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Kyoto University
Finite support iteration of c.c.c forcing notions and Parametrized ◯-principles

南 裕明 (Hiroaki Minami)
神戸大学自然科学研究科
(Graduate School of Science and Technology, Kobe University)

概 要
We present several models which satisfy some ◯-like principles by using the $\omega_2$-stage finite support iteration of Suslin forcing notions.

1 Introduction

In [10] Jensen showed $V = L$ implies Suslin's Hypothesis doesn't hold. To prove this he introduced the ◯-principle:

◊ There exists a sequence $\langle A_\alpha \subset \alpha : \alpha < \omega_1 \rangle$ such that for all $X \subset \omega_1$ the set 
\{ $\alpha < \omega_1 : X \cap \alpha = A_\alpha$ \} is stationary.

In [9] Hrušák gave a partial solution to a question of J. Roitman who asked whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$ and answered a question of Brendle who asked whether $\mathfrak{a} = \omega_1$ in any model obtained by adding a single Laver real. To prove those he introduced the ◯-like principle ◯$_\mathfrak{a}$:

◊$_\mathfrak{a}$ There exists a sequence $\langle g_\alpha : \omega \leq \alpha < \omega_1 \rangle$ such that $g_\alpha$ is a function from $\alpha$ to $\omega$ and for every $f : \omega_1 \rightarrow \omega$ there is an $\alpha \geq \omega$ with $f \restriction \alpha \leq^* g_\alpha$.

In [16] Moore, Hrušák, and Džamonja provided a broad framework of "parametrized ◯-principles" and they presented the following methods to construct parametrized ◯-principles:

Theorem 1.1. Let $\mathbb{C}(\omega_1)$ and $\mathbb{B}(\omega_1)$ be the Cohen and random algebras corresponding to the product space $2^{\omega_1}$ with its usual topological and measure theoretic structures. The orders $\mathbb{C}(\omega_1)$ and $\mathbb{B}(\omega_1)$ force ◯(non($\mathcal{M}$)) and ◯(non($\mathcal{N}$)) respectively.
Theorem 1.2. Suppose that $\{Q_\alpha : \alpha < \omega_2\}$ is a sequence of Borel partial orders such that for each $\alpha < \omega_2$ $Q_\alpha$ is equivalent to $\wp(2)^+ \times Q_\alpha$ as a forcing notion and let $\mathcal{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathcal{P}_{\omega_2}$ is proper and $(A, B, E)$ is a Borel invariant then $\mathcal{P}_{\omega_2}$ forces $\langle A, B, E \rangle \leq \omega_1$ iff $\mathcal{P}_{\omega_2}$ forces $\diamondsuit(A, B, E)$.

In [15] by using $\omega_1$-stage finite support iteration of c.c.c forcing notions, several models were presented which satisfy some parametrized $\diamondsuit$-principles while others fail. The purpose of this paper is to provide several models satisfying some parametrized $\diamondsuit$-principles by using $\omega_2$-stage finite support iteration of Suslin forcing notions.

2 Definition and properties of Parametrized Diamonds

In [20] Vojtáš introduced a framework to describe many cardinal invariants.

Definition 2.1. [20][16] The triple $(A, B, E)$ is an invariant if

1. $|A|, |B| \leq |\mathbb{R}|$,
2. $E \subset A \times B$,
3. For each $a \in A$ there exists $b \in B$ such that $(a, b) \in E$ and
4. For each $b \in B$ there exists $a \in A$ such that $(a, b) \notin E$.

We will write $aEb$ instead of $(a, b) \in E$. If $A$ and $B$ are Borel subsets of some Polish spaces and $E$ is a Borel subset of their product, we call the triple $(A, B, E)$ Borel invariant.

Borel invariants were introduced in [3]. In the present paper we are interested only in Borel invariants.

Definition 2.2. Suppose $(A, B, E)$ is an invariant. Then its evaluation is defined by

$$\langle A, B, E \rangle = \min\{|X| : X \subset B \text{ and } \forall a \in A \exists b \in X \ (aEb)\}.$$ 

If $A = B$, we will write $(A, E)$ and $\langle A, E \rangle$ instead of $(A, B, E)$ and $\langle A, B, E \rangle$.

Example 2.3. The following Borel invariants $(\mathcal{N}, \not\supset)$, $(\mathcal{N}, \subset)$, $(\mathcal{R}, \mathcal{M}, \epsilon)$, $(\mathcal{M}, \mathcal{R}, \not\supset)$, $\langle \omega^\omega, <^* \rangle$, $\langle \omega^\omega, \not\simeq^* \rangle$ and $\langle [\omega]^\omega, \text{split by} \rangle$ have the evaluations $\text{add}(\mathcal{N})$, $\text{cov}(\mathcal{N})$, $\text{cov}(\mathcal{M})$, $\text{non}(\mathcal{M})$, $\mathfrak{d}$, $\mathfrak{b}$ and $\mathfrak{s}$ respectively.
Definition 2.4. Suppose $A$ is a Borel subset in some Polish space. Then $F: 2^{<\omega_1} \rightarrow A$ is Borel if for every $\alpha < \omega_1$ $F \upharpoonright 2^\alpha$ is a Borel function.

In [7] the principle “weak diamond principle” was introduced by Devlin and Shelah. This was the starting point for the parametrized diamond principles introduced by Moore, Hrušák and Džamonja [16].

