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Kyoto University
Cofinal types around $\mathcal{P}_\kappa\lambda$ and the tree property for directed sets

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Abstract

Generalizing a result of Todorcević, we prove the existence of directed sets $D, E$ such that $D \not\geq \mathcal{P}_\kappa\lambda$ and $E \not\geq \mathcal{P}_\kappa\lambda$ but $D \times E \geq \mathcal{P}_\kappa\lambda$ in the Tukey ordering. As an application, we show that the tree property for directed sets introduced by Hinnion is not preserved under products. Most of the results appear in [14].

1 Introduction

Any notion of convergence, described in terms of sequences, nets or filters, involves directed sets, or at least a particular kind of them. In general, directed sets are considered to express the type of convergence. Tukey defined an ordering on the class of all directed sets [17]. This ordering, now called Tukey ordering, was studied by Schmidt [15], Isbell [14],[12], Todorcević [16] and others. In particular, the directed sets of the form $\mathcal{P}_\kappa\lambda$ are of interest, because they possess some nice properties. In section 4 we generalize the directed sets $\mathcal{D}(S)$ introduced by Todorcević to $\mathcal{D}_\kappa(S)$, where $\kappa$ is an arbitrary infinite regular cardinal. With these directed sets, we show (Theorem 4.8) that there exist directed sets $D, E$ such that $D \not\geq \mathcal{P}_\kappa\lambda$ and $E \not\geq \mathcal{P}_\kappa\lambda$ but $D \times E \geq \mathcal{P}_\kappa\lambda$ in the Tukey ordering.

The notion of tree property for infinite cardinals (the nonexistence of an Aronszajn tree) is well known, and is related to a large variety of set theoretic statements. The tree property for directed sets was invented by Hinnion [10], and studied by Esser and Hinnion [8],[9]. It is a generalization of the usual tree property for infinite cardinals and especially, for $\mathcal{P}_\kappa\lambda$, it is closely related with the mild ineffability if $\kappa$ is strongly inaccessible (see Corollary 7.5). By an application of the result mentioned above, we show (Theorem 8.1) that there exist two directed sets $D, E$ for which $\text{add}(D) = \text{add}(E)$ is weakly compact, and both $D$ and $E$ have the tree property but $D \times E$ does not. It was an open problem whether such $D, E$ exist [8].
2 Directed sets and cofinal types

By classifying directed sets into isomorphism types, and further identifying a directed set with its cofinal subset, we arrive at the notion of cofinal type. On the other hand, the same equivalence relation is deduced from a quasi-ordering on the class of all directed sets. First we state the definitions.

**Definition 2.1** Let \( \langle D, \leq_D \rangle, \langle E, \leq_E \rangle \) be directed sets. A function \( f: E \to D \) which satisfies

\[
\forall d \in D \exists e \in E \forall e' \geq_E e \ [f(e') \geq_D d]
\]

is called a *convergent function*. If such a function exists we write \( D \leq E \) and say \( E \) is *cofinally finer than \( D \)*. \( \leq \) is transitive and is called the *Tukey ordering* on the class of directed sets. A function \( g: D \to E \) which satisfies

\[
\forall e \in E \exists d \in D \forall d' \in D \ [g(d') \leq_E e \to d' \leq_D d]
\]

is called a *Tukey function*.

If there exists a directed set \( C \) into which \( D \) and \( E \) can be embedded cofinally, we say \( D \) is *cofinally similar with \( E \)*. In this case we write \( D \equiv E \). \( \equiv \) is an equivalence relation, and the equivalence classes with respect to \( \equiv \) are the *cofinal types*.

The following propositions give the connection between the definitions. For the proofs, consult [16]

**Proposition 2.2** For directed sets \( D \) and \( E \), the following are equivalent.

(a) \( D \leq E \).
(b) There exists a Tukey function \( g: D \to E \).
(c) There exist functions \( g: D \to E \) and \( f: E \to D \) such that

\[
\forall d \in D \exists e \in E \forall e' \in E \ [g(d) \leq_E e \to d \leq_D f(e)].
\]

**Proposition 2.3** For directed sets \( D \) and \( E \), the following are equivalent.

(a) \( D \equiv E \).
(b) \( D \leq E \) and \( E \leq D \).

So we can regard \( \leq \) as an ordering on the class of all cofinal types.

One should always keep in mind the distinction between the unbounded and the cofinal subsets of a directed set.

**Proposition 2.4** For directed sets \( D \) and \( E \),

(i) \( f: E \to D \) is convergent \iff \( \forall C \subseteq E \) cofinal \( [f(C) \) cofinal].
(ii) \( g: D \to E \) is Tukey \iff \( \forall X \subseteq D \) unbounded \( [g(X) \) unbounded].

With two or more directed sets, we can form the product of these, to which we will always give the product ordering.
Proposition 2.5 For directed sets $D$ and $E$, $D \times E$ is the least upper bound of $\{D, E\}$ in the Tukey ordering.

The next two cardinal functions are the most basic ones, being taken up in various contexts (mostly on a particular kind of directed sets).

Definition 2.6 For a directed set $D$,
\[
\text{add}(D) \overset{\text{def}}{=} \min\{|X| \mid X \subseteq D \text{ unbounded}\},
\text{cof}(D) \overset{\text{def}}{=} \min\{|C| \mid C \subseteq D \text{ cofinal}\}.
\]

These are the additivity and the cofinality of a directed set. $\text{add}(D)$ is only well-defined for $D$ without maximum. In the sequel, any statement referring to $\text{add}(D)$ presupposes that $D$ has no maximum.

Proposition 2.7 For a directed set $D$ (without maximum),
\[
\aleph_0 \leq \text{add}(D) \leq \text{cof}(D) \leq |D|.
\]
Furthermore, $\text{add}(D)$ is regular and $\text{add}(D) \leq \text{cf(cof}(D))$. Here $\text{cf}$ is the cofinality of a cardinal, which is the same as the additivity of it.

