Weak Kurepa trees and weak diamonds

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Abstract

We consider combinatorial statements which fit between the Kurepa and the weak Kurepa hypotheses. We also formulate weak diamonds and consider their relations to these statements.

Introduction

Two weak forms of the diamond principle $\diamondsuit$ and $\tilde{\diamondsuit}$ are introduced in [W]. It is shown that (see p.110 of [W] for more information)

- $\diamondsuit$ implies $\tilde{\diamondsuit}$.
- The Kurepa hypothesis (KH) also implies $\tilde{\diamondsuit}$.
- $\tilde{\diamondsuit}$ in turn implies $\tilde{\tilde{\diamondsuit}}$.
- $\tilde{\tilde{\diamondsuit}}$ negates the saturation of the non-stationary ideal on $\omega_1$.
- $\tilde{\tilde{\diamondsuit}}$ implies the weak Kurepa hypothesis (wKH), too.
- $\diamondsuit$ persists in the sense that if $\diamondsuit$ holds in a transitive model of ZFC which correctly computes $\omega_2$, then $\diamondsuit$ holds in the universe.

The following are dealt in this note.

(1) We give an equivalent statements to $\tilde{\diamondsuit}$ and $\tilde{\tilde{\diamondsuit}}$.
(2) Our equivalent to $\tilde{\tilde{\diamondsuit}}$ is seemingly more demanding than the original $\tilde{\diamondsuit}$. As a result, we get what we call stat-wKH which rather directly negates the saturation of the non-stationary ideal on $\omega_1$.
(3) We formulate same types of weak Kurepa hypotheses as stat-wKH and consider weak diamonds to investigate the situation between KH and these wKH.
(4) We provide more information on these weak diamonds. For example, we get a new fragment of $\diamondsuit$ different from $\tilde{\diamondsuit}$.
(5) We describe as many forcing constructions as we know of to separate these new combinatorial statements.

Though claims we make are within the reaches of established facts and forcing techniques, so-far-possibly-implicit points of view on KH, wKH and $\diamondsuit$ are examined.
§1. The KH, $\mathcal{D}$, $\mathcal{D}^*$ and the wKH

1.1 Definition. ([W]) $\mathcal{D}$ holds, if there exist $\omega_2$-many subsets $\langle A_\beta \mid \beta < \omega_2 \rangle$ of $\omega_1$ and $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ with each $T_\alpha$ countable and the following is stationary in $\omega_2$

$$\{ \beta_Y \mid Y \subset \mathcal{P}(\omega_1) \text{ is countable}, \langle T_\alpha \mid \alpha < \omega_1 \rangle \text{ guesses } Y \}$$

where,

$$\beta_Y = \sup \{ \beta + 1 \mid A_\beta \in Y \}$$

and

$\langle T_\alpha \mid \alpha < \omega_1 \rangle$ guesses $Y$, if the following is cofinal in $\omega_1$

$$\{ \alpha < \omega_1 \mid E \cap \alpha \in T_\alpha \text{ for all } E \in Y \}$$

We record the following for the sake of clarity.

1.2 Proposition. (1) For $S \subseteq \{ \beta < \omega_2 \mid \text{cf}(\beta) = \omega \}$, the following are equivalent

- $S$ is stationary in $\omega_2$.
- $\{ X \in [\omega_2]^\omega \mid \bigcup X \in S \}$ is stationary in $[\omega_2]^\omega$.

(2) For $S^* \subseteq [\omega_2]^\omega$, if $S^*$ is stationary in $[\omega_2]^\omega$, then $\{ \bigcup X \mid X \in S^* \}$ is stationary in $\omega_2$.

(The converse is false in some cases.)

In the manner we show the above on these two notions of stationary sets, we may show

1.3 Proposition. $\mathcal{D}$ holds iff there exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each $b_\beta$ is a function from $\omega_1$ into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each $S_\alpha$ is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \rightarrow 2$.
- The following is stationary in $[\omega_2]^\omega$:

$$\{ X \in [\omega_2]^\omega \mid \exists A \subseteq \omega_1 \exists B \subseteq X \text{ such that } \bigcup A = \omega_1, \bigcup B = \bigcup X, \forall (\alpha, \beta) \in A \times B \ b_\beta[\alpha \in S_\alpha] \}$$

Proof. Let $\langle A_\beta \mid \beta < \omega_2 \rangle$ and $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ satisfy $\mathcal{D}$. For each $\beta < \omega_2$, let $b_\beta : \omega_1 \rightarrow 2$ be the characteristic function of $A_\beta$. For each $\alpha < \omega_1$, let $S_\alpha = \{ \chi_\alpha \mid a \in T_\alpha \cap \mathcal{P}(\alpha) \}$, where $\chi_\alpha : \alpha \rightarrow 2$ is the characteristic function of $a$. Given $\varphi : <\omega \omega_2 \rightarrow \omega_2$, find $Y \subset \mathcal{P}(\omega_1)$ such that $\beta_Y$ is a limit ordinal, $\beta_Y$ is $\varphi$-closed and $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ guesses $Y$.

Let

$$A = \{ \alpha < \omega_1 \mid \forall E \in Y \ E \cap \alpha \in T_\alpha \}$$
and

\[ B = \{ \beta < \omega_2 \mid A_\beta \in Y \} \.]

Let \( X \in [\omega_2]^{\omega} \) be the \( \varphi \)-closure of \( B \). Then \( X \) is \( \varphi \)-closed, \( \bigcup A = \omega_1, \bigcup B = \bigcup X \) and for all \((\alpha, \beta) \in A \times B\), we have \( b_\beta[\alpha] \in S_\alpha \).

