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$\mathrm{P\scriptstyle RINC}(\kappa, \lambda)$, $\mathrm{C}^s(\kappa)$, $\mathrm{HP}(\kappa)$ etc. and variants of the bounding number

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Abstract

We consider the combinatorial principles $\mathrm{P\scriptstyle RINC}(\kappa, \lambda)$, $\mathrm{C}^s(\kappa)$, $\mathrm{HP}(\kappa)$ etc. and variants of the bounding number and their values under these principles.

0 Introduction

In this note, we consider the combinatorial principles $\mathrm{P\scriptstyle RINC}(\kappa, \lambda)$, $\mathrm{C}^s(\kappa)$, $\mathrm{HP}(\kappa)$ etc. and variants of the bounding number and their values under these principles.

In Section 1, we review the combinatorial principles we consider in this note. In Section 2, some cardinal invariants (called here $\mathfrak{b}'$, $\mathfrak{b}''$, $\mathfrak{b}^\dagger$) are introduced which are all variants of the bounding number $\mathfrak{b}$ and are defined similarly to the shrinkability of bounding families $\mathfrak{b}^*$ of Kada and Yuasa (see [10]). We give basic inequalities among them together with $\mathfrak{b}$ and $\mathfrak{b}^*$. In Section 3, we prove some restrictions the combinatorial principles in section 1 impose on the values of the cardinal invariants introduced in section 2. These results will be used in [1] and another forthcoming paper [4] to simplify the arguments to decide the constellations of the principles and the values of the cardinal invariants in generic extensions.

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1 The combinatorial principles

We begin with definition and basic properties of the principles we are going to consider in this note.

For cardinals κ_1, κ_2, $\kappa_1 << \kappa_2$ (or $\kappa_2 >> \kappa_1$) denotes the assertion "κ_2 is regular and $2^{\kappa_1} < \kappa_2$" while $\kappa_1 <<< \kappa_2$ (or $\kappa_2 >>> \kappa_1$) denotes "κ_2 is regular and $|\,\mathcal{H}((\kappa_1)^+)\,| \leq \kappa_2$".

Let κ be a regular cardinal and $\lambda \leq \kappa$. Our first principle is named after PRINC of S. Shelah [11] which is $\mathrm{PRINC}(\aleph_2, \aleph_2)$ in our notation.

$\mathrm{PRINC}(\kappa, \lambda)$: For any $\chi >> \kappa$ and $x \in \mathcal{H}(\chi)$, there is an $N \prec \mathcal{H}(\chi)$ such that
(0) $x \in N$,
(I) $|\,N\,| \leq N \cap \kappa \in \kappa$,
(II) $\forall a \in [\omega]^{\aleph_0}\ \ \exists P \in \left(\left[[\omega]^{\aleph_0}\right]^{<\min(|\,N\,|^+, \lambda)} \cap N \right)\ \ \forall b \in N$
$(a \subseteq b\ \rightarrow\ \exists c \in P\,(a \subseteq c \subseteq b))$.

Note that P in (II) is a subset of N as $|\,P\,| \leq |\,N\,|$ and $|\,N\,| \subseteq N$ by $N \models$ (I). Note also that the definition of $\mathrm{PRINC}(\kappa, \lambda)$ is only relevant for regular κ: Suppose that κ is singular and $\kappa = \sup_{i<\mu} \kappa_i$ for $\mu = \mathrm{cf}(\kappa) < \kappa$. If $N \prec \mathcal{H}(\chi)$ for some $\chi >> \kappa$, $N \models (0)$ for $x = \kappa$ and $N \models$ (I), then $\mu \in N$ by $\kappa \in N$ and elementarity. Hence there is a sequence $\langle \kappa_i : i < \mu \rangle \in N$ as above. Since $\mu \subseteq N$ by $N \models$ (I), $\kappa_i \in N$ for all $i < \mu$. Hence, again by $N \models$ (I), $\kappa_i \subseteq N$ for all $i < \mu$. Thus $\kappa \subseteq N$ and $|\,N\,| \geq \kappa$. But this is a contradiction to $|\,N\,| \in \kappa$ which follows from $N \models$ (I).

Lemma 1.1 *If λ as above is a successor cardinal then* (II) *in the definition of* $\mathrm{PRINC}(\kappa, \lambda)$ *may be replaced by*

(II') $\forall a \in [\omega]^{\aleph_0}\ \exists P \in \left[[\omega]^{\aleph_0}\right]^{<\lambda} \cap N\ \forall b \in N\ (a \subseteq b\ \rightarrow\ \exists c \in P\,(a \subseteq c \subseteq b))$.

Proof. Let us call the principle $\mathrm{PRINC}'(\kappa, \lambda)$ which is obtained from $\mathrm{PRINC}(\kappa, \lambda)$ by replacing (II) by (II') above. It is clear that $\mathrm{PRINC}'(\kappa, \lambda)$ follows from $\mathrm{PRINC}(\kappa, \lambda)$.