Proposition 2.8 For directed sets $D$ and $E$, $D \leq E$ implies
\[
\text{add}(D) \geq \text{add}(E) \text{ and } \text{cof}(D) \leq \text{cof}(E).
\]

From the above proposition we see that these cardinal functions are invariant under cofinal similarity.

Example 2.9 (see [1, chapter 2]) Let $\mathcal{M}, \mathcal{N}$ be respectively the meager ideal and the null ideal, each ordered by inclusion. $\langle^\omega \omega, \leq^*\rangle$ is the eventual dominance order on the reals. We have $\langle^\omega \omega, \leq^*\rangle \leq \mathcal{M} \leq \mathcal{N}$ in the Tukey ordering, and thus
\[
\aleph_1 \leq \text{add}(\mathcal{N}) \leq \text{add}(\mathcal{M}) \leq b \leq \mathfrak{d} \leq \text{cof}(\mathcal{M}) \leq \text{cof}(\mathcal{N}) \leq 2^{\aleph_0}.
\]

Proposition 2.10 For directed sets $D$ and $E$,
\[
\text{add}(D \times E) = \min\{\text{add}(D), \text{add}(E)\},
\text{cof}(D \times E) = \max\{\text{cof}(D), \text{cof}(E)\}.
\]

3 The width of a directed set

In the following, $\kappa$ is always an infinite regular cardinal. If $P$ is partially ordered set, we use the notation $X_{\leq a} = \{x \in X \mid x \leq a\}$ for $X$ a subset of $P$ and $a \in P$.

The cofinal type of $\mathcal{P}_{\kappa}\lambda$ is an interesting topic by itself (see [16]). As usual, $\mathcal{P}_{\kappa}\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$ is ordered by inclusion.
Lemma 3.1 \(\text{add}(\mathcal{P}_{\kappa}\lambda) = \kappa\), and \(\lambda \leq \text{cof}(\mathcal{P}_{\kappa}\lambda) \leq \lambda^{<\kappa}\). In particular, if \(\kappa\) is strongly inaccessible, then \(\text{cof}(\mathcal{P}_{\kappa}\lambda) = \lambda^{<\kappa}\).

Proof For the last statement, notice that in general for a cofinal \(C \subseteq \mathcal{P}_{\kappa}\lambda\), \(\mathcal{P}_{\kappa}\lambda = \bigcup_{x \in C} P_x\), and thus \(\lambda^{<\kappa} \leq 2^{<\kappa} \cdot |C|\). \(\square\)

Lemma 3.2 For a directed set \(D\), if \(\text{add}(D) \geq \kappa\) and \(\text{cof}(D) \leq \lambda\), then \(D \leq \mathcal{P}_{\kappa}\lambda\).

It turns out that the following cardinal function, which seems to be a natural one, gives a suitable formulation of Theorem 7.1.

Definition 3.3 The \textit{width} of a directed set \(D\) is defined by

\[\text{wid}(D) \overset{\text{def}}{=} \sup\{|X|^+ \mid X \text{ is a thin subset of } D\},\]

where 'a thin subset of \(D\)' means

\[\forall d \in D[|X_{\leq d}| < \text{add}(D)].\]

Example 3.4 Let \(\kappa, \lambda, \mu\) be regular with \(\lambda^{<\kappa} = \lambda\) and \(\lambda \leq \mu\). Then for the directed set \(\mu \times \mathcal{P}_{\kappa}\lambda\) ordered by

\[\langle \alpha, x \rangle \leq \langle \beta, y \rangle \iff \alpha \leq \beta \land x \subseteq y\]

we have

\[
\begin{align*}
\text{add}(\mu \times \mathcal{P}_{\kappa}\lambda) & = \kappa, \\
\text{wid}(\mu \times \mathcal{P}_{\kappa}\lambda) & = \lambda^+, \\
\text{cof}(\mu \times \mathcal{P}_{\kappa}\lambda) & = \mu.
\end{align*}
\]

The second equation can be verified using Proposition 4.1.

Fix \(D\) and put \(\kappa := \text{add}(D)\).

Lemma 3.5 For any cardinal \(\lambda \geq \kappa\), the following are equivalent.

(a) \(D\) has a thin subset of size \(\lambda\).
(b) \(D \geq \mathcal{P}_{\kappa}\lambda\).
(c) There exists an order-preserving function \(f : D \rightarrow \mathcal{P}_{\kappa}\lambda\) with \(f[D]\) cofinal in \(\mathcal{P}_{\kappa}\lambda\).

Proof (a) \(\Rightarrow\) (b) Let \(X \subseteq D\) be a thin subset of size \(\lambda\). Define

\[
\begin{align*}
f : D & \rightarrow \mathcal{P}_{\kappa}X \\
\psi & \psi \\
d & \mapsto X_{\leq d}
\end{align*}
\]

Then \(f\) is (order-preserving and) convergent.
(b) $\Rightarrow$ (c) If $f : D \to \mathcal{P}_\kappa \lambda$ is convergent, define

$$g : D \to \mathcal{P}_\kappa \lambda$$
$$\psi \mapsto \bigcap_{d' \geq d} f(d')$$

Then $g$ is convergent and also order-preserving.

(c) $\Rightarrow$ (a) For such $g$ as above, pick for each $\alpha \in \lambda$ a $d_\alpha \in D$ such that $g(d_\alpha) \ni \alpha$, and put $X := \{d_\alpha \mid \alpha \in \lambda\}$. It is readily seen that $X$ is thin. Furthermore $|X| = \lambda$ since $\bigcup_{d \in X} g(d) = \lambda$.

**Corollary 3.6**

$$\text{wid}(D) = \sup \{\lambda^+ \mid D \supseteq \mathcal{P}_\kappa \lambda\}$$
$$= \sup \{\lambda^+ \mid \exists f : D \to \mathcal{P}_\kappa \lambda \text{ order-preserving with } f[D] \text{ cofinal in } \mathcal{P}_\kappa \lambda\}.$$  

The next inequality is checked easily.