Conversely, for each \( \beta < \omega_2 \), let \( A_\beta = \{ i < \omega_1 \mid b_\beta(i) = 1 \} \). For each \( \alpha < \omega_1 \), let \( T_\alpha = \{ (i < \alpha \mid \sigma(i) = 1) \mid \sigma \in S_\alpha \} \). Let \( C \subseteq \omega_2 \) be a club. Take \( X \in [\omega_2]^{\omega} \), \( A \subseteq \omega_1 \) and \( B \subseteq X \) such that \( \bigcup X \in C \), \( \bigcup A = \omega_1 \), \( \bigcup B = \bigcup X \) and for all \((\alpha, \beta) \in A \times B\), we have \( b_\beta[\alpha] \in S_\alpha \). We may assume \( \bigcup X \) is a limit ordinal. Let \( Y = \{ A_\beta \mid \beta \in B \} \). Then \( \beta_Y = \bigcup X \in C \) and \((T_\alpha \mid \alpha < \omega_1)\) guesses this \( Y \).

\[ \square \]

The following is almost verbatim from [W].

1.4 Definition. ([W]) \( \tilde{\diamond} \) holds, if there exist \( \langle b_\beta \mid \beta < \omega_2 \rangle \) and \( \langle S_\alpha \mid \alpha < \omega \rangle \) such that

- Each \( b_\beta \) is a function from \( \omega_1 \) into \( 2 \) and if \( \beta_1 \neq \beta_2 \), then \( b_{\beta_1} \neq b_{\beta_2} \).
- Each \( S_\alpha \) is countable and if \( \sigma \in S_\alpha \), then \( \sigma : \alpha \rightarrow 2 \).
- The following is stationary in \([\omega_2]^{\omega} \).

\[ \{ X \in [\omega_2]^{\omega} \mid \exists \alpha \geq X \cap \omega_1 \exists B \subseteq X \text{ such that } \bigcup B = \bigcup X, \forall \beta \in B b_\beta[\alpha] \in S_\alpha \} \]

Here is our equivalent statement to \( \tilde{\diamond} \).

1.5 Proposition. \( \tilde{\diamond} \) holds iff there exist \( \langle b_\beta \mid \beta < \omega_2 \rangle \) and \( \langle S_\alpha \mid \alpha < \omega \rangle \) such that

- Each \( b_\beta \) is a function from \( \omega_1 \) into \( 2 \) and if \( \beta_1 \neq \beta_2 \), then \( b_{\beta_1} \neq b_{\beta_2} \).
- Each \( S_\alpha \) is countable and if \( \sigma \in S_\alpha \), then \( \sigma : \alpha \rightarrow 2 \).
- The following is stationary in \([\omega_2]^{\omega} \).

\[ \{ X \in [\omega_2]^{\omega} \mid \exists \alpha \geq X \cap \omega_1 \exists B \subseteq X \text{ such that } \bigcup B = \bigcup X, \forall \beta \in B b_\beta[\alpha] \in S_\alpha \} \]

We record a well-known lemma, say, from [B] and [W].

1.6 Lemma. Let \( \theta \) be a regular cardinal with \( \theta \geq \omega_2 \) and \( N \) be a countable elementary substructure of \( H_\theta \). By this we mean \( (N, \in) \) is an elementary substructure of \( (H_\theta, \in) \) with \( |N| = \omega \) and may simply denote \( N < H_\theta \). Define

\[ N^* = \{ f(N \cap \omega_1) \mid f \in N \}. \]

Then
• $(N^*, \in)$ is a countable elementary substructure of $(H_\theta, \in)$.
• $N \subset N^*$, $N \cap \omega_1 \in N^*$ and so $N \cap \omega_1 < N^* \cap \omega_1 < \omega_1$.
• However, $\sup(N \cap \omega_2) = \sup(N^* \cap \omega_2)$.

1.7 Corollary. Let $\theta$ be a regular cardinal with $\theta \geq \omega_2$. Then given any countable elementary substructure $N$ of $H_\theta$, we may automatically construct its canonical extensions $\langle N_i \mid i < \omega_1 \rangle$. By this we mean

• $N_0 = N$.
• Each $N_i$ is a countable elementary substructure of $H_\theta$.
• $N_{i+1} = N_i^*$.
• For limit $i$, we set $N_i = \bigcup\{N_k \mid k < i\}$.

Therefore,

• $\langle N_i \cap \omega_1 \mid i < \omega_1 \rangle$ forms a club in $\omega_1$.
• However, $\sup(N_i \cap \omega_2) = \sup(N \cap \omega_2)$ constantly for all $i < \omega_1$.

Isomorphic-types of the canonical extensions are considered via $\varphi_{AC}$ in [W].

Proof to the equivalence of $\tilde{\varphi}$.

Fix $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ so that $\tilde{\varphi}$ is witnessed. We show

1.7.1 Claim. The following $N \in [H_{\omega_2}]^\omega$ are stationary in $[H_{\omega_2}]^\omega$.

• $N \prec H_{\omega_2}$.
• $\exists f \in N \cap \omega_1 \omega_1$ with $\forall \alpha < \omega_1 \ f(\alpha) \geq \alpha$ such that $\exists B \subset N \cap \omega_2$ with $\bigcup B = \bigcup(N \cap \omega_2)$, $\forall \beta \in B \ b_\beta \upharpoonright f(N \cap \omega_1) \in S_{f(N \cap \omega_1)}$.

Then by the Fodor's Lemma,

1.7.2 Claim. $\exists f_0 \in \omega_1 \omega_1 \ \forall \alpha < \omega_1 \ f_0(\alpha) \geq \alpha$ and the following is stationary in $[H_{\omega_2}]^\omega$.

$\{N \in [H_{\omega_2}]^\omega \mid N \prec H_{\omega_2}, \exists B \subset N \cap \omega_2 \text{ with } \bigcup B = \bigcup(N \cap \omega_2), \forall \beta \in B \ b_\beta \upharpoonright f_0(N \cap \omega_1) \in S_{f_0(N \cap \omega_1)}\}$

Therefore, for each $\alpha < \omega_1$, may define $S^*_\alpha$ by

$S^*_\alpha = S_{f_0(\alpha)}[\alpha]$.

Then $S^*_\alpha \subset \alpha 2$, $S^*_\alpha$ is countable and the following is stationary in $[H_{\omega_2}]^\omega$.

$\{N \in [H_{\omega_2}]^\omega \mid \exists B \subset N \cap \omega_2 \text{ with } \bigcup B = \bigcup(N \cap \omega_2), \forall \beta \in B \ b_\beta \upharpoonright f_0(N \cap \omega_1) \in S_{f_0(N \cap \omega_1)}\}$
So we would be done, if we provide a proof to 1.7.1 Claim.