Suppose that $\lambda = \mu^+$. We show that $\mathrm{PRINC}(\kappa, \lambda)$ follows from $\mathrm{PRINC}'(\kappa, \lambda)$. Assume that $\mathrm{PRINC}'(\kappa, \lambda)$ holds. For $\chi >> \kappa$ and $x \in \mathcal{H}(\chi)$, let $x' = \langle x, \mu \rangle$. By $\mathrm{PRINC}'(\kappa, \lambda)$, there is an $N \prec \mathcal{H}(\chi)$ such that $|\,N\,| < \kappa$, $x' \in N$ and $N \models$(I), (II). By $\mu \in N$ and $N \models$ (I), it follows that $\mu \subseteq N$. Hence $|\,N\,|^+ \geq \lambda$ and $\lambda = \min(|\,N\,|^+, \lambda)$. Since $x \in N$, this shows that $N \models$(0), (I), (II) for this x.

□ (Lemma 1.1)

Lemma 1.2 *For regular κ and $\lambda \leq \kappa$ the following are equivalent:*

(a) $\mathrm{PRINC}(\kappa, \lambda)$.

(b) *For any $\chi >> \kappa$ and $x \in \mathcal{H}(\chi)$,*

$$\begin{aligned} \mathcal{S}_{\chi,x} = \{N \prec \mathcal{H}(\chi) : \ & (0) \quad x \in N, \\ & (\mathrm{I}) \quad |N| \leq N \cap \kappa \in \kappa, \\ & (\mathrm{II}) \quad \forall a \in [\omega]^{\aleph_0}\ \exists P \in \left(\left[[\omega]^{\aleph_0}\right]^{<\min(|N|^+,\lambda)} \cap N \right) \\ & \qquad \forall b \in N \ (a \subseteq b \ \rightarrow \ \exists c \in P \,(a \subseteq c \subseteq b)) \ \} \end{aligned}$$

is stationary in $[\mathcal{H}(\chi)]^{<\kappa}$.

Proof. (b) $\Rightarrow$ (a) is clear. For (a) $\Rightarrow$ (b), suppose that $\chi >> \kappa$, $x \in \mathcal{H}(\chi)$ and $\mathcal{C} \subseteq [\mathcal{H}(\chi)]^{<\kappa}$ is a club set. We show that $\mathcal{S}_{\chi,x} \cap \mathcal{C} \neq \emptyset$. Without loss of generality, we may assume that

(1.1) $\quad |M| = |M \cap \kappa|$

for all $M \in \mathcal{C}$.

Let $\chi' >>> \chi$. By $\mathrm{PRINC}(\kappa, \lambda)$, there is an $\tilde{N} \in \mathcal{S}_{\chi'}$ such that x, $\mathcal{C} \in \tilde{N}$. We have $\mathcal{H}(\chi) \in \tilde{N}$. By elementarity, it follows that

(1.2) $\quad \tilde{N} \cap \mathcal{H}(\chi) \in \mathcal{S}_{\chi,x}$.

On the other hand:

Claim 1.2.1 $\tilde{N} \cap \mathcal{H}(\chi) = \bigcup(\mathcal{C} \cap \tilde{N})$.

$\vdash$ For $M \in \mathcal{C} \cap \tilde{N}$, $\sup(M \cap \kappa) \in \tilde{N}$. Hence $M \cap \kappa \subseteq \tilde{N} \cap \mathcal{H}(\chi)$ by $\tilde{N} \models$ (I). Hence by (1.1), it follows that $M \subseteq \tilde{N} \cap \mathcal{H}(\chi)$. This shows that $\tilde{N} \cap \mathcal{H}(\chi) \supseteq \bigcup(\mathcal{C} \cap \tilde{N})$. For the other inclusion, suppose $x \in \tilde{N} \cap \mathcal{H}(\chi)$. then there is an $M \in \mathcal{C}$ such that $\{x\} \subseteq M$ (i.e. $x \in M$). By elementarity, there is such an $M \in \mathcal{C} \cap \tilde{N}$. Hence $\tilde{N} \cap \mathcal{H}(\chi) \subseteq \bigcup(\mathcal{C} \cap \tilde{N})$. $\dashv$ (Claim 1.2.1)

Since $\mathcal{C}$ is closed, it follows that

(1.3) $\quad \tilde{N} \cap \mathcal{H}(\chi) \in \mathcal{C}$.

Thus $\mathcal{S}_{\chi,x} \cap C \neq \emptyset$.

$\square$ (Lemma 1.2)

Let κ be regular and $\lambda \leq \kappa$. The following principle SEP(κ, λ) is derived from SEP of Juhász and Kunen [7]. In Fuchino and Geschke [5], it is shown that SEP$(\aleph_2, \aleph_1)$ in the notation below is equivalent to SEP of Juhász and Kunen [7].

SEP(κ, λ): For any $\chi >> \kappa$ and $x \in \mathcal{H}(\chi)$, there is an $N \prec \mathcal{H}(\chi)$ such that
(0) $x \in N$,
(I) $|N| \leq N \cap \kappa \in \kappa$ and $[N]^{<\lambda} \cap N$ is cofinal in $[N]^{<\lambda}$,
(II) $\mathcal{P}(\omega) \cap N \leq_\lambda \mathcal{P}(\omega)$

where $P \leq_\lambda Q$ for partial orderings P and Q means that P is a subordering of Q and for all $q \in Q$, $P \restriction q = \{p \in P : p \leq q\}$ has a cofinal subset of size $< \lambda$ and $P \uparrow q = \{p \in P : q \leq p\}$ has a coinitial subset of size $< \lambda$. $\mathcal{P}(\omega)$ is seen here as a partial ordering (or even a Boolean algebra) with respect to the canonical ordering $\subseteq$ on it.

