{p}_+\hat{\chi}} & 0\\ 0 & 1\end{bmatrix}\right)|\nu,y\rangle\\
&=\mathrm{Det}\langle\mu,x|\left(\begin{bmatrix}1 & 0\\ 0 & 1-\hat{p}_-\hat{\chi}\end{bmatrix}\begin{bmatrix}1-\hat{p}_+\hat{\chi} & 0\\ 0 & 1\end{bmatrix}\right.\\
&\qquad\left.+\begin{bmatrix}\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi} & \left(-\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}+\hat{p}\hat{J}\hat{p}\right)\hat{\chi}\\ -|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{\chi} & |i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}\hat{\chi}\end{bmatrix}\right)|\nu,y\rangle\\
&=\mathrm{Det}\langle\mu,x|\left(I_2+\begin{bmatrix}\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|-\hat{p}_+ & -\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}+\hat{p}\hat{J}\hat{p}\\ -|i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j| & |i\rangle\langle i|\mathcal{P}_N\hat{J}_N^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}-\hat{p}_-\end{bmatrix}\hat{\chi}\right)|\nu,y\rangle.
\end{aligned}
$$

Here we have used the facts that

$$\mathrm{Det}\langle\mu,x|\begin{bmatrix}1 & 0\\ 0 & 1-\hat{p}_-\hat{\chi}\end{bmatrix}|\nu,y\rangle=1,$$

and

$$\mathrm{Det}\langle\mu,x|\begin{bmatrix}1-\hat{p}_+\hat{\chi} & 0\\ 0 & 1\end{bmatrix}|\nu,y\rangle=1,$$

which are consequences of definitions (2.2) of the operators $\hat{p}_+$ and $\hat{p}_-$. Then we arrive at

(2.9) $$\left\{\Psi_{N,T}(\mathbf{f};\boldsymbol{\theta})\right\}^2=\mathrm{Det}\left(I_2\delta_{\mu\nu}\delta(x-y)+\begin{bmatrix}\widetilde{S}^{\mu,\nu}(x,y) & \widetilde{I}^{\mu,\nu}(x,y)\\ D^{\mu,\nu}(x,y) & \widetilde{S}^{\nu,\mu}(y,x)\end{bmatrix}\chi_\nu(y)\right),$$

where

(2.10)
$$
\begin{aligned}
D^{\mu,\nu}(x,y) &= -\langle\mu,x|i\rangle\langle i|\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N|j\rangle\langle j|\nu,y\rangle,\\
S^{\mu,\nu}(x,y) &= \langle\mu,x|\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N|j\rangle\langle j|\nu,y\rangle,\\
I^{\mu,\nu}(x,y) &= -\langle\mu,x|\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N|j\rangle\langle j|\hat{J}\hat{p}|\nu,y\rangle,
\end{aligned}
$$

and

(2.11)
$$
\begin{aligned}
\widetilde{S}^{\mu,\nu}(x,y) &= S^{\mu,\nu}(x,y)-\langle\mu,x|\hat{p}_+|\nu,y\rangle\\
&=\begin{cases}\langle\mu,x|\hat{p}\hat{J}|i\rangle\langle i|\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N|j\rangle\langle j|\nu,y\rangle, & \text{if } \mu\geq\nu,\\ -\langle\mu,x|\hat{p}\hat{J}|i\rangle\langle i|\left(\hat{J}^{\triangle}-\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N\right)|j\rangle\langle j|\nu,y\rangle, & \text{if } \mu<\nu,\end{cases}
\end{aligned}
$$

$$\widetilde{I}^{\mu,\nu}(x,y) = I^{\mu,\nu}(x,y) + \langle \mu, x|\hat{p}\hat{J}\hat{p}|\nu, y\rangle$$
$$= \langle \mu, x|\hat{p}\hat{J}|i\rangle\langle i|\Big(\hat{J}^{\triangle} - \mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N\Big)|j\rangle\langle j|\hat{J}\hat{p}|\nu, y\rangle,$$

where we have used the equality (2.6) for $\hat{A} = \hat{J}$.