Definition 2.5. [16](Parametrized diamond principle)
Suppose $(A, B, E)$ is a Borel invariant. Then $\diamondsuit(A, B, E)$ is the following statement:

$\diamondsuit(A, B, E)$ For all Borel $F: 2^{<\omega_1} \rightarrow A$ there exists $g: \omega_1 \rightarrow B$ such that for every $f: \omega_1 \rightarrow 2$ the set $\{\alpha \in \omega_1 : F(f \upharpoonright \alpha) \in Eg(\alpha)\}$ is stationary.

The witness $g$ for a given $F$ in this statement will be called $\diamondsuit(A, B, E)$-sequence for $F$.

$\diamondsuit(A, B, E)$ and $\diamondsuit$ are related as follows:

Proposition 2.6. [16] Let $(A, B, E)$ be a Borel invariant. $\diamondsuit$ implies $\diamondsuit(A, B, E)$.

$\diamondsuit(A, B, E)$ and $\langle A, B, E \rangle$ are related as follows:

Proposition 2.7. [16] Suppose $(A, B, E)$ is a Borel invariant and $\diamondsuit(A, B, E)$ holds. Then $\langle A, B, E \rangle \leq \omega_1$ holds.

If two Borel invariants $(A_1, B_1, E_1), (A_2, B_2, E_2)$ are comparable in the Borel Tukey order, then $\diamondsuit(A_1, B_1, E_1)$ and $\diamondsuit(A_2, B_2, E_2)$ are related as follows:

Definition 2.8. (Borel Tukey ordering [3]) Given a pair of Borel invariants $(A_1, B_1, E_1)$ and $(A_2, B_2, E_2)$, we say that $(A_1, B_1, E_1) \leq_{B}^{T} (A_2, B_2, E_2)$ if there exist Borel maps $\phi: A_1 \rightarrow A_2$ and $\psi: B_2 \rightarrow B_1$ such that $(\phi(a), b) \in E_2$ implies $(a, \psi(b)) \in E_1$.

Proposition 2.9. [16] Let $(A_1, B_1, E_1)$ and $(A_2, B_2, E_2)$ be Borel invariants. Suppose $(A_1, B_1, E_1) \leq_{B}^{T} (A_2, B_2, E_2)$ and $\diamondsuit(A_2, B_2, E_2)$ holds. Then $\diamondsuit(A_1, B_1, E_1)$ holds.

Concerning $\leq_{B}^{T}$, we know the following diagram holds.

(Cichoń’s diagram)

$$(\mathbb{R}, \mathcal{N}, \in) \leftarrow (\mathcal{M}, \mathbb{R}, \not\supset) \leftarrow (\mathcal{M}, \subset) \leftarrow (\mathcal{N}, \subset)$$

$$(\omega^{\omega}, \not\supset) \leftarrow (\omega^{\omega}, \leq_{*})$$

$$(\mathcal{N}, \not\supset) \leftarrow (\mathcal{M}, \not\supset) \leftarrow (\mathbb{R}, \mathcal{M}, \in) \leftarrow (\mathcal{N}, \mathbb{R}, \not\supset)$$
(The direction of the arrow is from larger to smaller in the Borel Tukey order. Hence the following holds:

\[ \Diamond (\text{cov}(\mathcal{N})) \leftarrow \Diamond (\text{non}(\mathcal{M})) \leftarrow \Diamond (\text{cof}(\mathcal{M})) \leftarrow \Diamond (\text{cof}(\mathcal{N})) \]

\[ \Diamond (\text{add}(\mathcal{N})) \leftarrow \Diamond (\text{add}(\mathcal{M})) \leftarrow \Diamond (\text{cof}(\mathcal{M})) \leftarrow \Diamond (\text{cov}(\mathcal{N})) \]

(The direction of the arrow is the direction of the implication.)

We call this diagram "Cichon's diagram for parametrized diamonds".

**Note** When we deal with Borel invariants in Cichon's diagram, we will use the standard notation for their evaluations to denote the Borel invariants themselves (e.g., we will use $\Diamond (\text{add}(\mathcal{N}))$ to denote $\Diamond (\mathcal{N}, \not\supset)$).

## 3 Construction of Parametrized Diamonds

By using $\omega_2$-stage finite support iteration of Suslin forcing notions we present several model which satisfies some parametrized $\Diamond$-principles.

### 3.1 Suslin forcing

Firstly we will introduce Suslin forcings and their properties.

**Definition 3.1.** [2, p.168] A forcing notion $\mathbb{P} = (\mathbb{P}, \leq_\mathbb{P})$ has a Suslin definition if $\mathbb{P} \subseteq \omega_\omega$, $\leq_\mathbb{P} \subseteq \omega_\omega \times \omega_\omega$ and $\perp_\mathbb{P} \subseteq \omega_\omega \times \omega_\omega$ are $\Sigma^1_1$.

$\mathbb{P}$ is Suslin if $\mathbb{P}$ is c.c.c and has a Suslin definition.

**Definition 3.2.** [2, p.168] Let $M \models \text{ZFC}^*$. A Suslin forcing $\mathbb{P}$ is in $M$ if all the parameters used in the definition of $\mathbb{P}$, $\leq_\mathbb{P}$ and $\perp_\mathbb{P}$ are in $M$.

For convenience we will interpret Suslin forcing notion in forcing extensions using its code rather than taking the ground model forcing notion.