Similarly to Lemma 1.2, the phrase "there is an $N \prec \mathcal{H}(\chi)$" in the definition of SEP(κ, λ) can be replaced by "there are stationary many $N \prec \mathcal{H}(\chi)$" where "stationary many" refers to stationarity in $[\mathcal{H}(\chi)]^{<\kappa}$ (see [5]). Here also, the case of singular κ is irrelevant for SEP(κ, λ) — see the argument after the definition of $\textsc{Princ}(\kappa, \lambda)$.

The next lemma follows immediately from the definitions of $\textsc{Princ}(\kappa, \lambda)$ and SEP(κ, λ).

Lemma 1.3 *Suppose that κ is regular and $\lambda \leq \lambda' \leq \kappa$.*
(1) *If* $\textsc{Princ}(\kappa, \lambda)$ *then* $\textsc{Princ}(\kappa, \lambda')$.
(2) *If* SEP(κ, λ) *then* SEP(κ, λ').
(3) *If* SEP(κ, λ) *then* $\textsc{Princ}(\kappa, \lambda)$.

For any set X, let

(1.4) $$((X))^n = \{\vec{x} \in X^n : \vec{x} \text{ is injective}\}$$

and

(1.5) $$((X))^{<\omega} = \bigcup_{n<\omega} ((X))^n.$$

Likewise, for sets $X_0, \ldots, X_{n-1}$, let

(1.6) $\quad ((X_0, \ldots, X_{n-1})) = \{\vec{x} \in X_0 \times \cdots \times X_{n-1} : \vec{x} \text{ is injective}\}$.

The following principle was introduced by I. Juhász, L. Soukup and Z. Szentmiklóssy in [8].

$C^s(\kappa)$: For any matrix $\langle a_{\alpha,n} : \alpha \in \kappa, n \in \omega \rangle$ of subsets of ω and $T \subseteq {}^{\omega>}\omega$, one of the following holds:

(c0) there is a stationary $S \subseteq \kappa$ such that $\bigcap_{i<\ell(t)} a_{\alpha_i,t(i)} \neq \emptyset$ for all $t \in T$ and $(\alpha_0, \ldots, \alpha_{\ell(t)-1}) \in ((S))^{<\omega}$;

(c1) there exist $t \in T$ and stationary $S_0, \ldots, S_{\ell(t)-1} \subseteq \kappa$ such that $\bigcap_{i<\ell(t)} a_{\alpha_i,t(i)} = \emptyset$ for all $(\alpha_0, \ldots, \alpha_{\ell(t)-1}) \in ((S_0, \ldots, S_{\ell(t)-1}))$.

It is easily seen that $\mathrm{SEP}(\kappa, \lambda)$ and hence also $\mathrm{P{\small RINC}}(\kappa, \lambda)$ holds for any κ, λ for regular κ and $\lambda \leq \kappa$ with $2^{\aleph_0} < \lambda$. Similarly we have the following:

Lemma 1.4 (I. Juhász, L. Soukup and Z. Szentmiklóssy [8]) (1) $C^s(\aleph_1)$ *does not hold.*

(2) $C^s(\kappa)$ *holds for any regular* $\kappa > 2^{\aleph_0}$.

Shelah proved the following Theorem 1.5 for the case $\kappa = \aleph_2$. Theorem 1.5 in its present form can be proved by a straightforward generalization of Shelah's original proof. Tadatoshi Miyamoto suggested to the author that the proof can be slightly simplified by using Lemma 1.2. The proof we give below is with this simplification.

Theorem 1.5 (S. Shelah [11]) *For regular* κ, $\mathrm{P{\small RINC}}(\kappa, \kappa)$ *implies* $C^s(\kappa)$.

Proof. Assume $\mathrm{P{\small RINC}}(\kappa, \kappa)$. Suppose that $\mathcal{A} = \langle a_{\alpha,n} : \alpha \in \kappa, n \in \omega \rangle$ is a matrix of subsets of ω and $T \subseteq {}^{\omega>}\omega$. Let $\chi >> \kappa$ and $\mathcal{S}_{\chi,\langle \mathcal{A},T \rangle}$ be defined as in Lemma 1.2,(b). For each $N \in \mathcal{S}_{\chi,\langle \mathcal{A},T \rangle}$, let $\delta_N = \kappa \cap N$. By $N \models$ (I), $\delta_N \in \kappa$.

Applying $N \models$ (II) to $\bigcup_{n\in\omega} a_{\delta_N,n} \times \{n\}$ (coded as a single subset of ω), we can find a set P_N for each $N \in \mathcal{S}_{\chi,\langle \mathcal{A},T \rangle}$ such that

$$(1.7) \quad \begin{array}{l} P_N \in \left[[\omega]^{\aleph_0}\right]^{<|N|^+} \cap N \text{ and} \\ \forall b \in N\ \forall n \in \omega\, (a_{\delta_N,n} \subseteq b \ \rightarrow\ \exists c \in P_N\, (a_{\delta_N,n} \subseteq c \subseteq b)) \end{array}$$

$\mathcal{S}_{\chi,\langle\mathcal{A},T\rangle}$ is stationary in $[\mathcal{H}(\chi)]^{<\kappa}$ by Lemma 1.2. Hence, by Fodor's lemma, there is a stationary $\mathcal{S} \subseteq \mathcal{S}_{\chi,\langle\mathcal{A},T\rangle}$ and P^* such that $P_N = P^*$ for all $N \in \mathcal{S}$. Let $S = \{\delta_N : N \in \mathcal{S}\}$. Then S is stationary in κ.