*Proof of 1.7.1 Claim.* (This part is based on [W])

Let $\varphi : <\omega \rightarrow H_{\omega_2}$ be a sufficiently large regular cardinal $\theta$ and a countable elementary substructure $M$ of $H_\theta$ with $\varphi \in M$. We may assume $X = M \cap \omega_2$ has a cofinal subset $B \subseteq X$ and there exists $\alpha \geq X \cap \omega_1$ such that

$$\forall \beta \in B \quad \varphi(\beta) \in \alpha \in S_\alpha.$$  

Construct the canonical extensions $\langle M_i \mid i < \omega_1 \rangle$ of $M$. Since $\langle M_i \mid i < \omega_1 \rangle$ forms a club in $\omega_1$ with $\alpha \geq M_0 \cap \omega_1$, there exists $i < \omega_1$ such that

$$M_i \cap \omega_1 \leq \alpha < M_{i+1} \cap \omega_1.$$  

By the definition of $M_{i+1}$ from $M_i$, we have $f \in M_i$ such that

$$f(M_i \cap \omega_1) = \alpha \geq M_i \cap \omega_1.$$  

We may assume that $f : \omega_1 \rightarrow \omega_1$ and that for all $\beta < \omega_1$, $f(\beta) \geq \beta$.

Let $N = M_i \cap H_{\omega_2}$. Since $H_{\omega_2} \in M_i \prec H_\theta$,

- $N$ is a countable elementary substructure of $H_{\omega_2}$.
- $f \in N$, as $\omega_1 \omega_1 \subset H_{\omega_2}$.
- $B \subseteq N \cap \omega_2$ and $\bigcup B = \bigcup (N \cap \omega_2)$.
- $\forall \beta \in B \quad \varphi(\beta) \in \alpha \in S_\alpha$.

Since $N$ is $\varphi$-closed, this completes the proof.  

\square

We go on to make

1.8 **Definition.** Let us *stat-weak Kurepa hypothesis (stat-wKH)* denote the following: There exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each $b_\beta$ is a function from $\omega_1$ into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each $S_\alpha$ is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \rightarrow 2$.
- For all $\beta < \omega_2$, $\{ \alpha < \omega_1 \mid b_\beta(\alpha) \in S_\alpha \}$ are stationary in $\omega_1$.

We may view stat-wKH as a sort of $\diamondsuit$. Namely, stat-wKH guesses some $\omega_2$-many subsets of $\omega_1$, while $\diamondsuit$ does all subsets of $\omega_1$. The weak diamond $\diamondsuit$ entails stat-wKH.

1.9 **Proposition.** $\diamondsuit$ implies stat-wKH.
Proof. It is just thinning. By our equivalent form of \(\diamondsuit\), we get \(\langle b_\beta \mid \beta < \omega_2 \rangle\) and \(\langle S_\alpha \mid \alpha < \omega_1 \rangle\) such that the following is stationary in \([\omega_2]^\omega\).

\[
\{X \in [\omega_2]^\omega \mid \exists \delta = X \cap \omega_1, \exists B \subseteq X \text{ with } \bigcup B = \bigcup X, \forall \beta \in B \ b_\beta[\delta] \in S_\delta\}
\]

1.9.1 Claim. \(\{ \beta < \omega_2 \mid \{ \alpha < \omega_1 \mid b_\beta[\alpha] \in S_\alpha\} \text{ is stationary in } \omega_1 \}\) is cofinal in \(\omega_2\).

Proof of Claim. Fix \(\eta < \omega_2\). Take a sufficiently large regular cardinal \(\theta\) and a countable elementary substructure \(M\) of \(H_\theta\) such that \(\langle b_\beta \mid \beta < \omega_2 \rangle, \langle S_\alpha \mid \alpha < \omega_1 \rangle, \eta \in M\). We may set \(\delta = M \cap \omega_1\) and assume that there exists \(B \subseteq M \cap \omega_2\) cofinal within \(M \cap \omega_2\) such that

\[
\forall \beta \in B \ b_\beta[\delta] \in S_\delta.
\]

Therefore, we may fix some \(\beta \in B\) such that \(\eta < \beta\) and \(b_\beta[\delta] \in S_\delta\).

1.9.1.1 Sub claim. \(\{ \alpha < \omega_1 \mid b_\beta[\alpha] \in S_\alpha\} \text{ is stationary in } \omega_1\).

Proof of sub claim. We make use of the elementarity of \(M\). Fix a club \(C \in M\). Then \(\delta \in C\) and so

\[
M \models \forall C \subseteq \omega_1 \text{ club } \exists \alpha \in C \ b_\beta[\alpha] \in S_\alpha.
\]

Therefore \(\{ \alpha < \omega_1 \mid b_\beta[\alpha] \in S_\alpha\}\) is really stationary in the universe.

\[\square\]

1.10 Proposition. The stat-wKH implies that there exists a family \(\mathcal{F}\) of almost disjoint stationary subsets of \(\omega_1\) with \(|\mathcal{F}| = \omega_2\). And so the non-stationary ideal on \(\omega_1\) is not saturated.

Proof. Let \(\langle b_\beta \mid \beta < \omega_2 \rangle\) and \(\langle S_\alpha \mid \alpha < \omega_1 \rangle\) be as in stat-wKH.

Let \(\langle \sigma^n_\alpha \mid n < \omega \rangle\) enumerate \(S_\alpha\). By thinning, say twice, we may assume that there exists \(n < \omega\) such that for all \(\beta < \omega_2\), the following \(T_\beta\) is stationary in \(\omega_1\).

\[
T_\beta = \{ \alpha < \omega_1 \mid b_\beta[\alpha] = \sigma^n_\alpha\}
\]

Now consider \(\mathcal{F} = \{ T_\beta \mid \beta < \omega_2 \}\). Then this \(\mathcal{F}\) works.

\[\square\]

The following is shown in \([W]\) by generic ultra-power constructions over set models of set theory.