**Definition 3.3.** Let $\mathbb{A}$ and $\mathbb{B}$ be forcing notions. Then $i : \mathbb{A} \rightarrow \mathbb{B}$ is a complete embedding if

1. $\forall a, a' \in \mathbb{A} (a \leq a' \rightarrow i(a) \leq i(a'))$,
(2) \( \forall a_1, a_2 \in A(a_1 \perp a_2 \leftrightarrow i(a_1) \perp i(a_2)) \),

(3) \( \forall A \subset A(A \text{ is a maximal antichain in } A \rightarrow i[A] \text{ is a maximal antichain in } B) \).

If there is complete embedding from \( A \) to \( B \), then we write \( A \leq B \).

Suslin forcing notion has the following good property:

**Lemma 3.4.** Assume \( A \leq B \) and \( \mathcal{P} \) is a Suslin forcing notion. Then \( A \ast \dot{\mathcal{P}} \leq B \ast \dot{\mathcal{P}} \).

**Proof.** Let \( i : A \rightarrow B \) be a complete embedding. Then define \( \bar{i} : A \ast \dot{\mathcal{P}} \rightarrow A \ast \dot{\mathcal{P}} \) by \( \bar{i}((a, \dot{p})) = (i(a), i_*(\dot{p})) \) where \( i_* \) is the class function from \( A \)-names to \( B \)-names induced by \( i \) (see [12, p. 222]). It is enough to show following claim.

**Claim 3.4.1.** If \( A \subset A \ast \dot{\mathcal{P}} \) is a maximal antichain, then \( i[A] \) is also a maximal antichain in \( B \ast \dot{\mathcal{P}} \).

**Proof of Claim.** Let \( A = \{(a_\alpha, \dot{p}_\alpha) : \alpha < \kappa\} \) be a maximal antichain of \( A \ast \dot{\mathcal{P}} \). Assume there exists \( (b, \dot{p}) \in B \ast \dot{\mathcal{P}} \) such that \( (b, \dot{p}) \) is compatible with all \( \bar{i}((a_\alpha, \dot{p}_\alpha)) \).

Let \( G \) be \( B \)-generic over \( V \) such that \( b \in G \) and let \( H = i^{-1}[G] \). Look at \( \{\bar{p}_\alpha[H] : i(a_\alpha) \in G\} = A' \in V[H] \).

**Subclaim 3.4.1.** \( V[H] \models A' \text{ is maximal antichain of } \mathcal{P} = \dot{\mathcal{P}}[H] \).

antichain: Suppose \( \alpha \neq \beta \) and \( i(a_\alpha), i(a_\beta) \in G \). Since \( (a_\alpha, \dot{p}_\alpha) \perp (a_\beta, \dot{p}_\beta), \dot{p}_\alpha[H] \perp \dot{p}_\beta[H] \).

maximality: Assume to the contrary, there exists \( p \in \mathcal{P} \) such that \( p \perp \dot{p}_\alpha[H] \) for any \( \dot{p}_\alpha[H] \in A' \). Then there exists \( a \in H \) such that

\[
\kappa \ni \forall \alpha < \kappa(a_\alpha \in H \rightarrow p \perp \dot{p}_\alpha).
\]

Hence \( (a, \dot{p}) \perp (a_\alpha, \dot{p}_\alpha) \). This is a contradiction to the maximality of \( A \).

Subclaim \( \Box \)

Since \( V[H] \models "A' \text{ is maximal antichain in } \mathcal{P}" \) and "\( A' \) is maximal antichain of \( \mathcal{P} \)" is a \( \Pi_1(A', \mathcal{P}, \leq, \perp) \)-formula, \( V[G] \models "A' = \{i_*(\dot{p}_\alpha)[G] : i(a_\alpha) \in G\} \text{ is maximal antichain of } \mathcal{P}" \) by \( \Pi_1 \)-absoluteness. But this is a contradiction to the fact \( V[G] \models \dot{p}[G] \perp i_*(\dot{p}_\alpha)[G] \) for \( i(a_\alpha) \in G \).

Claim \( \Box \)

Hence \( A \ast \dot{\mathcal{P}} \leq B \ast \dot{\mathcal{P}} \). 

\( \square \)
Corollary 3.5. Let \( \langle Q_\alpha : \alpha < \kappa \rangle \) be a sequence of Suslin forcing notions. Let \( P_\kappa \) be the finite support iteration of \( \langle P_\alpha, Q_\alpha : \alpha < \kappa \rangle \) where \( \Vdash_{P_\alpha} Q_\alpha = Q_\alpha^{V_{P_\alpha}} \). If \( A \prec B \), then \( A \ast P_\kappa \prec B \ast P_\kappa \).

Proof. We shall show that if \( A \) is a maximal antichain of \( A \ast P_\kappa \), then \( \hat{i}[A] \) is also a maximal antichain of \( B \ast P_\kappa \) where \( i : A \ast P_\kappa \to B \ast P_\kappa \) is induced by the complete embedding \( i : A \to B \). It is enough to prove the following claim.

Claim 3.5.1. Let \( A \subseteq A \ast P_\kappa \). If for each \( p \in A \ast P_\kappa \) there exists \( q \in A \) such that \( q \| p \), then for each \( r \in B \ast P_\kappa \) there exists \( q \in A \) such that \( \hat{i}(q) \| r \).

Proof of Claim. We shall show this by induction on \( \kappa \).

The successor step is as in Lemma 3.4.

Limit step. Let \( \kappa \) be a limit ordinal and for \( \alpha < \kappa \) the induction hypothesis holds.