**Lemma 3.7**

$$\text{add}(D)^+ \leq \text{wid}(D) \leq \text{cof}(D)^+.$$  

**Lemma 3.8** \text{wid}(D) is never singular.

**Proof** Assume $\lambda := \text{wid}(D) > \text{cf}(\lambda)$ for a directed set $D$ with $\text{add}(D) = \kappa$. Fix a sequence of ordinals $\langle \theta_\alpha \mid \alpha < \text{cf}(\lambda)\rangle$ converging up to $\lambda$. Then there are convergent order-preserving mappings $f_\alpha : D \to \mathcal{P}_\kappa \theta_\alpha$ for all $\alpha < \text{cf}(\lambda)$. Fix also a convergent order-preserving $g : D \to \mathcal{P}_\kappa \text{cf}(\lambda)$. Consider

$$h : D \to \mathcal{P}_\kappa \lambda$$
$$\psi \mapsto \bigcap_{\alpha \in g(d)} f_\alpha(d).$$

$h$ is order-preserving and convergent. Hence we have a contradiction. 

However, the next proposition will show that $\text{wid}(D)$ can be a limit cardinal. For example, that for any strongly inaccessible $\lambda$ there is a directed set $D$ such that $\text{wid}(D) = \lambda$.

**Proposition 3.9** Let $\kappa$ be regular and let $\lambda$ be strongly $\kappa^+$-inaccessible (i.e. $\lambda$ is regular and $\forall \mu < \lambda [\mu^\kappa < \lambda]$). Then there exists a directed set $D$ such that $\text{add}(D) = \kappa$ and $\text{wid}(D) = \lambda$.

**Proof** Consider

$$D = \prod_{\kappa \leq \alpha < \lambda} \mathcal{P}_\kappa \alpha.$$
I.e. $D$ is the set of functions $f$ such that $\text{dom}(f) \subseteq \lambda \setminus \kappa$, $|\text{dom}(f)| \leq \kappa$, and for all $\alpha \in \text{dom}(f)$, $f(\alpha) \in \mathcal{P}_{\kappa}\alpha$. The order is given by

$$f \leq_D g \iff \text{dom}(f) \subseteq \text{dom}(g) \land \forall \alpha \in \text{dom}(f) \ [f(\alpha) \subseteq g(\alpha)].$$

Since $\text{add}(D) = \kappa$ and $\mathcal{P}_{\kappa}\alpha \leq D$ for each $\alpha \in \lambda \setminus \kappa$, $\text{wid}(D) \geq \lambda$. To show that equality holds, let $\langle f_\alpha \mid \alpha < \lambda \rangle$ be a sequence of distinct elements in $D$. By the $\Delta$-system lemma there are $d \subseteq \lambda \setminus \kappa$ and $A \subseteq \lambda$ such that $|A| = \lambda$ and $\text{dom}(f_\alpha) \cap \text{dom}(f_\beta) = d$ for distinct $\alpha, \beta \in A$. Then by noting that $|\prod_{\alpha \in d}^{(\kappa^+)} \mathcal{P}_{\kappa}\alpha| < \lambda$, there is a $g \in D$ which bounds $\kappa$-many $f_\alpha$'s.

4 The directed sets $D_\kappa(S)$

One notices at once that if $\text{add}(D) = \text{add}(E)$, then $\text{wid}(D \times E) \geq \max\{\text{wid}(D), \text{wid}(E)\}$. But unlike $\text{add}$ and $\text{cof}$, the width of finite products cannot be computed easily. In this section we show that there are directed sets $D, E$ such that $\text{add}(D) = \text{add}(E)$ and $\text{wid}(D \times E) > \max\{\text{wid}(D), \text{wid}(E)\}$.

Before that, we will take a look at the case $\text{add}(D) \neq \text{add}(E)$.

**Proposition 4.1** If $\text{add}(D) < \text{add}(E)$, then $\text{wid}(D \times E) = \text{wid}(D)$.

This is proved by the next lemma.

**Lemma 4.2** Let $\kappa := \text{add}(D) < \text{add}(E)$. Then

$$\mathcal{P}_\kappa \lambda \leq D \times E \iff \mathcal{P}_\kappa \lambda \leq D$$

for any cardinal $\lambda \geq \kappa$.

**Proof** ($\Leftarrow$) Let $X \subseteq D \times E$ be a thin subset of size $\lambda$, and let $p : D \times E \to D$ be the projection. Put $Y := p[X]$. Then $Y$ is thin and $|Y| = \lambda$, since for each $d \in Y$, $|p^{-1}[Y_{\leq d}]| < \kappa$.

($\Rightarrow$) Trivial, using transitivity of $\leq$. $\square$

Now we turn to our main results on cofinal types.

**Definition 4.3** Let $\kappa, \lambda$ be both regular with $\kappa < \lambda$. We define the following directed set, where the ordering is given by inclusion. For $S \subseteq E^\lambda_\kappa = \{\alpha \in \lambda \mid \text{cf}\alpha = \kappa\}$,

$$D_\kappa(S) \overset{\text{def}}{=} \{x \subseteq S \mid |x| \leq \kappa \text{ and } \forall y \subseteq x \ [\text{otp}y = \kappa \to \sup y \in x]\}.$$ 

Here, otp denotes the order type of a set of ordinals.

Todorčević [16] defined and studied these directed sets for $\kappa = \omega$. Note that by letting $S := \{\alpha \in E^2_\omega \mid \alpha \text{ is not a limit point of elements of } E^\lambda_\kappa\}$, we have $D_\kappa(S) = \mathcal{P}_\kappa S \cong \mathcal{P}_\kappa \lambda$.