Now, if $\bigcap_{i<n} a_{\alpha_i,t(i)} \neq \emptyset$ for all $t \in T$, $n = |\, t \,|$ and $\langle\alpha_0, \ldots , \alpha_{n-1}\rangle \in ((S))^n$, then $(\mathcal{A}, T) \models$ (c0) and we are done.

Otherwise, there are $t^* \in T$, $n^* = |\, t^* \,|$ and $\langle\alpha^*_0, \ldots \alpha^*_{n^*-1}\rangle \in ((S))^{n^*}$ such that

(1.8) $\quad \bigcap_{i<n^*} a_{\alpha^*_i,t^*(i)} = \emptyset.$

Claim 1.5.1 *There are* $c_0, \ldots , c_{n^*-1} \in P^*$ *such that*

(1.9) $\quad c_i \in P^*$ *and* $a_{\alpha^*_i,t^*(i)} \subseteq c_i$

for $i < n^*$ *and* $\bigcap_{i<n^*} c_i = \emptyset$.

$\vdash$ Without loss of generality, we may assume that $\alpha^*_0 < \alpha^*_1 < \cdots < \alpha^*_{n^*-1}$. For $i < n^*$, let $\alpha^*_i = \delta_{N_i}$ for some $N_i \in \mathcal{S}$. We take $c^*_{n-1}, c^*_{n-2}, \ldots , c^*_0 \in P^*$ in turn by downward induction so that at k'th step we have

(1.10) $\quad \bigcap_{i<k} a_{\alpha^*_i,t^*(i)} \cap \bigcap_{k\leq i<n^*} c_i = \emptyset.$

To see that this is possible, assume that (1.9) for all $k \leq i < n^*$ and (1.10) hold — for $k = n^*$ this is just our assumption (1.8). We show that then we can choose an appropriate c_{k-1}. Let

$$b_k = \omega \setminus \left(\bigcap_{i<k-1} a_{\alpha^*_i,t^*(i)} \cap \bigcap_{k\leq i<n^*} c_i \right).$$

Then $a_{\alpha^*_{k-1},t^*(k)} \subseteq b_k$. Since $\alpha^*_0 < \cdots < \alpha^*_{k-2} < \alpha^*_{k-1} = \delta_{N_{k-1}} = \kappa \cap N_{k-1}$, $\alpha^*_0, \ldots , \alpha^*_{k-2} \in N_{k-1}$. Hence by $\mathcal{A} \in N_{k-1}$ we have $a_{\alpha^*_i,t^*(i)} \in N_{k-1}$ for $i < k - 1$. Also $c_k, \ldots , c_{n^*-1} \in P^* \subseteq N_{k-1}$ by induction hypothesis. It follows that $b_k \in N_{k-1}$. By $N_{k-1} \models$ (II), we can find a $c_{k-1} \in P_{N_{k-1}} = P^*$ such that $a_{\alpha^*_{k-1},t^*(k)} \subseteq c_{k-1} \subseteq b_k$. This c_{k-1} is as desired. $\dashv$ (Claim 1.5.1).

Let $S_i = \{\alpha \in S : a_{\alpha,t^*(i)} \subseteq c_i\}$ for $i < n^*$. The next claim shows that these S_i, $i < n^*$ witness $(\mathcal{A}, T) \models$ (c1) and so we are done also in this case.

Claim 1.5.2 *S_i is stationary for $i < n^*$.*

$\vdash$ Let $i < n^*$. As in the proof of the previous claim, let $\alpha_i^* = \delta_{N_i}$, for some $N_i \in \mathcal{S}$. By definition of S_i, we have $\alpha_i^* \in S_i$.

By $c_i \in P^* = P_N \subseteq N_i$, $\mathcal{A} \in N_i$ and by elementarity, we have $S_i \in N_i$. For any club $C \subseteq \kappa$, $C \in N_i$, we have $\alpha_i^* = \delta_{N_i} = \kappa \cap N_i \in C$. Hence $\mathcal{H}(\chi) \models C \cap S_i \neq \emptyset$. By elementarity $N_i \models C \cap S_i \neq \emptyset$. It follows that $N_i \models S_i$ is stationary. Thus, again by elementarity, we conclude that S_i is really stationary. $\dashv$ (Claim 1.5.2)

$\square$ (Theorem 1.5)

The following principle was introduced in [1]. We shall call a subset A of $\mathcal{H}(\aleph_1)$ definable if there are a formula φ and $a \in \mathcal{H}(\aleph_1)$ such that $A = \{x \in \mathcal{H}(\aleph_1) : \mathcal{H}(\aleph_1) \models \varphi(x, a)\}$. Note that for any $n \in \mathbb{N}$, $A \subseteq \mathbb{R}^n$ is projective if and only if it is definable in our sense.