In [15] Rains introduced Fredholm pfaffian, denoted here by PF, and proved a useful equality

$$\Big\{\mathrm{PF}(J_2 + K)\Big\}^2 = \mathrm{Det}(I_2 + J_2^{-1}K), \tag{2.12}$$

for any antisymmetric 2×2 matrix kernel K, where

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.13}$$

Then (2.9) implies the Fredholm pfaffian representation of the multi-time characteristic function

$$\Psi_{N,T}(\mathbf{f};\boldsymbol{\theta}) = \mathrm{PF}\left(J_2\delta_{\mu\nu}\delta(x-y) + \begin{bmatrix} D^{\mu,\nu}(x,y) & \widetilde{S}^{\nu,\mu}(y,x) \\ -\widetilde{S}^{\mu,\nu}(x,y) & -\widetilde{I}^{\mu,\nu}(x,y) \end{bmatrix} \chi_\nu(y)\right). \tag{2.14}$$

3 Pfaffian Process

Let

$$A^{\mu,\nu}(x,y) = \begin{bmatrix} D^{\mu,\nu}(x,y) & \widetilde{S}^{\nu,\mu}(y,x) \\ -\widetilde{S}^{\mu,\nu}(x,y) & -\widetilde{I}^{\mu,\nu}(x,y) \end{bmatrix}, \tag{3.1}$$

and construct $2\sum_{\mu=1}^{M+1} N_\mu \times 2\sum_{\mu=1}^{M+1} N_\mu$ antisymmetric matrices

$$A\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \cdots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right) = \left(A^{\mu,\nu}\left(x_i^{(\mu)}, x_j^{(\nu)}\right)\right)_{1\le i\le N_\mu, 1\le j\le N_\nu, 1\le\mu,\nu\le M+1},$$

for $N_m = 1, 2, \cdots, N, 1 \le m \le M+1$. By the definition of Fredholm pfaffian [15] and the equality (2.14), we can establish the following statement.

Theorem 1 *The non-colliding system of Brownian motions* $\mathbf{X}(t)$ *is a pfaffian process in the sense that any multi-time correlation function is given by a pfaffian*

$$\rho_{N,T}\left(t_1, \{\mathbf{x}_{N_1}^{(1)}\}; t_2, \{\mathbf{x}_{N_2}^{(2)}\}; \ldots; t_{M+1}, \{\mathbf{x}_{N_{M+1}}^{(M+1)}\}\right) \tag{3.2}$$
$$= \mathrm{Pf}\left[A\left(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \cdots, \mathbf{x}_{N_{M+1}}^{(M+1)}\right)\right]. \tag{3.3}$$

Remark. The pfaffian processes considered here may be regarded as the continuous space-time version of the pfaffian point processes and pfaffian Schur processes studied by Sasamoto and Imamura [16] and by Borodin and Rains [1]. The processes studied in [4] are also pfaffian processes, since the 'quaternion determinantal expressions', in the sense of Dyson and Mehta [3, 12, 13], of correlation functions are readily transformed to pfaffian expressions.

Let $H_i(x)$ be the i-th Hermite polynomial

$$\begin{aligned} H_i(x) &= e^{x^2}\left(-\frac{d}{dx}\right)^i e^{-x^2} \\ &= i!\sum_{j=0}^{[i/2]}(-1)^j\frac{(2x)^{i-2j}}{j!(i-2j)!}, \end{aligned}$$

where $[a]$ denotes the greatest integer not greater than a. The Hermite polynomials satisfy the orthogonal relations

$$\int_{\mathbb{R}} dx\, e^{-x^2}H_i(x)H_j(x) = 2^i i!\sqrt{\pi}\delta_{ij}, \qquad i,j=0,1,2,\cdots. \tag{3.4}$$

Set

$$c_1=\sqrt{\frac{t_1(2T-t_1)}{T}}, \qquad z_1=\sqrt{\frac{2T-t_1}{t_1}},$$

and

$$\alpha_{ij}=\begin{cases} 2^{-i}c_1^i\delta_{ij}, & \text{if } i \text{ is even}, \\ 2^{-i}c_1^i\Big\{\delta_{ij}-2(i-1)\delta_{i-2\,j}\Big\}, & \text{if } i \text{ is odd}. \end{cases} \tag{3.5}$$

Now we specify polynomials $\{M_i(x)\}$ as

$$M_i(x)=b_i z_1^{-i}\sum_{j=0}^{i}\alpha_{ij}H_j\left(\frac{x}{c_1}\right)z_1^j, \quad i=0,1,2,\cdots \tag{3.6}$$

with $b_i=\Big\{r_{[i/2]}\Big\}^{-1/2}$, where $r_i=\frac{1}{\pi}\Gamma(i+1/2)\Gamma(i+1)\left(\frac{t_1^2}{T}\right)^{2i+1/2}$. Set

$$J_N=I_{N/2}\otimes J_2,$$

where $I_{N/2}$ denotes the unit matrix with size $N/2$ and J_2 is given by (2.13), and let J be the semi-infinite matrix obtained as the $N\to\infty$ limit of J_N. By the orthogonality of Hermite polynomials (3.4), we can show through (2.3) with the choice (3.6) that [14, 7]

$$\langle i|\hat{J}|j\rangle = \langle i|\mu,x\rangle\langle\mu,x|\hat{J}|\nu,y\rangle\langle\nu,y|j\rangle = J_{ij}, \qquad i,j=1,2,\cdots.$$