The following statements mimic Lemmas 1, 2, 3 and Theorems 4, 6 in [16], but because of the assumption on cardinal arithmetic, they are not full generalizations.
Lemma 4.4 Let \( \omega \leq \kappa < \lambda \), where \( \kappa \) is regular and \( \lambda \) is strongly \( \kappa^+ \)-inaccessible, and let \( S, S' \subseteq E^\lambda_\kappa \) with \( S \) unbounded in \( \lambda \). Then

\[
D_\kappa(S) \leq D_\kappa(S') \quad \text{implies} \quad S' \setminus S \text{ is nonstationary in } \lambda.
\]

Proof Let \( f : D_\kappa(S) \to D_\kappa(S') \) be a Tukey function. Without loss of generality, \( f \) depends only on its values for singletons, i.e. \( f(x) = \bigcup_{\alpha \in x} f(\{\alpha\}) \) for all nonempty \( x \in D_\kappa(S) \). By the \( \Delta \)-system lemma we obtain an \( A \subseteq S \) of size \( \lambda \) and a \( d \subseteq S' \) such that

\[
\forall \alpha, \beta \in A \ [ \alpha \neq \beta \to f(\{\alpha\}) \cap f(\{\beta\}) = d],
\]

\[
\forall \alpha \in A \ [ \min(\{\alpha\} \cap d) > \sup d],
\]

and

\[
\forall \alpha, \beta \in A \ [ \alpha < \beta \to \sup(f(\{\alpha\}) \setminus d) < \min(f(\{\beta\}) \setminus d)].
\]

Next, put

\[
C_0 = \{ \alpha \in \lambda \mid \text{there exists a strictly increasing sequence } \langle \alpha_\xi | \xi < \kappa \rangle \text{ such that } \\
\alpha = \sup\{\alpha_\xi | \xi < \kappa\} = \sup_{\xi < \kappa} f(\{\alpha_\xi\}) \}
\]

and let \( C \) be the topological closure of \( C_0 \) in \( \lambda \) (with respect to the order topology). \( C_0 \) is closed for \( \kappa \)-sequences and also unbounded in \( \lambda \), and thus \( C \) becomes a club. For our aim, we demonstrate that \( C \cap (S' \setminus S) = \emptyset \). Suppose there were a \( \gamma \in C \cap (S' \setminus S) \). Then \( \gamma \in C_0 \), so fix a sequence \( \langle \alpha_\xi | \xi < \kappa \rangle \) witnessing it. But \( \gamma \in S' \setminus S \) implies that \( \{\alpha_\xi | \xi < \kappa\} \) is unbounded in \( D_\kappa(S) \) and that \( \{\gamma\} \cup \bigcup_{\xi < \kappa} f(\{\alpha_\xi\}) \) is an upper bound of \( \{f(\alpha_\xi) | \xi < \kappa\} \) in \( D_\kappa(S') \). This contradicts the assumption that \( f \) is Tukey.

\]
Consider

\[ f : \mathcal{P}_\kappa \lambda \rightarrow D_\kappa(S) \times D_\kappa(S') \]

\[ x \mapsto \langle \{ \alpha_\xi | \xi \in x \}, \{ \beta_\xi | \xi \in x \} \rangle. \]

We show that this function is Tukey. First note that \( X \subseteq \mathcal{P}_\kappa \lambda \) is unbounded iff \( | \cup X | \geq \kappa \). If \( X \) is such, then \( f[X] = \{ \{ \alpha_\xi | \xi \in x \}, \{ \beta_\xi | \xi \in x \} \} | x \in X \) is also unbounded, since there exists a \( \gamma \in C \) which is a limit of two strictly increasing \( \kappa \)-sequences consisting of \( \alpha_\xi (\xi \in \cup X) \) and \( \beta_\xi (\xi \in \cup X) \) respectively.

\[ \square \]

**Corollary 4.7** ([14]) Under the same notations and assumptions as above,

\[ D_\kappa(S) \geq \mathcal{P}_\kappa \lambda \quad \text{iff} \quad S \text{ is nonstationary in } \lambda. \]

**Proof** Just take \( S = S' \) in Lemma 4.6.

\[ \square \]

**Theorem 4.8** ([14]) Let \( \kappa, \lambda \) be infinite regular cardinals with \( \kappa^+ < \lambda \) and \( \lambda \) strongly \( \kappa^+ \)-inaccessible. Then there exist directed sets \( D_1 \) and \( D_2 \) such that

\[ D_i \not\geq \mathcal{P}_\kappa \lambda \quad \text{for } i = 1, 2 \]

but

\[ D_1 \times D_2 \equiv \mathcal{P}_\kappa \lambda. \]

**Proof** To prove the Theorem, let \( A \) be any unbounded nonstationary subset of \( E_\kappa^\lambda \). Split \( E_\kappa^\lambda \setminus A \) into two disjoint stationary sets \( S'_1 \) and \( S'_2 \). Then apply Lemma 4.6 to \( D_\kappa(S'_1 \cup A) \times D_\kappa(S'_2 \cup A) \). That \( D_i \leq \mathcal{P}_\kappa \lambda \) \( (i = 1, 2) \) is clear from Lemma 3.2.

\[ \square \]

We will call such a pair \( D_1, D_2 \) of directed sets a **Tukey decomposition** of \( \mathcal{P}_\kappa \lambda \).

**Remark 4.9** We note that, in view of Lemma 4.2, the above \( D_1 \) and \( D_2 \) must satisfy \( \text{add}(D_1) = \text{add}(D_2) \). Besides, \( D_1 \) and \( D_2 \) must have different cofinal types, because \( D \times D \equiv D \) for any directed set \( D \). (This follows from Proposition 2.5, or from the fact that the diagonal \( \{ \{ d, d \} | d \in D \} \) is cofinal in \( D \times D \).)

## 5 The tree property for directed sets

**Definition 5.1** (\( \kappa \)-**tree**) ([8]) Let \( D \) denote a directed set. A triple \( \langle T, \leq_T, s \rangle \) is said to be a **\( \kappa \)-tree** on \( D \) if the following holds.