HP(κ): For any $f : \kappa \rightarrow \mathcal{P}(\omega)$ and any definable $A \subseteq ((\mathcal{P}(\omega)))^{<\omega}$, one of the following holds:

(h0) there is a stationary $S \subseteq \kappa$ such that $((f''S))^{<\omega} \setminus \{\emptyset\} \subseteq A$;

(h1) there are $n \in \omega \setminus \{0\}$ and stationary $S_0, \ldots, S_{n-1} \subseteq \kappa$ such that $((f''S_0, \ldots, f''S_{n-1})) \cap A = \emptyset$.

Remark. $\mathcal{P}(\omega)$ in the definition of HP(κ) above can be replaced by ${}^{\omega}\omega$, $(\mathcal{P}(\omega))^n$, $({}^{\omega}\omega)^n$ etc. since these spaces can be coded as definable subsets of $\mathcal{P}(\omega)$.

Lemma 1.6 ([1]) *For a regular cardinal* κ, HP(κ) *implies* $\mathrm{C}^s(\kappa)$.

By Lemma 1.4, (1), $\mathrm{C}^s(\aleph_1)$ does not hold. Hence $\aleph_2$ is the least non-trivial setting of κ for $\mathrm{C}^s(\kappa)$. For $\kappa = \aleph_2$, the combinatorial principles introduced above together with some other principles can be put together in the following diagram[1)]:

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1) In the following diagram, "WFN" is the assertion "$\mathcal{P}(\omega)$ has the weak Freese-Nation property" (see [6]). "SEP" is SEP($\aleph_2, \aleph_1$). For "IP($\aleph_2$)" see [1].

$$\begin{array}{ccc}
\text{WFN} & & \\
\downarrow & & \\
\text{SEP} & & \\
\downarrow & & \\
\textsc{Princ}(\aleph_2,\aleph_1) & & \text{IP}(\aleph_2) \\
\downarrow & & \downarrow \\
\textsc{Princ}(\aleph_2,\aleph_2) & & \text{HP}(\aleph_2) \\
& \searrow \quad \swarrow & \\
& \text{C}^s(\aleph_2) &
\end{array}$$

Sometimes it is more convenient to consider the following variant of PRINC:

PRINC$^+(\kappa,\lambda)$: For any $\chi >> \kappa$ and $x \in \mathcal{H}(\chi)$, there is an $N \prec \mathcal{H}(\chi)$ such that
(0) $x \in N$,
(I) $|N| \leq N \cap \kappa \in \kappa$,
(II$^+$) $\forall \mu < \lambda \, \forall A \in [\mu]^\mu \ \exists \tilde{P} \in \left([[\mu]^\mu]^{<\min(|N|^+,\lambda)} \cap N \right)$
$\forall B \in N \ (A \subseteq B \ \rightarrow \ \exists C \in \tilde{P}\,(A \subseteq C \subseteq B))$.

The following is immediate from the definition:

Lemma 1.7 (1) *For any regular κ,* PRINC$^+(\kappa,\aleph_1)$ *if and only if* PRINC$(\kappa,\aleph_1)$.
(2) *For any regular κ and λ with $\lambda \leq \kappa$,* PRINC$^+(\kappa,\lambda)$ *implies* PRINC(κ,λ).

2 Cardinal invariants and cardinal spectra connected to the bounding number

For a partial ordering $P = \langle P, \leq \rangle$. The following sets are introduced in [12]:

$$\mathfrak{S}^{\uparrow}(P) = \{\mathrm{cf}(C) : C \subseteq P,\ C \text{ is an unbounded chain}\}$$

$$\mathfrak{S}^{h}(P) = \{|X| : X \subseteq P,\ \forall B \subseteq X \ (B \text{ is bounded in } P \leftrightarrow |B| < |X|)\}$$

$$\mathfrak{S}(P) = \{|X| : X \subseteq P, X \text{ is unbounded in } P, \forall B \in [X]^{<|X|} (B \text{ is bounded in } P)\}$$

Clearly, we have

(2.1) $\mathfrak{S}^{\uparrow}(P) \subseteq \mathfrak{S}^{h}(P) \subseteq \mathfrak{S}(P)$.

For $P = \langle {}^{\omega}\omega, \leq^* \rangle$, we shall simply write $\mathfrak{S}^{\uparrow}$, $\mathfrak{S}^{h}$ and $\mathfrak{S}$ in place of $\mathfrak{S}^{\uparrow}(\langle {}^{\omega}\omega, \leq^* \rangle)$, $\mathfrak{S}^{h}(\langle {}^{\omega}\omega, \leq^* \rangle)$ and $\mathfrak{S}(\langle {}^{\omega}\omega, \leq^* \rangle)$, respectively.

Recall the definition of the following cardinal invariants:

$$\mathfrak{b} = \min\{|X| : X \subseteq {}^{\omega}\omega \text{ is unbounded with respect to } \leq^*\}$$
$$\mathfrak{b}^* = \min\{\kappa : \forall X \subseteq {}^{\omega}\omega \ (X \text{ is unbounded } \rightarrow \exists X' \in [X]^{\leq\kappa}(X' \text{ is unbounded}))\}$$

$\mathfrak{b}$ and $\mathfrak{b}^*$ can be characterized in terms of $\mathfrak{S}^{\uparrow}$, $\mathfrak{S}^{h}$ and $\mathfrak{S}$ as follows:

Lemma 2.1 (1) $\mathfrak{b} = \min \mathfrak{S}^{\uparrow} = \min \mathfrak{S}^{h} = \min \mathfrak{S}$.