Let \( A \subseteq A \ast P_\kappa \) such that for each \( p \in A \ast P_\kappa \) there exists \( q \in A \) such that \( p \| q \). Assume to the contrary there exists \( p \in B \ast P_\kappa \) such that \( p \perp \hat{i}(q) \) for any \( q \in A \). Let \( \alpha = \sup \{ \beta < \kappa : \Vdash_{P_\alpha} p(\beta) \neq 1 \} < \kappa \). Since for each \( r \in A \ast P_\kappa \), there exists \( q \in A \) such that \( r \| q \), for each \( r' \in A \ast P_\alpha \), there exists \( q \in A \) such that \( q \| r \). By induction hypothesis there exists \( q \in A \) such that \( p \| q \) and \( \hat{i}(r) \| \alpha \) where \( \hat{i}_\alpha : A \ast P_\alpha \to B \ast P_\alpha \) is induced by \( i \). By \( \hat{i}_\alpha(q \| \alpha) = \hat{i}(q) \| \alpha \), \( p \| q \| i(q) \| \alpha \). So \( p \| \hat{i}(q) \). It is a contradiction.

Claim \( \Box \)

Let \( \langle R_\alpha : \alpha < \kappa \rangle \) be a sequence of Suslin forcing notions where all parameters appear in the ground model. Let \( P_\kappa \) be the finite support iteration of \( \langle P_\alpha, Q_\alpha : \alpha < \kappa \rangle \) where \( \Vdash_{P_\alpha} Q_\alpha = Q_\alpha^{V_{P_\alpha}} \). Let \( I \subseteq \kappa \). Recursively define \( P_\gamma \) by

(i) \( P_\gamma \) is given. Then \( P_{\gamma+1}^I = P_\gamma \ast \hat{Q}_\alpha \) where

\[
\Vdash_{P_\gamma} \hat{Q}_\alpha = \begin{cases} \{ R_\alpha^{V_{P_\gamma}} \} & \alpha \in I \\ \{1\} & \text{otherwise} \end{cases}
\]

(ii) Suppose \( \alpha \) is a limit ordinal and \( P_\alpha^I \) is given for \( \beta < \alpha \). Define \( P_\beta^I \) as the finite support iteration of \( \langle P_\beta^I, Q_\beta^I : \beta < \alpha \rangle \)

Put \( P_I := P_\gamma^I \).

Lemma 3.6. \( P_I \prec P_\kappa \).

Proof. We shall show for \( \alpha \leq \kappa \) \( P_\gamma^I \prec P_\alpha\) by the induction on \( \alpha \leq \kappa \). Successor step. Suppose \( P_\gamma^I \prec P_\alpha \). If \( \alpha \notin I \), it is clear that \( P_{\alpha+1}^I \prec P_\alpha \). If \( \alpha \in I \),
then $\mathbb{P}_{\alpha+1} \triangleleft \mathbb{P}_{\alpha+1}$ is proved as in Lemma 3.4.

Limit step. Let $\alpha$ be a limit ordinal and for $\beta < \alpha$ the induction hypothesis holds. Define $i : \mathbb{P}_{\alpha} \rightarrow \mathbb{P}_{\alpha}$ by $i(p) = i_{\beta}(p)$ if $p \in \mathbb{P}_{\beta}$ for some $\beta < \alpha$ where $i_{\beta} : \mathbb{P}_{\beta} \rightarrow \mathbb{P}_{\beta}$ is the complete embedding. It is enough to prove the following claim.

Claim 3.6.1. Let $A \subset \mathbb{P}_{\alpha}$. If for each $p \in \mathbb{P}_{\alpha}$ there exists $q \in A$ such that $q \| p$, then for each $r \in \mathbb{P}_{\alpha}$ there exists $q \in A$ such that $i(q) \| r$.

Proof of Claim. Let $A \subset \mathbb{P}_{\alpha}$ such that for each $p \in \mathbb{P}_{\alpha}$ there exists $q \in A$ such that $q \| p$. Let $r \in \mathbb{P}_{\alpha}$. Since $\mathbb{P}_{\alpha}$ is the finite support iteration of $\langle \mathbb{P}_{\beta}, Q_{\beta} : \beta < \alpha \rangle$, there is $\beta < \alpha$ such that $r \in \mathbb{P}_{\beta}$. Since for each $p \in \mathbb{P}_{\alpha}$ there exists $q \in A$ such that $q \| p$, for each $p' \in \mathbb{P}_{\beta}$ there exists $q \in A$ such that $q \upharpoonright \beta \| p'$. By induction hypothesis there exists $q \in A$ such that $i_{\beta}(q \upharpoonright \beta) = i(q) \upharpoonright \beta \| r$. So $i(q) \| r$. Hence for each $r \in \mathbb{P}_{\alpha}$ there exists $q \in A$ such that $i(q) \| r$.

Claim \[ \square \]

Lemma \[ \square \]

For $\mathbb{P}_{\kappa}$-name $\dot{x}$ for a real, there is following property.

Lemma 3.7. Let $\mathbb{P}_{\kappa}$ is the $\kappa$-stage finite support iteration of Suslin forcing notions. If $\dot{x}$ is $\mathbb{P}_{\kappa}$-name for a real. Then there exists countable $I \subset \kappa$ such that $\dot{x}$ is $\mathbb{P}_{I}$-name.