1.11 Corollary. \([W]\) \(\diamondsuit\) implies the non-stationary ideal on \(\omega_1\) is not saturated.

1.12 Definition. Let us cof-weak Kurepa hypothesis (cof-wKH) denote the following:
There exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each $b_\beta$ is a function from $\omega_1$ into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each $S_\alpha$ is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \rightarrow 2$.
- For all $\beta < \omega_2$, $\{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ are cofinal in $\omega_1$.

Therefore, stat-wKH implies cof-wKH. We return to this in the next section.

1.13 **Proposition.** The cof-wKH implies wKH. I.e, there exists a sub tree $T$ of $\omega_2$ such that $|T| = \omega_1$ and there are at least $\omega_2$-many cofinal branches through $T$.

**Proof.** We argue as in the previous proposition. Let $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be as in cof-wKH.

Let $\langle \sigma_n^\alpha \mid n < \omega \rangle$ enumerate $S_\alpha$. By thinning, say twice, we may assume that there exists $n < \omega$ such that for all $\beta < \omega_2$, the following $E_\beta$ is cofinal in $\omega_1$.

$$E_\beta = \{\alpha < \omega_1 \mid b_\beta \lceil \alpha = \sigma_n^\alpha\}$$

Let $T = \{\sigma_n^\alpha \pa | \pa \leq \alpha < \omega_1\}$. Then this $T$ works. The $b_\beta$ provide cofinal branches through $T$.

\[\square\]

1.14 **Corollary.** ([W]) $\tilde{\diamond}$ implies wKH.

Since KH implies $\tilde{\diamond}$ by [W], we conclude

1.15 **Corollary.** The following are all equiconsistent.

1.15 (1) There exists a strongly inaccessible cardinal.

1.15 (2) Either wKH, cof-wKH, stat-wKH, $\tilde{\diamond}$, $\tilde{\boxdot}$ or KH gets negated.

§2. **Weak Kurepa Trees**

We recap stat-wKH and cof-wKH in this section and generalize them.

2.1 **Definition.** Let $\square$ be either cof, stat, club, or coint. Let us $\square$-weak Kurepa hypothesis ($\square$-wKH) denote the following:

There exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each $b_\beta$ is a function from $\omega_1$ into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each $S_\alpha$ is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \rightarrow 2$.
- For each $\beta < \omega_2$, either $\{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ is cofinal, stationary, contains a club, or is coinitial in $\omega_1$, respectively.
We view KH, $\diamondsuit$, $\hat{\diamondsuit}$, stat-wKH, cof-wKH and wKH along this generalization and record the following.

### 2.2 Proposition

1. KH iff coint-wKH.

- The coint-wKH implies club-wKH.
- The club-wKH implies stat-wKH.
- The stat-wKH implies cof-wKH.
- The cof-wKH implies wKH.

(3) The club-wKH implies $\diamondsuit$.

(2) $\langle b_\beta | \beta < \omega_2 \rangle \subset \omega_1$. If $\alpha \geq \alpha_0$, we have $b_\beta | \alpha \in S_\alpha$.

**Proof.** For (1): Suppose $T$ is a Kurepa tree. We may assume $T \subset \omega_1$. Let $\{b_\beta | \beta < \omega_2 \} \subset \omega_1$ be one-to-one such that $b_\beta | \alpha \in T_\alpha$ for all $\beta < \omega_2$ and $\alpha < \omega_1$. Let $S_\alpha = T_\alpha$ for all $\alpha < \omega_1$. Then $S_\alpha$ is countable and $b_\beta | \alpha \in S_\alpha$ for every possible combination. Hence we certainly have coint-wKH.

Conversely, let $\langle b_\beta | \beta < \omega_2 \rangle$ and $\langle S_\alpha | \alpha < \omega_1 \rangle$ be witnesses to coint-wKH. By thinning, we may assume that there exists $\alpha_0 < \omega_1$ such that for all $\beta < \omega_2$ and all $\alpha \geq \alpha_0$, we have $b_\beta | \alpha \in S_\alpha$.

Let $T = \{ b_\beta | \beta < \omega_2, \alpha < \omega_1 \}$. If $\alpha \geq \alpha_0$, then $T_\alpha \subset S_\alpha$ which is countable. If $\alpha < \alpha_0$, then $T_\alpha \subset S_{\alpha_0} \setminus \alpha$ which is also countable. Each $b_\beta$ provide different cofinal branch $\{ b_\beta | \alpha < \omega_1 \}$. Hence $T$ is a Kurepa tree.

For (2): First three are trivial by definition and we have seen the fourth.

For (3): Since we have seen the last two items, we consider the first item. Let $\langle b_\beta | \beta < \omega_2 \rangle$ and $\langle S_\alpha | \alpha < \omega_1 \rangle$ be witnesses to club-wKH. Let $E_\beta = \{ \alpha < \omega_1 | b_\beta | \alpha \in S_\alpha \}$. Then for all $X \in [\omega_2]^\omega$, we set $A = \bigcap E_\beta$ and $B = X$ so that $\bigcup A = \omega_1$, $\bigcup B = \bigcup X$ and for all $(\alpha, \beta) \in A \times B$, we have $b_\beta | \alpha \in S_\alpha$. Hence we certainly have $\diamondsuit$.

**2.3 Proposition.** The club-wKH implies the transversal hypothesis (TH). Namely, there exists a family $\mathcal{F}$ of almost disjoint functions from $\omega_1$ into $\omega$ with $|\mathcal{F}| = \omega_2$.

**Proof.** We must observe that there exist $\omega_2$-many functions $g_\beta : \omega_1 \longrightarrow \omega$ such that if $\beta_1 \neq \beta_2$, then there exists $\alpha_{\beta_1\beta_2} < \omega_1$ such that for all $\alpha$ with $\alpha_{\beta_1\beta_2} \leq \alpha < \omega_1$, we have $g_{\beta_1}(\alpha) \neq g_{\beta_2}(\alpha)$.