(2) $\mathfrak{b}^* = \sup \mathfrak{S}$.

Proof. (1): $\mathfrak{b} \leq \min \mathfrak{S} \leq \min \mathfrak{S}^{h} \leq \min \mathfrak{S}^{\uparrow}$ is clear by definition. It is also easily seen that there is an increasing sequence in $\langle {}^{\omega}\omega, \leq^* \rangle$ of length $\mathfrak{b}$. Hence $\min \mathfrak{S}^{\uparrow} \leq \mathfrak{b}$.

(2): For any cardinal κ, we have

$$\begin{aligned}\kappa < \mathfrak{b}^* &\Leftrightarrow \exists X \subseteq {}^{\omega}\omega \ (X \text{ is bounded and } \forall X' \in [X]^{\leq\kappa} \ (X' \text{ is bounded}))\\ &\Leftrightarrow \exists \lambda > \kappa \ (\lambda \in \mathfrak{S})\\ &\Leftrightarrow \kappa < \sup \mathfrak{S}.\end{aligned}$$

□ (Lemma 2.1)

On analogy of Lemma 2.1, (2), let

$$\mathfrak{b}' = \sup \mathfrak{S}^{\uparrow}, \qquad \mathfrak{b}'' = \sup \mathfrak{S}^{h}.$$

By Lemma 2.1 and (2.1), we have

$$\mathfrak{b} \leq \mathfrak{b}' \leq \mathfrak{b}'' \leq \mathfrak{b}^*.$$

Let

$$\mathfrak{D}\mathfrak{O} = \{\mathrm{cf}(otp(\langle X, R \restriction X \rangle)) : X \subseteq {}^{\omega}\omega,\ R \text{ is a projective binary relation and } R \cap X^2 \text{ well orders } X\}$$

and

$$\mathfrak{b}^{\dagger} = \sup \mathfrak{D}\mathfrak{O}.$$

Lemma 2.2 (1) $\mathfrak{S}^\dagger \subseteq \mathfrak{D}\mathfrak{O}$.

(2) $\mathfrak{S}^h \cap Reg \subseteq \mathfrak{D}\mathfrak{O}$. *In particular, if* $2^{\aleph_0} < \aleph_\omega$ *then* $\mathfrak{S}^h \subseteq \mathfrak{D}\mathfrak{O}$.

Proof. (1): This is clear by definition.

(2): For $\kappa \in \mathfrak{S}^h \cap Reg$, let $X \in [{}^\omega\omega]^\kappa$ be as in the definition of $\mathfrak{S}^h$. Then we can construct a_α, b_α, $\alpha < \kappa$ inductively such that

(2.2) $\quad a_\alpha \in X$ and $b_\alpha \in {}^\omega\omega$ for $\alpha < \kappa$.

(2.3) $\quad a_\alpha \leq^* b_\beta \Leftrightarrow \alpha < \beta$.

(2.3) is possible since, at the β'th step in the inductive construction, we have $|\{a \in X \,:\, a \leq^* b_\alpha \text{ for some } \alpha < \beta\}| < \kappa$. Note that we need here that κ is regular.

Let $Y = \{\langle a_\alpha, b_\alpha\rangle \,:\, \alpha < \kappa\}$ and let R be the binary relation defined by

$$\langle a, b\rangle \; R \; \langle c, d\rangle \;\Leftrightarrow\; a \leq^* d.$$

for $\langle a, b\rangle$, $\langle c, d\rangle \in ({}^\omega\omega)^2$ Clearly R is projective and orders Y in order type κ. $\square$ (Lemma 2.2)

Corollary 2.3 (1) $\mathfrak{b}' \leq \mathfrak{b}^\dagger$,

$$\begin{array}{ccccccccc} & & & & \mathfrak{b}^\dagger & & & & \\ & & & & \leq & & & & \\ \mathfrak{b} & \leq & \mathfrak{b}' & \leq & \mathfrak{b}'' & \leq & \mathfrak{b}^* & \leq & \mathfrak{d} \end{array}$$

(2) *If* $\mathfrak{S}^h \cap Reg$ *is cofinal in* $\mathfrak{S}^h$ *then* $\mathfrak{b}'' \leq \mathfrak{b}^\dagger$. *In particular, if* $\mathfrak{b}''$ *is regular*[2] *then we have* $\mathfrak{b}'' \leq \mathfrak{b}^\dagger$.

$$\begin{array}{ccccccccc} & & & & & & \mathfrak{b}^\dagger & & \\ & & & & & & \leq & & \\ \mathfrak{b} & \leq & \mathfrak{b}' & \leq & \mathfrak{b}'' & \leq & \mathfrak{b}^* & \leq & \mathfrak{d} \end{array}$$

$\square$

[2] Since $\aleph_0 < \mathfrak{b}'' \leq 2^{\aleph_0}$, this is the case if $2^{\aleph_0} < \aleph_\omega$.