3.2 Niceness

In this paper we will force $\diamondsuit(A, B, E)$ for Borel invariants $(A, B, E)$ which satisfy the following properties:

There exist $\langle E_{n} : n \in \omega \rangle$ and $\langle U^{n} : n \in \omega \rangle$ such that

(0) $E_{n}$ is a Borel set for $n \in \omega$,

(1) $E = \bigcap_{n \in \omega} E_{n}$,

(2) $E_{n+1} \subset E_{n}$,

(3) $U^{n} : A \rightarrow \wp(A)$ such that $U^{n}(x)$ is a Borel set

(4) $xE_{n}y$ implies that there exists $m \geq n$ such that $U^{m}(x) \subset \{ z \in A : zE_{n}y \}$.

(5) $U^{m}(x) \subset \{ z \in A : zE_{n}y \}$ is absolute with parameters $x$, $y$, $U^{m}$ and $E_{n}$.
Example

(i) For \( \langle 2^\omega, 2^\omega, \exists n (\star \upharpoonright I_n = \star' \upharpoonright I_n) \rangle \) let \( x E_n y \) if \( \exists m \geq n (x \upharpoonright I_m = y \upharpoonright I_m) \) and \( U^n(x) = [x \upharpoonright I_n] := \{ y \in 2^\omega : y \upharpoonright I_n = x \upharpoonright I_n \} \). Then \( \langle E_n : n \in \omega \rangle \) and \( \langle U^n : n \in \omega \rangle \) satisfy (0)-(5).

(ii) For \( \langle \omega^\omega, \not\exists^* \rangle \) let \( x E_n y \) if \( \exists m \geq n (x(m) < y(n)) \) and \( U^n(x) = \bigcup_{m \leq x(n)} [\langle n, m \rangle] \).

Then \( \langle E_n : n \in \omega \rangle \) and \( \langle U^n : n \in \omega \rangle \) satisfy (0)-(5).

(iii) Let \( \text{LOC} = \{ \phi : \phi : \omega \to [\omega]^\omega \text{ where } |\phi(n)| \leq (n+1)^2 \text{ for } n \in \omega \} \). If \( \phi \in \text{LOC} \), we call \( \phi \) slalom. Then for \( f \in \omega^\omega \) and \( \phi \in \text{LOC} \phi \equiv f \) if \( \forall^\infty n \ (f(n) \in \phi(n)) \). For \( \langle \text{LOC}, \omega^\omega, \not\exists^* \rangle \) let \( \phi E_n f \) if \( \exists m \geq n (f(m) \not\in \phi(m)) \) and \( U^n(\phi) = \bigcup_{s \leq \phi(n)} [\langle n, s \rangle] \). Then \( \langle E_n : n \in \omega \rangle \) and \( \langle U^n : n \in \omega \rangle \) satisfy (0)-(5).

For a Borel invariant \( \langle A, B, E \rangle \) with \( \langle U^n : n \in \omega \rangle \) and \( \langle E_n : n \in \omega \rangle \) which satisfies (0)-(5), we will define the notion \( \langle A, B, E \rangle \)-nice and show that the \( \omega_2 \)-stage finite support iteration of some Suslin forcing notions forces parametrized \( \square \)-principles.

Definition 3.8. Let \( \langle A, B, E \rangle \) be a Borel invariant with \( \langle E_n : n \in \omega \rangle \) and \( \langle U^n : n \in \omega \rangle \) satisfying (0)-(5). Let \( \mathbb{P} \) be a forcing notion and \( \mathcal{Q} \) be a Suslin forcing notion or finite support iteration of Suslin forcing notions.

Then \( \mathcal{Q} \) is \( \langle A, B, E \rangle \)-nice for \( \mathbb{P} \) if for all \( \mathcal{Q} \)-names \( \dot{x} \) for an element of \( A \) for each \( (p, \dot{q}) \in \mathbb{P} \times \mathcal{Q} \) there exists \( x \in A \cap V \) such that for all \( r \leq p \) for all but finitely many \( n \) there exists \( q^t \in \mathcal{Q} \) such that \( (1, q^t) \vdash (r, \dot{q}) \) and \( q^t \models_{\mathcal{Q}} \dot{x} \in U^n(x) \).

There are following examples of niceness.

Proposition 3.9. Suppose \( I \) is countable subset of some ordinal \( \kappa \). Then

(1) \( \mathbb{D}_I \) is \( \langle 2^\omega, N, \in \rangle \)-nice for \( \mathbb{D}_\omega \)

(2) \( \mathbb{E}_I \) is \( \langle \omega^\omega, \not\exists^* \rangle \)-nice for \( \mathbb{E}_\omega \).

(3) \( \mathbb{E}_I \) is \( \langle 2^\omega, N, \in \rangle \)-nice for \( \mathbb{E}_\omega \) and \( \langle \omega^\omega, \not\exists^* \rangle \)-nice for \( \mathbb{E}_\omega \).

(4) \( \langle \mathbb{B} * \mathbb{D} \rangle_I \) is \( \langle \text{LOC}, \omega^\omega, \not\exists^* \rangle \)-nice for \( \langle \mathbb{B} * \mathbb{D} \rangle_{\omega_1} \).

Proof.
We shall show only \( |I| = 1 \). The General case is similar but more complicated.
(1). Let \( \langle I_n : n \in \omega \rangle \) be a partition of \( \omega \) such that \( I_0 = \{0\}, I_1 = \{1,2\}, \ldots, I_{n+1} = \{\max(I_n) + 1, \ldots, \max(I_n) + n + 1\} \). For \( x \in \omega^\omega \) let
\[
A_x = \{ y \in \omega^\omega : \exists \infty n \in \omega (x \upharpoonright I_n = y \upharpoonright I_n) \}.
\]
Then \( A_x \) is null. So if \( \Diamond(\omega^\omega, \omega^\omega, \exists \infty n (*I_n = *' \upharpoonright I_n)) \) holds, then \( \Diamond(\text{cov}(N)) \) holds. So instead of showing that \( D \) is \((\omega^\omega, N, \in), \) nice for \( D_{\omega_1} \), we shall show \( D \) is \((\omega^\omega, \omega^\omega, \exists \infty n (*I_n = *' \upharpoonright I_n)) \)-nice for \( D_{\omega_1} \).