To this end, let $\{ \sigma_n^\alpha | n < \omega \}$ enumerate $S_\alpha$. Then let $f_{\beta}(\alpha) = \text{the least } n \text{ such that } b_\beta | \alpha = \sigma_n^\alpha$, if applicable. Then if $\beta_1 \neq \beta_2$, then $\{ \alpha < \omega_1 | f_{\beta_1}(\alpha) \neq f_{\beta_2}(\alpha) \}$ contains a
club. Now we may resort to a trick due to Jensen to produce \( g_{\beta} \). See the proof of Lemma 1 on p. 72 of [D].

\[ \square \]

When I gave a talk on this at the Set Theory Seminar, Nagoya university, 17th, Dec. 2004, T. Sakai provided an idea for a direct proof on the spot. Accordingly, I record the following based on his idea.

**Proof.** Let us fix \( \langle e_{\alpha} \mid \alpha < \omega_{1} \rangle \) so that \( e_{\alpha} : \omega \rightarrow \alpha + 1 \) onto. Let \( \langle \beta \mid \beta < \omega_{2} \rangle \) and \( \langle S_{\alpha} \mid \alpha < \omega_{1} \rangle \) be as in club-wKH. Let \( C_{\beta} \subset \{ \alpha < \omega_{1} \mid b_{\beta} | \alpha \in S_{\alpha} \} \) be a club and \( \langle a_{n}^\alpha \mid n < \omega \rangle \) enumerate \( S_{\alpha} \).

For each \( \beta \), let us define \( g_{\beta} : \omega_{1} \rightarrow \omega \times \omega \) so that for any \( \alpha \geq \min C_{\beta} \), if \( \delta = \max (C_{\beta} \cap (\alpha + 1)) \), then \( g_{\beta}(\alpha) = (n, m) \), where

\[
n = \text{the least } n \text{ s.t. } e_{\alpha}(n) = \delta, \\
m = \text{the least } m \text{ s.t. } a_{m}^\delta = b_{\beta}[\delta].
\]

Let \( \beta_{1}, \beta_{2} < \omega_{2} \) with \( \beta_{1} \neq \beta_{2} \). Pick \( \alpha^{*} < \omega_{1} \) so that \( [\alpha_{\beta_{1} \beta_{2}}, \alpha^{*}] \cap (C_{\beta_{1}} \cap C_{\beta_{2}}) \neq \emptyset \), where if \( \alpha' \geq \alpha_{\beta_{1} \beta_{2}} \), then \( b_{\beta_{1}}[\alpha' \neq b_{\beta_{2}}[\alpha' \cdot \\

**2.3.1 Claim.** If \( \alpha \geq \alpha^{*} \), then \( g_{\beta_{1}}(\alpha) \neq g_{\beta_{2}}(\alpha) \).

**Proof.** Let \( g_{\beta_{1}}(\alpha) = (n_{1}, m_{1}), g_{\beta_{2}}(\alpha) = (n_{2}, m_{2}), \delta_{1} = e_{\alpha}(n_{1}) \) and \( \delta_{2} = e_{\alpha}(n_{2}) \).

**Case 1.** \( n_{1} \neq n_{2} \): Then \( g_{\beta_{1}}(\alpha) \neq g_{\beta_{2}}(\alpha) \).

**Case 2.** \( n_{1} = n_{2} \): Then let \( \delta_{2} = \delta_{1} = \delta \in C_{\beta_{1}} \cap C_{\beta_{2}} \). We have \( b_{\beta_{1}}[\delta] = a_{m_{1}}^\delta, b_{\beta_{2}}[\delta] = a_{m_{2}}^\delta \) and \( \delta \geq \alpha_{\beta_{1} \beta_{2}} \). Then \( m_{1} \neq m_{2} \) and so \( g_{\beta_{1}}(\alpha) \neq g_{\beta_{2}}(\alpha) \).

\[ \square \]

We interpolated the following well-known.

**2.4 Corollary.** KH implies TH.

We provide a characterization of weak Kurepa trees along the line of \( \square \)-wKH, where \( \square \) is either coint, club, stat, or cof.

**2.5 Proposition.** The following are equivalent.

(1) The wKH holds.

(2) There exist \( \langle b_\beta \mid \beta < \omega_2 \rangle \) and \( \langle S_\alpha \mid \alpha < \omega_1 \rangle \) such that

- Each \( b_\beta \) is a function from \( \omega_1 \) into 2 and if \( \beta_1 \neq \beta_2 \), then \( b_{\beta_{1}} \neq b_{\beta_{2}} \).
- Each \( S_\alpha \) is countable and if \( \sigma \in S_\alpha \), then \( \sigma : \alpha \rightarrow 2 \).
- For all \( \beta < \omega_2 \), there exist \( f_\beta : \omega_1 \rightarrow \omega_1 \) such that for all \( \alpha < \omega_1 \), we have \( \alpha \leq f_\beta(\alpha) \) and \( b_\beta[\alpha] \in S_{f_\beta(\alpha)}[\alpha] \).
Proof. (1) implies (2): Let $T$ be a weak Kurepa tree. Let $\langle b_{\beta} \mid \beta < \omega_{2} \rangle$ be a one-to-one enumeration of functions from $\omega_{1}$ to 2 such that $b_{\beta}[\alpha] \in T_{\alpha}$ for all possible combinations of $(\alpha, \beta)$. Let $\langle \sigma_{i} \mid i < \omega_{1} \rangle$ enumerate $\{b_{\beta}[\alpha] \mid \beta < \omega_{2}, \alpha < \omega_{1}\} \subseteq T$. For each $\alpha' < \omega_{1}$, let $S_{\alpha'} \subseteq \alpha'$ be countable so that for any $i < \omega_{1}$, if $\sigma_{i}$ satisfies $|\sigma_{i}| \leq \alpha'$, then there exists $\tau \in S_{\alpha'}$ with $\sigma_{i} \subseteq \tau$. We claim these $\langle b_{\beta} \mid \beta < \omega_{2} \rangle$ and $\langle S_{\alpha'} \mid \alpha' < \omega_{1} \rangle$ work. To see this, let $\beta < \omega_{2}$ and $\alpha < \omega_{1}$. Let $\sigma_{i} = b_{\beta}[\alpha]$. Then take $\alpha' < \omega_{1}$ so large that $i, \alpha \leq \alpha'$. Since $i \leq \alpha'$ and $|\sigma_{i}| = \alpha \leq \alpha'$, we have $\tau \in S_{\alpha'}$ with $\sigma_{i} \subseteq \tau$ and so $b_{\beta}[\alpha] \in S_{\alpha'}[\alpha]$. Let $f_{\beta}(\alpha) = \alpha'$.