3 $\mathfrak{b}$, $\mathfrak{b}'$, $\mathfrak{b}''$, $\mathfrak{b}^*$ and $\mathfrak{b}^\dagger$ under the combinatorial principles

Proposition 3.1 (1) *$\kappa \in \mathfrak{S}^h \cap Reg$ implies $\neg$*PRINC*(κ', κ') for all $\kappa' \leq \kappa$.*

(2) *If $\mathfrak{S}^h \cap Reg$ is cofinal in $\mathfrak{S}^h$ then* PRINC*(κ, κ) implies $\mathfrak{b}'' < \kappa$. In particular, if $2^{\aleph_0} < \aleph_\omega$, then* PRINC*$(\kappa, \kappa)$ implies $\mathfrak{b}'' < \kappa$.*

(3) *If there is a $\leq^*$-chain of length κ then $\neg\mathrm{C}^s(\kappa)$. In particular, $\kappa \in \mathfrak{S}^\uparrow$ implies $\neg\mathrm{C}^s(\kappa)$.*

(4) *$\mathrm{C}^s(\kappa)$ implies $\mathfrak{b}' < \kappa$.*

(5) PRINC*$(\kappa, \mathfrak{b})$ implies $\mathfrak{b}^* < \kappa$.*

(6) *$\kappa \in \mathfrak{S}$ implies $\neg$*PRINC*$(\kappa, \mathfrak{b})$.*

(7) *$\kappa \in \mathfrak{DD} \rightarrow \neg\mathrm{HP}(\kappa)$.*

(8) *$\mathrm{HP}(\kappa)$ implies $\mathfrak{b}^\dagger < \kappa$.*

Proof. (1): Since $\neg$PRINC(κ', κ') for all singular κ', we may assume that κ' is regular. Let $X \in [{}^\omega\omega]^\kappa$ be a witness of $\kappa \in \mathfrak{S}^h$ and let a_α, $b_\alpha \in {}^\omega\omega$ for $\alpha < \kappa$ satisfy (2.2) and (2.3) above. Suppose that χ is sufficiently large and let $N \prec \mathcal{H}(\chi)$ be such that

(3.1) $\quad \kappa'$, $\langle a_\alpha : \alpha < \kappa \rangle$, $\langle b_\alpha : \alpha < \kappa \rangle \in N$ and

(3.2) $\quad |N| \leq \kappa' \cap N < \kappa'$.

We show that N does not satisfy (II) in the definition of PRINC(κ', κ'). Suppose, for contradiction, that $N \models$ (II). Then we have[3]:

(3.3) $\quad$ for any $f \in {}^\omega\omega$ there is a $P \in N$ such that $P \subseteq N$, $|P| < \kappa'$ and $\forall g \in {}^\omega\omega \cap N$ $(g \leq^* f \rightarrow \exists h \in P$ $(g \leq^* h \leq^* f))$.

Let

(3.4) $\quad \alpha^* = \sup \kappa \cap N$.

Let P be as in (3.3) for $f = b_{\alpha^*}$. Let

(3.5) $\quad S = \{\alpha < \kappa : a_\alpha \leq^* h$ for some $h \in P\}$.

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[3] If $g \in {}^\omega\omega \cap N$ is such that $g \leq^* f$, then there is $g' \in {}^\omega\omega \cap N$ such that $g' \Delta g$ is finite and $g' \leq f$. Note that we have $s_{g'} \in N$ and $s_{g'} \subseteq s_f$ where we let $x_h = \{\langle m, n \rangle \in \omega^2 : n \leq h(m)\}$ for $h \in {}^\omega\omega$. Hence we obtain (3.3) applying (II) in the definition of PRINC(κ', κ') to $\bigcup_{n \in \omega} x_{f_n} \times \{n\} \subseteq \omega \times \omega$ where $\{f_n : n \in \omega\}$ enumerates finite modifications of f.

By $a_\alpha \in X$ and since κ is regular, we have $|S| < \kappa$. Hence, by $S \in N$ and by elementarity, there is $\alpha \in (\kappa \cap N) \setminus S$. But since $a_\alpha \leq b_{\alpha^*}$ by (3.4) and (2.3), this is a contradiction.

(2): This follows from (1).

(3): Assume that there is a $\leq^*$-chain of length κ in ${}^\omega\omega$. Then there is a sequence $\langle b_\alpha : \alpha < \kappa \rangle$ of subsets of ω such that $b_\alpha \subseteq^* b_\beta$ and $b_\beta \not\subseteq^* b_\alpha$ for all $\alpha < \beta < \kappa$. For $\alpha < \kappa$ and $n \in \omega$, let

$$(3.6) \qquad a_{\alpha,n} = \begin{cases} \omega \setminus b_\alpha, & \text{if } n = 0 \\ b_\alpha \setminus n, & \text{otherwise} \end{cases}$$

and $\mathcal{A} = \langle a_{\alpha,n} : \alpha < \kappa, n \in \omega \rangle$. Let $T = \{\langle 0, n \rangle : n \in \omega\}$. Then it is easy to see that $(\mathcal{A}, T) \not\models$ (c0) and $(\mathcal{A}, T) \not\models$ (c1).

(4): This follows from (3).