Let \( \dot{x} \) be a \( D \)-name such that \( \Vdash_{D} \dot{x} \in \omega^\omega \). Let \( \langle p, \dot{q} \rangle \in D_{\omega_1} \ast D \). For \( s \in \omega^{<\omega} \) define \( D_s \subset D \) by \( p \in D_s \) if there exists \( f \in \omega^\omega \) such that \( p = \langle s, f \rangle \). Then \( D = \bigcup_{s \in \omega^{<\omega}} D_s \).

Without loss of generality we can assume \( p \Vdash_{D_{\omega_1}} \dot{q} = \langle \dot{s}, \dot{f} \rangle \) for some \( s \in \omega^{<\omega} \). Then define \( x_s \in \omega^\omega \cap V \) so that \( \forall m \in \omega \forall p \in D_s \neg p \Vdash x_s \upharpoonright I_m \neq \dot{x} \upharpoonright I_m \). Let \( r \leq p \) and \( m \in \omega \). Define \( \langle r_n : n \in \omega \rangle, f \in \omega^\omega \cap V \) so that
(\begin{enumerate}
\item \( r_0 \leq r, r_{n+1} \leq r_n \) and
\item \( r_n \) decides \( f(n) \) and \( r_n \Vdash \dot{f}(n) = f(n) \).
\end{enumerate}

Let \( q' \leq \langle s, f \rangle \) such that \( q' \Vdash \dot{x} \upharpoonright \omega^\omega = U^m(x_s) \).

**Claim 3.9.1.** \( \langle 1, q' \rangle \Vdash (r, \langle s, \dot{f} \rangle) \).

**Proof of Claim.** Let \( q' = \langle t, g \rangle \). Then \( r \upharpoonright t \Vdash \dot{f} \upharpoonright t = f \upharpoonright t \). So \( \langle r \upharpoonright t, \langle s, \dot{f} \rangle \rangle \leq \langle r, \langle s, \dot{f} \rangle \rangle \). Hence \( \langle 1, q' \rangle \Vdash (r, \langle s, \dot{f} \rangle) \).

Claim \( \Box \) (1) \( \square \)

(2). \( (\omega^\omega, N, \in) \)-niceness is shown as (1).

\( (\omega^\omega, \not \in) \)-niceness: For \( s \in \omega^{<\omega} \) and \( k \in \omega \) let \( E_{s,k} = \{ p \in E : p = \langle s, F \rangle \text{ and } |F| = k \} \). Then \( E = \bigcup_{s \in \omega^{<\omega}, k \in \omega} E_{s,k} \). Let \( \dot{x} \) be a \( E \)-name such that \( \Vdash_{E} \dot{x} \in \omega^\omega \). Let \( \langle p, \dot{q} \rangle \in E_{\omega_1} \ast \overline{E} \). Without loss of generality we can assume \( p \Vdash_{E_{\omega_1}} \dot{q} \in E_{s,k} \). Then define \( x_{s,k} \in \omega^\omega \cap V \) by
\[
x_{s,k}(i) = \min\{ j : \forall p \in E_{s,k} \neg(p \Vdash \dot{x} > j) \}.
\]

For \( j < k \) let \( \dot{f}_j \) be a \( E_{\omega_1} \)-name such that \( p \Vdash E_{\omega_1} \langle \dot{s}, \dot{F} \rangle \text{ and } \dot{F} = \{ \dot{f}_j : j < k \} \).

Let \( r \leq p \) and \( m \in \omega \). Then define \( \langle r_n : n \in \omega \rangle \) and \( \{ f_i : i < k \} \in \omega^\omega \cap V \) so that
(i) $r_0 \leq r, r_{n+1} \leq r_n$ and

(ii) $r_m$ decides $f_j \upharpoonright m$ for $j < k$ and $r_m \Vdash_{E_{n+1}} f_j \upharpoonright m = f_j \upharpoonright m$ for $j < m$.

Let $F = \{ f_j ; j < k \}$ and $q' \leq \langle s, F \rangle$ such that $q' \Vdash_E x(m) < x_{s,k}(m)$. Then

$q' \Vdash_E x(m) \in \bigcup_{i < x_{s,k}(m)} [\langle m, i \rangle] = U^m(x_{s,k}).$

Claim 3.9.2. $(r, q) \| (1, q')$.

Proof of Claim. Let $q' = \langle t, G \rangle$. Since $r_{|t|} \Vdash_{E_{n+1}} f_j \upharpoonright |t| = f_j \upharpoonright |t|$ for $j < k$.

$r_{|t|} \Vdash_{E_{n+1}} q' \| q$. So $(1, q') \| (r, q)$.