1) \( \langle T, \leq_T \rangle \) is a partially ordered set.
2) \( s : T \rightarrow D \) is an order preserving surjection.
3) For all \( t \in T \), \( s[T_{\leq_t}] \cong D_{\leq s(t)} \) (order isomorphism).
4) For all $d \in D$, $|s^{-1}\{d\}| < \kappa$. We call $s^{-1}\{d\}$ the level $d$ of $T$.

Note that under conditions 1)2)4), condition 3) is equivalent to 3'):

3') (downwards uniqueness principle) $\forall t \in T \forall d' \leq_D s(t) \exists! t' \leq_T t [s(t') = d']$.

We write $t \downarrow d$ for this unique $t'$.

If a $\kappa$-tree $(T, \leq_T, s)$ satisfies in addition

5) (upwards access principle) $\forall t \in T \forall d' \geq_D s(t) \exists_T t \geq_T t [s(t') = d']$,

then it is called a $\kappa$-arbor on $D$.

If $D$ is an infinite regular cardinal $\kappa$, a ‘$\kappa$-tree on $\kappa$’ coincides with the classical ‘$\kappa$-tree’. Moreover, an ‘arbor’ is a generalization of a ‘well pruned tree’.

**Definition 5.2 (tree property) ([8])** Let $(D, \leq_D)$ be a directed set and $(T, \leq_T, s)$ a $\kappa$-tree on $D$. $f: D \rightarrow T$ is said to be a faithful embedding if $f$ is an order embedding and satisfies $s \circ f = \text{id}_D$. If for each $\kappa$-tree $T$ on $D$ there is a faithful embedding from $D$ to $T$, we say that $D$ has the $\kappa$-tree property. If $D$ has the add($D$)-tree property, we say simply $D$ has the tree property.

We note that in [8] the tree property in our definition is called ‘weakly ramifiable’, and a $\kappa$-arbor is called $\kappa$-arborescence.

Classically, $\kappa$ has the tree property (as a cardinal) if $\kappa$ carries no Aronszajn tree, which is, in our words, a $\kappa$-tree on $\kappa$ into which there is no faithful embedding.

**Proposition 5.3 ([8])** Let $D$ be a directed set and let $\kappa = \text{add}(D)$. $D$ has the tree property iff for any $\kappa$-arbor on $D$ there is a faithful embedding into it.

In [8], Esser and Hinno posed the question whether the tree property for directed sets with the same additivity is preserved under products. In fact, for the case $\text{add}(D) \neq \text{add}(E)$, a positive result was given.

**Proposition 5.4 ([8])** Let $D, E$ be directed sets and $\text{add}(D) < \text{add}(E)$. If $D$ has the tree property, then $D \times E$ also has the tree property.

**Proof** Put $\kappa := \text{add}(D \times E) = \text{add}(D)$. Let $(T, \leq_T, s)$ be an arbitrary $\kappa$-tree on $D \times E$. We have to find a faithful embedding $f: D \times E \rightarrow T$.

First, for each $d \in D$, $T_d := s^{-1}[\{d\} \times E]$ is a $\kappa$-tree on $\{d\} \times E (\equiv E)$. Now we have $\kappa < \text{add}(E)$ and hence there exists a faithful embedding into $T_d$, and moreover the number of faithful embeddings is less than $\kappa$ (see [8]). Let $F_d$ be the set of all faithful embeddings from $\{d\} \times E$ to $T_d$, and let $\overline{g}: D \times E \rightarrow \bigcup_{d \leq_D d} T_d$ denote the faithful embedding which is generated by $g \in F_d$. Define

$$T_* := \bigcup_{d \in D} \{g \mid g \in F_d\},$$

$$\overline{g} \leq_* \overline{g'} \iff g \subseteq g',$$

$$s_*^{-1}\{d\} := \{g \mid g \in F_d\},$$

so that $(T_*, \leq_*, s_*)$ becomes a $\kappa$-tree on $D$. Since we are assuming that $D$ has the tree property, we get a faithful embedding $f_*: D \rightarrow T_*$. Define $f(d,e)$ to be $(f_*(d))(e)$, and this completes the proof.
So we may concentrate on the case \( \text{add}(D) = \text{add}(E) \).

The following proposition gives the connection between our problem and the Tukey ordering. It is implicit in [10] but we give a direct proof. This has the advantage that the related statements in [10] can now be obtained as corollaries.

**Proposition 5.5** If \( E \) has the tree property, \( D \leq E \) in the Tukey ordering and \( \text{add}(D) = \text{add}(E) \), then \( D \) also has the tree property.

**Proof** Let \( \kappa := \text{add}(D) = \text{add}(E) \), and let \( (T, \leq_T, s) \) be an arbitrary \( \kappa \)-arbor on \( D \). We have to construct a corresponding \( \kappa \)-arbor on \( E \).

Fix a pair of functions \( g: D \to E \) and \( f: E \to D \) such that

\[
\forall d \in D \forall e \in E \ [g(d) \leq_E e \to d \leq_D f(e)]
\]

(see Proposition 2.2). Define a \( \kappa \)-arbor \( (T', \leq', s') \) on \( E \) so that

\[
s'^{-1}\{e\} = \{ \langle e, T_{\leq t} \cap s^{-1}g^{-1}[E_{\leq e}] \rangle \mid t \in s^{-1}\{f(e)\} \} \quad \text{for } e \in E,
\]

and

\[
\langle e_1, A \rangle \leq' \langle e_2, B \rangle \iff e_1 \leq_E e_2 \land A \subseteq B \quad \text{for } \langle e_1, A \rangle, \langle e_2, B \rangle \in T'.
\]