(2) implies (1): Let $T = \{b_{\beta}[\alpha] \mid \beta < \omega_{2}, \alpha < \omega_{1}\}$. Then for each $\beta < \omega_{2}$, $\{b_{\beta}[\alpha] \mid \alpha < \omega_{1}\}$ is a cofinal branch through $T$. For each $\alpha < \omega_{1}$, we have $T_{\alpha} \subseteq \bigcup\{S_{\alpha'}[\alpha] \mid \alpha \leq \alpha', \alpha' < \omega_{1}\}$ which is at most of size $\omega_{1}$. Hence $T$ is a weak Kurepa tree.

The following is also from the Set Theory Seminar, Nagoya university, and due to S. Fuchino and T. Sakai.

2.6 Note. The following are equivalent.

(1) The CH holds.
(2) There exists $\langle S_{\alpha} \mid \alpha < \omega_{1} \rangle$ such that $S_{\alpha} \subseteq \alpha$, $|S_{\alpha}| \leq \omega$ and for all $b \in \omega_{1} 2$ and $\alpha < \omega_{1}$, there exist $\alpha' < \omega_{1}$ such that $\alpha \leq \alpha'$ and $b[\alpha] \in S_{\alpha'}[\alpha]$.
(3) Same as above with $|S_{\alpha}| = 1$.

Along the lines of guessing all subsets of $\omega_{1}$, we have the three principles $\Diamond$, $\Diamond^{*}$ and $\Diamond^{+}$. Now we are tempted to consider the following $\Diamond^{(\text{coint})}$.

2.7 Note. However, $\Diamond^{(\text{coint})}$ is false, where $\Diamond^{(\text{coint})}$ denotes that there exists $\langle S_{\beta} \mid \alpha < \omega_{1} \rangle$ such that $S_{\alpha} \subseteq \alpha$, $|S_{\alpha}| \leq \omega$ and for all $b \in \omega_{1} 2$, $\{\alpha < \omega_{1} \mid b[\alpha] \in S_{\alpha}\}$ are coinitial in $\omega_{1}$.

§3. Weak Diamonds

We formulate weak diamonds and investigate their impacts on the situation between $\text{wKH}$ and $\text{KH}$.

3.1 Definition. Let $\square$ denote either cof, stat, club or coint. We denote $\overline{\Phi}(\square)$, if for any $F: <\omega_{1} 2 \rightarrow \omega_{1}$ and any $\langle b_{\beta} \mid \beta < \omega_{2} \rangle$ (no need to be one-to-one) such that each $b_{\beta}$ is a member of $\omega_{1} 2$, there exists $g: \omega_{1} \rightarrow \omega_{1}$ such that for each $\beta < \omega_{2}$, we have either $\{\alpha < \omega_{1} \mid F(b_{\beta}[\alpha]) < g(\alpha)\}$ is cofinal, stationary, contains a club, or is coinitial in $\omega_{1}$, respectively.

So for example, $\overline{\Phi}(\text{stat})$ claims that given any coloring of the nodes of the tree $<\omega_{1} 2$ by countable ordinals, if we fix at most $\omega_{2}$-many cofinal branches and concentrate on the nodes in $\{b_{\beta}[\alpha] \mid \beta < \omega_{2}, \alpha < \omega_{1}\}$, then there exists a uniform coloring $g: \omega_{1} \rightarrow \omega_{1}$ such that $g$ correctly bounds each $\langle \alpha \mapsto F(b_{\beta}[\alpha]) \mid \alpha < \omega_{1}\rangle$ stationary often.
We also formulate a stronger diamond along the line of $\Phi(\square)$.

3.2 Definition. Let $\square$ denote either cof, stat, club or coint. We denote $\Phi(\square)$, if for any $F : <\omega_1 \rightarrow \omega_1$, there exists $g : \omega_1 \rightarrow \omega_1$ such that for any $b : \omega_1 \rightarrow 2$, we have either $\{\alpha < \omega_1 \mid F(b_{\beta}[\alpha] < g(\alpha))\}$ is cofinal, stationary, contains a club, or is coinitial in $\omega_1$, respectively.

Therefore, given any coloring of $<\omega_1 2$ with countable ordinals, the principle $\Phi(\text{stat})$ provides a uniform coloring $g$ which correctly bounds every possible cofinal branch's coloring as often as a stationary subset of $\omega_1$.

3.3 Definition. We denote $(<^*)$, if for any $\langle f_{\beta} \mid \beta < \omega_2 \rangle$ such that for each $\beta$, $f_{\beta}$ is a function from $\omega_1$ into $\omega_1$, there exists $f : \omega_1 \rightarrow \omega_1$ such that for every $\beta < \omega_2$, we have $f_{\beta} <^* f$. By this we mean that $\{\alpha < \omega_1 \mid f_{\beta}(\alpha) < f(\alpha)\}$ is coinitial in $\omega_1$.

3.4 Proposition. Let $\square$ denote either cof, stat, club or coint.

1. The wKH combined with $\Phi(\square)$ implies $\square$-wKH.
2. $(<^*)$ implies $\Phi(\square)$.

Proof. For (1): Let $T$ be a weak Kurepa tree. Then $T$ has at least $\omega_2$-many cofinal branches. So let $\langle b_{\beta} \mid \beta < \omega_2 \rangle$ be a one-to-one enumeration such that for all $\langle \alpha, \beta \rangle \in \omega_1 \times \omega_2$, $b_{\beta}[\alpha] \in T_{\alpha}$. Now let us fix $F : <\omega_1 \rightarrow \omega_1$ so that $F[T]$ is one-to-one. Then by $\Phi(\square)$, get $g : \omega_1 \rightarrow \omega_1$ such that for all $\beta < \omega_2$, we have $\{\alpha < \omega_1 \mid F(b_{\beta}[\alpha] < g(\alpha))\}$ are $\square$ in $\omega_1$.