Claim (3) \( \square \)

(4) By [11] we can assume $A := (\mathbb{B} * \mathbb{D})_I$ is Boolean Algebra with strictly positive finitely additive measure $\mu$. Let $\phi$ is $A$-name such that $\Vdash_A \phi \in \text{LOC}$. For each $n \in \omega$ define $k_n \in \omega$ so that $\mu([k_n \in \phi(n)]) < \frac{1}{n}$. Then define $\phi \in \text{LOC} \cap V$ by $\phi(n) = \{ k_n \}$. Let $(p, \dot{q}) \in (\mathbb{B} * \mathbb{D})_{\omega_1}$. Without loss of generality we can assume $p \Vdash_{(\mathbb{B} * \mathbb{D})_{\omega_1}} \mu(\dot{q}) > \frac{1}{k}$. Let $r \leq q$. Since $\mu([k_n \not\in \phi(n)]) \geq 1 - \frac{1}{k}$ for $n \geq k$, $r \Vdash_{(\mathbb{B} * \mathbb{D})_{\omega_1}} \mu(\dot{q} \cap [k_n \not\in \phi]) \geq 0$ for $n \geq k$. Since $[\phi \in U^n(\phi)] = [k_n \not\in \phi(n)]$, $r \Vdash_{(\mathbb{B} * \mathbb{D})_{\omega_1}} \mu(\dot{q} \cap [\phi \in U^n(\phi)]) > 0$. Hence $(r, q) \| (1, [\phi \in U^n(\phi)]).

\( \square \)

If $Q$ is $(A, B, E)$-nice for $P$, then elements of $A \cap V^Q$ have a following property.

Theorem 3.10. [Minami] Let $(A, B, E)$ be a Borel invariant with $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ which satisfy (0)-(5). Let $P$ be a forcing notion such that there exists $P$-name $\dot{r}$ for an element of $B$ such that $\Vdash_P \exists x E \dot{r}$ for $x \in A \cap V$ and let $Q$ be a Suslin forcing notion or the finite support iteration of Suslin forcing notions. If $Q$ is $(A, B, E)$-nice for $P$ and $\dot{x}$ is a $Q$-name for an element of $A \cap V^P$, then $\Vdash_{P \ast Q} \exists E \dot{r} \dot{x}$.

Proof. Suppose $Q$ is $(A, B, E)$-nice for $P$. Let $\dot{r}$ be a $P$-name for an element of $B \cap V^P$ such that $\Vdash_P \exists x E \dot{r}$ for $x \in A \cap V$. Let $\dot{x}$ be a $Q$-name for an element of $A \cap V^Q$. It suffices to show for each $(p, \dot{q}) \in P \ast Q$ there exists $(r, \dot{s}) \leq (p, \dot{q})$ such that

$(r, \dot{s}) \Vdash \exists E \dot{r} \dot{x}$.

Let $(p, \dot{q}) \in P \ast Q$. Since $P$ is $(A, B, E)$-nice for $Q$, there exists $x \in A \cap V$ such that

$\forall r \leq_P p \forall \in \mathbb{N} \exists q \in Q( (1, q') \| (r, \dot{q}) \text{ and } q' \Vdash_{Q} \dot{x} \in U^n(x) )$. 
Let $r \leq p$ and $n \in \omega$ such that $r \forces \text{"}x \in A\text{"}^\beta$ and if $m \geq n$, there exists $q' \in \mathcal{Q}(1, q')\parallel (q, \dot{q})$ and $q' \forces \exists \dot{x} \in U^m(x)$. Since $r \forces \exists \dot{x} \in A\text{"}^\beta$, there exists $m \geq n$ such that

$$r \forces \exists \dot{x} \in U^m(x) \subset \{z \in A : z \in A\text{"}^\beta\}.$$ 

Pick $q' \in \mathcal{Q}$ such that $(1, q')\parallel (q, \dot{q})$ and $q' \forces \exists \dot{x} \in U^m(x)$. Let $(p', \dot{q}^*) \leq (1, q'), (r, \dot{q})$. Then

$$(p', \dot{q}^*) \forces \exists \dot{x} \in U^m(x) \subset \{z \in A : z \in A\text{"}^\beta\}.$$ 

Hence $(p', \dot{q}^*) \forces \exists \dot{x} \in A\text{"}^\beta$. Therefore $\vdash \exists \dot{x} \in A\text{"}^\beta$.

\[\square\]

**Theorem 3.11.** Let $(A, B, E)$ be a Borel invariant with $(A, n \in \omega)$ and $(U^n : n \in \omega)$ satisfying (0)-(5). Let $\mathbb{P}_{\omega_2}$ be a $\omega_2$-stage finite support iteration of Suslin forcing notion and

1. for all $\beta < \omega_2$ there exists a $\mathbb{P}_{\beta+\omega_1}$-name $\dot{r}$ for an element of $A$ such that $\mathbb{P}_{\beta+\omega_1} \forces \text{"}x \in A\text{"}^\beta$ for $x \in A \cap V^{\mathbb{P}_\beta}$.
2. for all $\beta < \omega_2$ for all $I$ countable subset of $\omega_2 \setminus (\beta + \omega_1)$ $V^{\mathbb{P}_\beta} \models \text{"}I \text{ is a } \mathbb{P}_{(\beta, \beta+\omega_1)} \text{-nice for } \mathbb{P}_{(\beta, \beta+\omega_1)}\text{"}$.

Then $\mathbb{P}_{\omega_2} \models \Diamond(A, B, E)$.

**Proof.** Let $F$ be a $\mathbb{P}_{\omega_2}$-name for a Borel function. Since $\mathbb{P}_{\omega_2}$ has c.c.c and $\mathbb{P}_{\omega_2}$ is the finite support iteration of $(\mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \omega_2)$ without loss of generality we can assume $F$ is in ground model. By (1) let $\dot{r}_\alpha$ be a $\mathbb{P}_{\omega_1}$-name such that $\mathbb{P}_{\omega_1} \models \exists \dot{x} \in A \cap V^{\mathbb{P}_\alpha}$ for $\alpha < \omega_1$. We shall show $\mathbb{P}_{\omega_2} \models \langle \mathbb{P}_I \text{ is } \mathbb{P}_{(\beta, \beta+\omega_1)} \text{-nice for } \mathbb{P}_{(\beta, \beta+\omega_1)} \rangle$.