We check that \( T' = \bigcup_{e \in E} s'^{-1}\{e\} \) is actually a \( \kappa \)-arbor on \( E \). It is straightforward that \( \leq' \) is transitive, that \( s' \) is order preserving, and that each level has size less than \( \kappa \). To prove the upwards access property, fix \( e_0, e \in E \) with \( e_0 \leq_E e \) and \( t_0 \in s^{-1}\{f(e_0)\} \) arbitrarily. Take some upper bound of \( \{f(e_0), f(e)\} \) in \( D \), say \( d^* \). By the upwards access property of \( T \), there is some \( t^* \in s^{-1}\{d^*\} \) with \( t^* \geq_T t_0 \). Then by the downwards uniqueness property of \( T \),

\[
\langle e_0, T_{\leq t_0} \cap s^{-1}g^{-1}[E_{\leq e_0}] \rangle \leq' \langle e, T_{\leq t^*} \cap s^{-1}g^{-1}[E_{\leq e}] \rangle \in s'^{-1}\{e\}.
\]

To prove downwards uniqueness, fix \( e_0 \leq_E e \) and \( t \in s^{-1}\{f(e)\} \) arbitrarily. Take an upper bound \( d^* \) of \( \{f(e_0), f(e)\} \) in \( D \). By the upwards access property of \( T \), we have a \( t^* \in s^{-1}\{d^*\} \) with \( t^* \geq_T t \). Put \( t_0 := t^* \downarrow f(e_0) \). Then

\[
\langle e_0, T_{\leq t_0} \cap s^{-1}g^{-1}[E_{\leq e_0}] \rangle = \langle e_0, T_{\leq t^*} \cap s^{-1}g^{-1}[E_{\leq e}] \rangle = \langle e, T_{\leq t} \cap s^{-1}g^{-1}[E_{\leq e}] \rangle \downarrow e_0.
\]

By assumption, there exists a faithful embedding \( \varphi: E \to T' \). From it we can deduce a faithful embedding from \( D \) into \( T \), by choosing the image to be exactly \( \bigcup \{A \mid \exists e \in E \ [\langle e, A \rangle = \varphi(e)] \} \).

Thus the tree property is a property applying to the cofinal type of a directed set.

**Remark 5.6** We note that this proposition does not hold if \( \text{add}(D) \neq \text{add}(E) \). \( D = \omega_1 \) and \( E = \mathcal{P}_\omega(\omega_1) \) is a counterexample.

**Corollary 5.7** ([8]) If \( D \) has the tree property, then \( \text{add}(D) \) has the tree property in the classical sense.
By Hechler’s theorem (see [4]), the eventual dominance order on the reals $\langle^\omega\omega, \leq^*\rangle$ can be consistently cofinally similar with any directed set which has $\text{add}(D) \geq \aleph_1$. Hence to obtain the following result we apply Hechler’s theorem by taking $(D, \leq_D) = (\kappa, \in)$. For (1), we let $\kappa = \omega_1$, and for (2), we let $\kappa$ be weakly compact.

**Theorem 5.8**

1. ZFC and ZFC + $\langle^\omega\omega, \leq^*\rangle$ does not have the tree property” are equiconsistent.
2. ZFC + $\exists$ weakly compact and ZFC + $\langle^\omega\omega, \leq^*\rangle$ has the tree property” are equiconsistent.

Since Hechler’s theorem holds with $\langle^\omega\omega, \leq^*\rangle$ replaced by $\mathcal{M}$ [2] or $\mathcal{N}$ [5], we have analogous results for $\mathcal{M}$ and $\mathcal{N}$.

6 Mild ineffability

Mild ineffability was introduced by DiPrisco and Zwicker, and studied by Carr [6] in detail. It can be viewed as a kind of tree property for $\mathcal{P}_{\kappa}\lambda$. We give the definition and an overview on the basic facts. In all the statements of section 6 and 7, the possibility of taking $\kappa = \omega$ is not excluded.

**Definition 6.1 (mild ineffability)** ([6]) $\mathcal{P}_{\kappa}\lambda$ is said to be mildly ineffable (or $\kappa$ is mildly $\lambda$-ineffable) iff for any given $\langle A_x \mid x \in \mathcal{P}_{\kappa}\lambda \rangle$ with $A_x \subseteq x$ for all $x$, there exists some $A \subseteq \lambda$ such that

$$\forall x \in \mathcal{P}_{\kappa}\lambda \exists y \in \mathcal{P}_{\kappa}\lambda [x \subseteq y \land A_y \cap x = A \cap x].$$

**Proposition 6.2** ([6]) For a cardinal $\kappa$, the following are equivalent:

(a) $\kappa$ is mildly $\kappa$-ineffable.
(b) $\kappa$ is strongly inaccessible and has the tree property.
(c) $\kappa$ is weakly compact.

**Proposition 6.3** ([6]) If $\kappa$ is mildly $\lambda$-ineffable and $\kappa \leq \lambda' \leq \lambda$, then $\kappa$ is mildly $\lambda'$-ineffable.

The relation between mild ineffability and strong compactness for pairs of cardinals $\kappa, \lambda$ is as follows.

**Proposition 6.4** ([6]) For cardinals $\kappa \leq \lambda$,

1. If $\kappa$ is mildly $2^{<\kappa}$-ineffable then $\kappa$ is $\lambda$-strongly compact.
2. If $\kappa$ is $\lambda$-strongly compact then $\kappa$ is mildly $\lambda$-ineffable.
Proof (1) Let \( \mathcal{P}(\mathcal{P}_\kappa \lambda) = \{X_\alpha \mid \alpha < 2^{\lambda^{<\kappa}}\} \). For each \( x \in \mathcal{P}_\kappa (2^{\lambda^{<\kappa}}) \), we put
\[
A_x := \{x \in x \cap \lambda \in X_\alpha\}.
\]

By the mild \( 2^{\lambda^{<\kappa}} \)-ineffability of \( \kappa \), there exists an \( A \subseteq 2^{\lambda^{<\kappa}} \) such that
\[
\forall x \in \mathcal{P}_\kappa (2^{\lambda^{<\kappa}}) \exists y \in \mathcal{P}_\kappa (2^{\lambda^{<\kappa}}) \ [x \subseteq y \land A_y \cap x = A \cap x].
\]
If we let \( \mathcal{U} := \{X_\alpha \mid \alpha \in A\} \), then one can check (by applying the above formula to suitable \( x \)'s) that \( \mathcal{U} \) is a \( \kappa \)-complete fine ultrafilter on \( \mathcal{P}_\kappa \lambda \).