Define $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ by

$$S_\alpha = \{\sigma \in \omega_1^2 \cap T \mid F(\sigma) < g(\alpha)\}.$$ 

Since $F[T]$ is one-to-one, $S_\alpha$ is countable. If $F(b_{\beta}[\alpha] < g(\alpha))$, then $b_{\beta}[\alpha] \in S_\alpha$ holds. Hence these $b_{\beta}$ and $S_\alpha$ work.

For (2): Let $F : <\omega_1 \rightarrow \omega_1$ and $\langle b_{\beta} \mid \beta < \omega_2 \rangle$ be given. Define $\langle f_{\beta} \mid \beta < \omega_2 \rangle$ by

$$f_{\beta}(\alpha) = F(b_{\beta}[\alpha]).$$

Then get $f : \omega_1 \rightarrow \omega_1$ such that for all $\beta < \omega_2$,

$$\{\alpha < \omega_1 \mid f_{\beta}(\alpha) < f(\alpha)\}$$

are coinitial. Hence $\{\alpha < \omega_1 \mid F(b_{\beta}[\alpha] < f(\alpha))\}$ is $\square$ in $\omega_1$.

\[\square\]

The following is a rendition from [We].

3.5 Corollary. If CH, $2^{\omega_1} = \omega_3$ and GMA($\sigma$-closed, $\aleph_1$-linked, well-met) hold, then KH holds.
Proposition 3.6. Let $\square$ denote either $\text{cof}$, stat, club or coint.

(1) $\Phi(\square)$ implies $\overline{\Phi}(\square)$.
(2) $\Phi(\text{cof})$ implies $2^\omega < 2^\omega_1$.
(3) CH + $\Phi(\text{stat})$ iff $\diamondsuit$.
(4) CH + $\Phi(\text{club})$ iff $\diamondsuit^*$.

Proof. For (1): Fix $F : <\omega_1 2 \rightarrow \omega_1$. Then $\Phi(\square)$ provides a uniform coloring $g : \omega_1 \rightarrow \omega_1$ which works for all $b : \omega_1 \rightarrow 2$. Hence $g$ works for any prefixed $(b_\beta | \beta < \omega_2)$ with each $b_\beta : \omega_1 \rightarrow 2$.

For (2): We follow [MHD]. Suppose not and let $H : \omega_2 \rightarrow \omega_1 \omega_1$ be a bijection. Define $F : <\omega_1 2 \rightarrow \omega_1$ by

$$F(\sigma) = H(\sigma\lceil\omega)(|\sigma|), \text{ if } |\sigma| \geq \omega.$$ 

Then get $g : \omega_1 \rightarrow \omega_1$ such that for all $b : \omega_1 \rightarrow 2$, \{\alpha < \omega_1 | F(b\lceil\alpha) < g(\alpha)\} are cofinal in $\omega_1$.

Take $b \in \omega_1$ with $H(b\lceil\omega) = g$. Then for each $\alpha \geq \omega$, we have

$$F(b\lceil\alpha) = H(b\lceil\omega)(\alpha) = g(\alpha).$$

Hence $\{\alpha < \omega_1 | F(b\lceil\alpha) = g(\alpha)\}$ is cointial in $\omega_1$. This is a contradiction.

For (3) and (4): We show (3), since (4) has a similar proof. Suppose CH and $\Phi(\text{stat})$. Let $F : <\omega_2 \rightarrow \omega_1$ be a bijection via CH. Apply, $\Phi(\text{stat})$. We have $g : \omega_1 \rightarrow \omega_1$ such that for all $b \in \omega_2$, \{\alpha < \omega_1 | F(b\lceil\alpha) < g(\alpha)\} are stationary in $\omega_1$.

For each $\alpha < \omega_1$, let

$$S_\alpha = \{\sigma \in \omega_2 | F(\sigma) < g(\alpha)\}.$$  

Then $S_\alpha$ is countable and for any $b \in \omega_2$, it holds that $\{\alpha < \omega_1 | b\lceil\alpha \in S_\alpha\}$ is stationary in $\omega_1$. Hence $\diamondsuit$ holds.

Conversely, suppose $\diamondsuit$. We know CH holds. To show $\Phi(\text{stat})$, let $\langle S_\alpha | \alpha < \omega_1 \rangle$ be a diamond sequence such that for any $b \in \omega_1$, it holds that $\{\alpha < \omega_1 | b\lceil\alpha \in S_\alpha\}$ is stationary in $\omega_1$.

Given $F : <\omega_2 \rightarrow \omega_1$, let $g : \omega_1 \rightarrow \omega_1$ be such that for all $\alpha < \omega_1$ and all $\sigma \in S_\alpha$, $F(\sigma) < g(\alpha)$. This is possible, as $|S_\alpha| \leq \omega$. Then for any $g : \omega_1 \rightarrow 2$, it certainly holds that $\{\alpha < \omega_1 | F(b\lceil\alpha) < g(\alpha)\}$ is stationary in $\omega_1$. Hence $\Phi(\text{stat})$ holds.

\[\square\]
It is known that $\diamondsuit$ negates the following CB.

3.7 Definition. The complete bounding (CB) holds, if for each $f \in \omega_1 \omega_1$ there exists $\gamma \in (\omega_1, \omega_2)$ and $(X_\alpha \mid \alpha < \omega_1)$ such that $X_\alpha$ are continuously increasing countable subsets of $\gamma$ with $\bigcup \{X_\alpha \mid \alpha < \omega_1\} = \gamma$ and for all $\alpha < \omega_1$, we have $f(\alpha) < o.t.(X_\alpha)$.

3.8 Proposition. $\overline{\Phi}(\text{stat})$ negates CB.

Proof. Define $F : \omega_1 \to \omega_1$ so that $F(\sigma) = \alpha$, if $\sigma$ codes a countable ordinal $\alpha$. And consider $(b_\gamma \mid \omega_1 < \gamma < \omega_2)$ such that $b_\gamma : \omega_1 \to \omega_1$ codes $\gamma$. We show the contrapositive.