(5): Suppose that $S \subseteq {}^\omega\omega$ is unbounded. By $\textsc{Princ}(\kappa, \mathfrak{b})$, there is $N \prec \mathcal{H}(\chi)$ such that

$$(3.7) \qquad S \in N,$$

$$(3.8) \qquad |N| \leq N \cap \kappa < \kappa,$$

$$(3.9) \qquad \forall a \in [\omega]^{\aleph_0} \ \exists P \in \left[[\omega]^{\aleph_0}\right]^{<\min(|N|^+, \,)} \cap N \ \ \forall b \in N \ (a \subseteq b \rightarrow \exists c \in P(a \subseteq c \subseteq b)).$$

By (3.8), it is enough to show the following:

Claim 3.1.1 *$S \cap N$ is unbounded in ${}^\omega\omega$.*

$\vdash$ Otherwise there is a $g \in {}^\omega\omega$ such that $f \leq^* g$ for all $f \in S \cap N$. There is an $P \in N$ such that

$$(3.10) \qquad |P| < \mathfrak{b},\ P \subseteq N \text{ and}$$

$$(3.11) \qquad \text{for any } f \in S \cap N \text{ there is } h \in P \text{ with } f \leq^* h \leq^* g$$

— see the footnote on page 11. By $P \in N$, (3.10) and by elementarity of N, there is $g' \in {}^\omega\omega \cap N$ such that $h \leq^* g'$ for all $h \in P$. By (3.11), $f \leq^* g'$ for all $f \in S \cap N$. It follows that $N \models S$ is bounded. By elementarity this is a contradiction. $\dashv$ (Claim 3.1.1)

(6): This follows from (5).

(7): Suppose that $\kappa \in \mathfrak{D}\mathfrak{D}$ and let $\langle X, R\rangle$ be such that $X \subseteq \mathcal{P}(\omega)$, R is a projective binary relation and $otp(\langle X, R \cap X^2\rangle) = \kappa$. Let $f : \kappa \to \mathcal{P}(\omega)$ be the mapping sending $\alpha < \kappa$ to the α'th element of X with respect to R. Let

$$A = R \cup \bigcup_{k\in\omega\setminus\{2\}} ((\mathcal{P}(\omega)))^k .$$

Then is easily seen that $\langle f, A\rangle \not\models$ (h0) and $\langle f, A\rangle \not\models$ (h1).

(8): This follows from (7) since $\mathfrak{D}\mathfrak{D}$ is downward closed. $\square$ (Proposition 3.1)

Corollary 3.2

(1) *Suppose that* $2^{\aleph_0} < \aleph_\omega$ *then* $\textsc{Princ}(\aleph_2, \aleph_2)$ *implies* $\mathfrak{b}'' = \aleph_1$.

(2) $\mathrm{C}^s(\aleph_2)$ *implies* $\mathfrak{b}' = \aleph_1$.

(3) $\textsc{Princ}(\aleph_2, \aleph_1)$ *implies* $\mathfrak{b} = \mathfrak{b}^* = \aleph_1$.

(4) $\mathrm{HP}(\aleph_2)$ *implies* $\mathfrak{b}^\dagger = \aleph_1$. $\square$

Let

$$\begin{aligned} shr(meager) = \min\{\kappa :& \forall X \subseteq \mathbb{R}\,(X \text{ is non-meager} \\ & \to \exists Y \subseteq X\,(|\,Y\,| \leq \kappa \wedge Y \text{ is non-meager}))\}. \end{aligned}$$

$shr(meager)$ as well as $\mathfrak{b}^*$ was studied in [10], [13] and [14]. In these papers it was shown that extended Cichón's diagram with these cardinal invariants looks like:

$$\begin{tikzcd}
cov(null) \arrow[ddd] & non(meager) \arrow[l] \arrow[d] & shr(meager) \arrow[l] \arrow[d] & cof(meager) \arrow[l] \arrow[d] & cof(null) \arrow[l] \arrow[ddd] \\
 & \mathfrak{b} \arrow[dd] & \mathfrak{b}^* \arrow[l] & \mathfrak{d} \arrow[l] \arrow[dd] & \\
 & & & & \\
add(null) & add(meager) \arrow[l] & & cov(meager) \arrow[ll] & non(null) \arrow[l]
\end{tikzcd}$$

Proposition 3.3 *Suppose that* κ *and* λ *are regular cardinals with* $\lambda \leq \mathfrak{b}$, κ . *Then* $\textsc{Princ}^+(\kappa, \lambda)$ *implies* $shr(meager) < \kappa$. *In particular,* $\textsc{Princ}(\aleph_2, \aleph_1)$ *implies* $shr(meager) = \aleph_1$.

Proof. Suppose that κ and λ are as above and that $\textsc{Princ}^+(\kappa, \lambda)$ holds. Then by Lemma 1.7 and Proposition 3.1, (5), we have $\mathfrak{b}^* < \kappa$. Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ and G be a $(V, \mathbb{P})$-generic filter. By [3], $V[G] \models \kappa \leq shr(meager)^V \leq \mathfrak{b}^*$. But it can be checked easily that $V[G] \models \textsc{Princ}^+(\kappa, \lambda)$. This is a contradiction to Lemma 1.7 and Proposition 3.1, (5). $\square$ (Proposition 3.3)

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