(2) Assume that there exists a \( \kappa \)-complete fine ultrafilter \( \mathcal{U} \) on \( \mathcal{P}_\kappa \lambda \). We are given \( \langle A_x \mid x \in \mathcal{P}_\kappa \lambda \rangle \) such that \( A_x \subseteq x \) for all \( x \). For each \( \alpha < \lambda \), put \( X_\alpha := \{x \in \mathcal{P}_\kappa \lambda \mid \alpha \in A_x\} \). Let \( A := \{\alpha < \lambda \mid X_\alpha \in \mathcal{U}\} \). We check that this is the required set. Let \( x \in \mathcal{P}_\kappa \lambda \) be arbitrary. Then \( X_\alpha \in \mathcal{U} \) for \( \alpha \in x \cap A \), and \( \mathcal{P}_\kappa \lambda \setminus X_\alpha \in \mathcal{U} \) for \( \alpha \in x \setminus A \). Put
\[
X := \bigcap\{X_\alpha \mid \alpha \in x \cap A\} \cap \bigcap\{\mathcal{P}_\kappa \lambda \setminus X_\alpha \mid \alpha \in x \setminus A\} \in \mathcal{U}.
\]

\( X \) is cofinal in \( \mathcal{P}_\kappa \lambda \) since \( \mathcal{U} \) is fine, so we can pick \( y \in X \) with \( y \supseteq x \), and thus \( A_y \cap x = A \cap x \).

Corollary 6.5 (GCH) Assume \( \kappa \) is not strongly compact. Let \( \lambda \) be the least cardinal such that \( \kappa \) is not \( \lambda \)-strongly compact, and let \( \mu \) be the least cardinal such that \( \kappa \) is not mildly \( \mu \)-ineffable. Assume that \( \lambda \) is regular. Then \( \mu = \lambda \) or \( \mu = \lambda^+ \).

Corollary 6.6 ([6]) For a cardinal \( \kappa \), \( \kappa \) is mildly \( \lambda \)-ineffable for all \( \lambda \geq \kappa \) iff \( \kappa \) is strongly compact.

7 Characterization of the tree property by mild ineffability

The next theorem is stated in [9, Theorem 3.3] with a different formulation. Using the cardinal width, we can state the theorem in a more convenient way.

Theorem 7.1 ([14], cf [9]) Let \( D \) be a directed set and let \( \kappa := \text{add}(D) \) be strongly inaccessible. The following are equivalent:

(a) \( D \) has the tree property.
(b) For all \( \lambda < \text{wid}(D) \), \( \mathcal{P}_\kappa \lambda \) has the tree property.
(c) For all \( \lambda < \text{wid}(D) \), \( \mathcal{P}_\kappa \lambda \) is mildly ineffable.

The proof we give here is a combination of the proofs in [14] and [7]. It enabled a good deal of simplification.
Definition 7.2 ([7]) Let \( \langle T, \leq_T, s \rangle \) be an arbor on a directed set \( D \). We define an equivalence relation on \( D \). For \( d_1, d_2 \in D \),

\[
d_1 \sim d_2 \iff \\
\forall d' \in D \ [d' \geq d_1, d_2 \rightarrow \forall t_1 \in s^{-1}\{d_1\} \exists! t_2 \in s^{-1}\{d_2\} \forall u \in s^{-1}\{d'\} \ [t_1 \leq_T u \iff t_2 \leq_T u]].
\]

In the above formula, we say that the \( t_1 \in s^{-1}\{d_1\} \) and the corresponding \( t_2 \in s^{-1}\{d_2\} \) are linked. Equivalent levels give the same partial information on how to take the faithful embedding. Notice that \( d_1 \sim d_2 \) does not imply that they are comparable.

Lemma 7.3 For the relation defined above,

\[
d_1 \sim d_2 \iff \\
\exists d' \in D \ [d' \geq d_1, d_2 \wedge \forall t_1 \in s^{-1}\{d_1\} \exists! t_2 \in s^{-1}\{d_2\} \forall u \in s^{-1}\{d'\} \ [t_1 \leq_T u \iff t_2 \leq_T u]].
\]

Thus \( \sim \) is in fact an equivalence relation on \( D \).

Proof (\( \Rightarrow \)) Trivial, since \( D \) is directed.

(\( \Leftarrow \)) Use upwards access and downwards uniqueness.

\[\square\]

Lemma 7.4 Assume that \( \kappa := \text{add}(D) \) is strongly inaccessible, and let \( \langle T, \leq_T, s \rangle \) be a \( \kappa \)-arbor on \( D \). If \( F \subseteq D \) is a set of representatives with respect to \( \sim \), then \( F \) is thin.

Proof Fix an arbitrary \( d_0 \in D \). For each element \( d \in D_{\leq d_0} \),

\[
P_d := \{ \{ u \in s^{-1}\{d_0\} \mid u \geq_T t \} \mid t \in s^{-1}\{d\} \}
\]

provides a partition of \( s^{-1}\{d_0\} \). By Lemma 7.3, we see that \( P_{d_1} = P_{d_2} \) if \( d_1 \sim d_2 \) for \( d_1, d_2 \in D_{\leq d_0} \). Since \( \kappa \) is strongly inaccessible, the number of partitions of \( s^{-1}\{d_0\} \) is less than \( \kappa \).

\[\square\]

Proof of 7.1 (a) \( \Rightarrow \) (b) Let \( \text{add}(D) \leq \lambda < \text{wid}(D) \). Then \( \mathcal{P}_\lambda \leq D \), so by Proposition 5.5 \( \mathcal{P}_\lambda \) has the tree property.