**Claim 3.11.1.** Let $\dot{f}$ be a $\mathbb{P}_{\omega_2}$-name such that $\mathbb{P}_{\omega_2} \models \exists \dot{x} \in A \cap \omega_1 < 2$. Then

$$\{\alpha \in \omega_1 : \dot{f} \models \alpha \text{ is } \mathbb{P}_I \text{-name where } I \cap \omega_1 \subset \alpha \text{ and } I \text{ is countable}\}$$

contains a club.

\[\square\]

Let $\dot{x} = F(\dot{f} \parallel \alpha)$ such that $\dot{x}$ is a $\mathbb{P}_I$-name, $I$ is countable and $I \cap \omega_1 \subset \alpha$. In $V^{\mathbb{P}_\alpha}$ we can assume $\dot{r}_\alpha$ is $\mathbb{P}_{(\alpha, \omega_1)}$-name and $\dot{r} \in \mathbb{P}_{(\alpha, \omega_1, \omega_2)}$-name. Hence to show $\mathbb{P}_{\omega_2} \models \langle \mathbb{P}_I | \alpha < \omega_1 \rangle \models \Diamond(A, B, E)\text{-sequence for } F$, it suffices to show that $\mathbb{P}_{\omega_2} \models \langle \dot{r} : \alpha < \omega_1 \rangle \text{ is } \mathbb{P}_I \text{-name for an element of } A \cap V^{\mathbb{P}_I}$.

By (2) $\mathbb{P}_I$ is $(A, B, E)$-nice for $\mathbb{P}_{\omega_1}$. By Theorem 3.10 $\vdash \dot{x} \in A\text{"}^\beta$. Hence $\langle \dot{r} : \alpha < \omega_1 \rangle \text{ is } \Diamond(A, B, E)\text{-sequence for } F$.

\[\square\]
Remark 3.11.2. Same argument holds for $\mathbb{P}_\kappa$ if $\text{cf}(\kappa) \geq \omega_2$.

Corollary 3.12. Each of the following are relatively consistent with ZFC:

(i) $c = \text{add}(\mathcal{M}) = \omega_2 + \Diamond(\text{cov}(\mathcal{N}))$ (see Diagram 1).

(ii) $c = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2 + \Diamond(b)$ (see Diagram 2).

(iii) $c = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \omega_2 + \Diamond(b) + \Diamond(\text{cov}(\mathcal{N}))$ (see Diagram 3).

(iv) $c = \text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \omega_2 + \Diamond(\text{add}(\mathcal{N}))$ (see Diagram 4).

Proof. (i) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (1) $V^{\mathbb{D}_{\omega_2}} \models \Diamond(\text{cov}(\mathcal{N}))$. Since $\mathbb{D}_{\omega_2}$ adds $\omega_2$-many dominating reals and Cohen reals, $V^{\mathbb{D}_{\omega_2}} \models c = b = \text{cov}(\mathcal{M}) = \omega_2$. Since $\text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\}$ (see [19], [14]),

$$V^{\mathbb{D}_{\omega_2}} \models \Diamond(\text{cov}(\mathcal{N})) + c = \text{add}(\mathcal{M}) = \omega_2.$$  

Cichoński's diagram for parametrized diamond looks as follows where a $\omega_2$ means the corresponding evaluation of Borel invariant is $\omega_2$ while parametrized diamonds principle for the others hold.

```
\Diamond(\text{cov}(\mathcal{N})) \quad \omega_2 \quad \omega_2 \quad \omega_2
```

Diagram 1.

(ii) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (2) $V^{\mathbb{B}_{\omega_2}} \models \Diamond(b)$. Since $\mathbb{B}_{\omega_2}$ adds $\omega_2$ many Cohen and random reals, $V^{\mathbb{B}_{\omega_2}} \models c = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{B}_{\omega_2}} \models \Diamond(b) + c = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2.$$  

```
\Diamond(b) \quad \omega_2 
```

Diagram 2.
(iii) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (3) $V^{E_{\omega_2}} \models \varnothing(\text{cov}(N)) + \varnothing(b)$. Since $E_{\omega_2}$ adds $\omega_2$ many Cohen and almost different reals, $\mathfrak{c} = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{E_{\omega_2}} \models \varnothing(\text{cov}(N)) + \varnothing(\text{cov}(\mathcal{M})) + \mathfrak{c} = \text{non}(\mathcal{M}) + \text{cov}(\mathcal{M}).$$

Diagram 3.

(iv) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (4) $V^{(B*D)_{\omega_2}} \models \varnothing(\text{add}(N))$. Since $(B*D)_{\omega_2}$ adds $\omega_2$ many random, Cohen and dominating reals, $\mathfrak{c} = \text{cov}(N) = \text{add}(\mathcal{M}) = \min\{b, \text{cov}(N)\} = \omega_2$. Hence

$$V^{(B*D)_{\omega_2}} \models \varnothing(\text{add}(N)) + \mathfrak{c} = \text{cov}(N) = \text{add}(\mathcal{M}) = \omega_2.$$ 

Diagram 4

\qed

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参考文献