Since $J_N^2=-I_N$ for any even $N\geq 2$, this implies that the matrix $\Big(\langle i|\hat{J}|j\rangle\Big)_{i,j=1,2,\cdots}$ is invertible and

$$\langle i|\hat{J}^{\triangle}|j\rangle = = -J_{ij}, \tag{3.7}$$

$$\begin{aligned} \langle i|\mathcal{P}_N(\mathcal{P}_N\hat{J}\mathcal{P}_N)^{\triangle}\mathcal{P}_N|j\rangle &= \langle i|\mathcal{P}_N\hat{J}^{\triangle}\mathcal{P}_N|j\rangle \\ &= \begin{cases} -J_{ij}=-(J_N)_{ij}, & \text{if } 1\leq i,j\leq N, \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{3.8}$$

for $i,j=1,2,\cdots$.

If we write

$$\langle \mu, x | i \rangle = b_i R_{i-1}^{(\mu)}(x),$$
$$\langle i | \hat{J}\hat{p} | \mu, x \rangle = -\langle \mu, x | \hat{p}\hat{J} | i \rangle = b_i \Phi_{i-1}^{(\mu)}(x),$$

$i = 1, 2, \cdots, \mu = 1, 2, \cdots, M+1, x \in \mathbb{R}$, then the functions (2.10) are written as

$$\begin{aligned} D^{\mu,\nu}(x,y) &= \sum_{i=0}^{N/2-1} \frac{1}{r_i} \Big[R_{2i}^{(\mu)}(x) R_{2i+1}^{(\nu)}(y) - R_{2i+1}^{(\mu)}(x) R_{2i}^{(\nu)}(y) \Big], \\ S^{\mu,\nu}(x,y) &= \sum_{i=0}^{N/2-1} \frac{1}{r_i} \Big[\Phi_{2i}^{(\mu)}(x) R_{2i+1}^{(\nu)}(y) - \Phi_{2i+1}^{(\mu)}(x) R_{2i}^{(\nu)}(y) \Big], \\ I^{\mu,\nu}(x,y) &= -\sum_{i=0}^{N/2-1} \frac{1}{r_i} \Big[\Phi_{2i}^{(\mu)}(x) \Phi_{2i+1}^{(\nu)}(y) - \Phi_{2i+1}^{(\mu)}(x) \Phi_{2i}^{(\nu)}(y) \Big], \end{aligned} \tag{3.9}$$

and Equations (2.11) become

$$\begin{aligned} \widetilde{S}^{\mu,\nu}(x,y) &= \begin{cases} \displaystyle\sum_{i=0}^{N/2-1} \frac{1}{r_i} \Big[\Phi_{2i}^{(\mu)}(x) R_{2i+1}^{(\nu)}(y) - \Phi_{2i+1}^{(\mu)}(x) R_{2i}^{(\nu)}(y) \Big], & \text{if } \mu \geq \nu, \\ \displaystyle -\sum_{i=N/2}^{\infty} \frac{1}{r_i} \Big[\Phi_{2i}^{(\mu)}(x) R_{2i+1}^{(\nu)}(y) - \Phi_{2i+1}^{(\mu)}(x) R_{2i}^{(\nu)}(y) \Big], & \text{if } \mu < \nu, \end{cases} \\ \widetilde{I}^{\mu,\nu}(x,y) &= \sum_{i=N/2}^{\infty} \frac{1}{r_i} \Big[\Phi_{2i}^{(\mu)}(x) \Phi_{2i+1}^{(\nu)}(y) - \Phi_{2i+1}^{(\mu)}(x) \Phi_{2i}^{(\nu)}(y) \Big]. \end{aligned} \tag{3.10}$$

Theorem 1 with the expressions (3.9), (3.10) of the elements of matrix (3.1) is equivalent with Theorem 3 reported in [7], although the latter was given in the form of quaternion determinant. The present argument will be generalized to discuss other non-colliding systems of diffusion particles reported in [10, 11].

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