(b) \( \Rightarrow \) (c) It suffices to show, for an arbitrary \( \lambda \), that the tree property for \( \mathcal{P}_\lambda \) implies its mild ineffability. Assume that \( \mathcal{P}_\lambda \) has the tree property. Suppose we are given a family \( \langle A_x \mid x \in \mathcal{P}_\lambda \rangle \) such that \( A_x \in 2^\lambda \) for \( x \in \mathcal{P}_\lambda \). Then

\[
\langle \{ A_y \mid y \supseteq x \} \mid x \in \mathcal{P}_\lambda \rangle
\]

is a \( \kappa \)-tree on \( \mathcal{P}_\lambda \) since \( \kappa \) is strongly inaccessible. Therefore we have a faithful embedding, which is the same as an \( A \in 2^{\lambda \cdot 2} \) such that

\[
\forall x \in \mathcal{P}_\lambda \exists y \in \mathcal{P}_\lambda \ [x \subseteq y \wedge A_y \models x = A \models x].
\]
(c) $\Rightarrow$ (a) Let $(T, \leq_T, s)$ be a $\kappa$-arbor on $D$. Our goal is to produce a faithful embedding $f : D \rightarrow T$. Fix a set of representatives $F \subseteq D$ with respect to the equivalence $\sim$ defined above.

Put $\lambda := |F|$. Then $T^* := s^{-1}[F]$ also has size $\lambda$. As we have $\lambda < \text{wid}(D)$, the assumption (c) says $\kappa$ is mildly $\lambda$-ineffable.

We define a family $\langle A_x | x \in \mathcal{P}_\kappa T^* \rangle$ to which we will apply the mild ineffability. For each $x \in \mathcal{P}_\kappa T^*$, pick an upper bound $d \in D$ of $s[x]$, and fix $t \in s^{-1}\{d\}$. For $v \in x$ we put $A_x(v) = 1$ if $v \leq_T t$, and $A_x(v) = 0$ otherwise. Then we get an $A \in T^*2$ such that

$$\forall x \in \mathcal{P}_\kappa T^* \exists y \in \mathcal{P}_\kappa T^* [x \subseteq y \land A_y \lceil x = A \lceil x].$$

It remains to derive the faithful embedding $f$ from $A$. For each $d \in F$, let $v_d$ be the unique $v \in s^{-1}\{d\}$ such that $A(v) = 1$. Then $d \mapsto v_d$ is an embedding from $F$ to $T^*$. To extend this map to all of $D$, let $d \in D$ be arbitrary and let $d \sim d^* \in F$ be the corresponding representative. Now $v_{d^*}$ is defined, and we can put $f(d)$ to be the unique $u \in s^{-1}\{d\}$ such that $u$ and $v_{d^*}$ are linked. One can verify that $f : D \rightarrow T$ is a faithful embedding.

\begin{corollary}
Corollary 7.5 Let $\kappa$ be strongly inaccessible and $\lambda \geq \kappa$. Then

$\mathcal{P}_\kappa \lambda$ has the tree property iff $\kappa$ is mildly $\lambda^{<\kappa}$-ineffable.
\end{corollary}

8 Application of the Tukey decomposition

\begin{theorem}
Theorem 8.1 Assume that $\kappa$ is weakly compact but not strongly compact, and that $\lambda > \kappa^+$ is the least cardinal such that $\kappa$ is not mildly $\lambda$-ineffable. Assume further that $\lambda$ is strongly $\kappa^+$-inaccessible. Then there exist directed sets $D_1$ and $D_2$ with $\text{add}(D_1) = \text{add}(D_2) = \kappa$ such that

$D_1$ and $D_2$ have the tree property

but

$D_1 \times D_2$ does not have the tree property.

\end{theorem}

\begin{proof}
By the Theorem 4.8, we have directed sets $D_1$ and $D_2$ such that $D_i \not\subseteq \mathcal{P}_\kappa \lambda$ for $i = 1, 2$ but $D_1 \times D_2 \equiv \mathcal{P}_\kappa \lambda$. Recalling how $D_1$ and $D_2$ were defined (or by Remark 4.9), we see that $\text{add}(D_1) = \text{add}(D_2) = \kappa$. By Theorem 7.1, $D_1$ and $D_2$ have the tree property but $D_1 \times D_2$ does not have the tree property.

At last, we discuss the consistency of the assumption in the above theorem.

We quote the following theorem.

\begin{theorem}
([13]) If $\lambda$ is regular and $\kappa$ is mildly $\lambda$-ineffable, then for each regular $\eta < \kappa$, any stationary set $S \subseteq E_\eta^\lambda$ is reflecting.
\end{theorem}
Here we call $S \subseteq E_{\eta}^\lambda$ reflecting iff there is a limit ordinal $\gamma < \lambda$ such that $S \cap \gamma$ is stationary in $\gamma$. Otherwise $S$ is called nonreflecting.

Assuming a strongly compact cardinal $\kappa$, we perform a forcing which destroys the mild $\lambda^+$-ineffability of $\kappa$ and which at the same time preserves the mild $\lambda$-ineffability. By Theorem 8.2 the standard forcing which adds a nonreflecting stationary subset (see [3, Definition 4.14]) serves our purpose. To be precise, define $P$ to be the forcing which consists of conditions $p \in <\lambda^+ 2$ (i.e. $p$ is a characteristic function for a subset of an ordinal $< \lambda^+$) such that if we let $S_p := p^{-1}\{1\}$, then $S_p \subseteq E_{\omega}^{\lambda^+}$ and for all limit ordinals $\gamma < \lambda^+$, $S_p \cap \gamma$ is nonstationary in $\gamma$. For $p, q \in P$, $p$ extends $q$ iff $p \supseteq q$. It is known [3] that $P$ preserves cardinals, cofinalities, and GCH, and that $P$ is $\lambda$-strategically closed.

This completes the proof.

**Theorem 8.3** If we assume the consistency of ZFC+$\exists$ strongly compact, then ZFC+$"$the tree property for directed sets is not always preserved under products"$"$ is consistent.

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**References**