Suppose CB. Fix any possible $g : \omega_1 \to \omega_1$. Then we have $\gamma$ and $X_\alpha$ with $g(\alpha) < o.t.(X_\alpha)$. Let $b = b_\gamma$. Take a sufficiently large regular cardinal $\theta$ and any countable elementary substructure $N$ of $H_\theta$ with $b \in N$. Let $\delta = N \cap \omega_1$. Now we transitive collapse $N$. Then

\[ b[\delta] \text{ codes } o.t.(N \cap \gamma). \]

Since $X_\delta = N \cap \gamma$, we have

\[ F(b[\delta]) = o.t.(N \cap \gamma) = o.t.(X_\delta) > g(\delta). \]

Hence $\{\alpha < \omega_1 \mid F(b[\alpha]) \leq g(\alpha)\}$ is non-stationary.

\[ \Box \]

3.9 Corollary. $\diamondsuit$ negates CB.

Proof. $\diamondsuit$ implies $\Phi(\text{stat})$. And $\Phi(\text{stat})$ implies $\overline{\Phi}(\text{stat})$.

\[ \Box \]

We know that $\diamondsuit$ iff CH + $\clubsuit$.

3.10 Question. (1) It is known, say by [W] and [F], that $\clubsuit$ negates the saturation of the non-stationary ideal on $\omega_1$. Is it ever holds that Con($\clubsuit$ + CB) ?

(2) We know $\diamondsuit(\text{coint})$ iff CH + $\Phi(\text{coint})$ but $\diamondsuit(\text{coint})$ is always false. Is it simply that $\Phi(\text{coint})$ is false ?

§4. Not Club-$\text{wKH}$ + Stat-$\text{wKH}$

We look at the standard model of set theory in which KH gets negated ([Si] and [K]).

4.1 Theorem. Let $\kappa$ be a strongly inaccessible cardinal and $\text{Lv}(\kappa, \omega_1)$ denote the Levy collapse which turns $\kappa$ into $\omega_2$. Then $\neg\text{club-wKH}$ holds in the generic extensions $V[\text{Lv}(\kappa, \omega_1)]$.

Since $\diamondsuit$ holds in $V[\text{Lv}(\kappa, \omega_1)]$, we have

4.2 Corollary. The following are all equiconsistent.
(1) Con(\text{There exists a strongly inaccessible cardinal}).
(2) Con(\neg\text{-club-wKH} + \diamondsuit).
(3) Con(\neg\text{-club-wKH} + \diamondsuit).
(4) Con(\neg\text{-club-wKH} + \text{stat-wKH}).
(5) Con(\neg\text{KH}).

\textbf{Proof of theorem.} We repeat the standard proof, due to Silver, for showing \neg\text{KH}. Then we notice that it actually shows \neg\text{-club-wKH}.

Here are some details. We first provide

\textbf{4.2.1 Claim.} Let $S_{\alpha} \subseteq \omega 2$ be countable for all $\alpha < \omega_{1}$. Let $\dot{b}$ and $\dot{C}$ be $\text{Lv}(\kappa, \omega_{1})$-names. Then $\models_{\text{Lv}(\kappa, \omega_{1})} \neg\text{\neg club-wKH}$ if $\dot{C}$ is a club in $\omega_{1}$ and $\dot{b} : \omega_{1} \rightarrow 2$ such that $\dot{b}[\alpha] \in S_{\alpha}$ for all $\alpha \in \dot{C}$, then $\dot{b} \in V$ holds.

\textbf{Proof.} By contradiction. Suppose $p \models_{\text{Lv}(\kappa, \omega_{1})} \neg\text{\neg club-wKH}$ is a club in $\omega_{1}$ and $\dot{b} : \omega_{1} \rightarrow 2$ such that $\dot{b}[\alpha] \in S_{\alpha}$ for all $\alpha \in \dot{C}$ and $p \models_{\text{Lv}(\kappa, \omega_{1})} \neg\phi$. We derive a contradiction.

To this end, let $N$ be a countable elementary substructure of $H_{\kappa^{+}}$ with $p, \kappa, \dot{b}, \dot{C} \in N$. Denote $\delta = N \cap \omega_{1}$.

\textbf{Construct} $\langle (p_{s}, b_{s}) \mid s < \omega \rangle$ by recursion on $|s|$ such that for each $s < \omega$, let $p_{0} = p$ and $b_{0} = \emptyset$.

- $p_{s} \in \text{Lv}(\kappa, \omega_{1}) \cap N$ and $b_{s} \in S_{b_{s}} \cup \{\emptyset\}$.
- $p_{s} \models_{\text{Lv}(\kappa, \omega_{1})} \{b_{s}\} \in \dot{C} \cup \{\emptyset\}$ and $b_{s} \subseteq \dot{b}$.
- $b_{s}^{\langle i \rangle} \subset b_{s}^{\langle i \rangle} \supseteq b_{s}^{\langle 0 \rangle}$ for $i = 0, 1$ and $b_{s}^{\langle 0 \rangle}$, $b_{s}^{\langle 1 \rangle}$ are incomparable. I.e., $b_{s}^{\langle 0 \rangle} \not\subseteq b_{s}^{\langle 1 \rangle}$ and $b_{s}^{\langle 1 \rangle} \not\subseteq b_{s}^{\langle 0 \rangle}$.
- $\langle p_{f}[n] \mid n < \omega \rangle$ is a $(\text{Lv}(\kappa, \omega_{1}), N)$-generic sequence for all $f \in \omega 2$.

Let $p_{f} = \bigcup\{p_{f}[n] \mid n < \omega\}$ and $b_{f} = \bigcup \{b_{f}[n] \mid n < \omega\}$ for each $f \in \omega 2$. Then $p_{f} \models_{\text{Lv}(\kappa, \omega_{1})} \delta = N[\dot{G}] \cap \omega_{1} \subseteq \dot{C}$ and $b_{f}[\delta] = b_{f} : \delta \rightarrow 2^\omega$ for all $f \in \omega 2$, where $\dot{G}$ denotes the canonical $\text{Lv}(\kappa, \omega_{1})$-name of the generic filters. Hence $p_{f} \models_{\text{Lv}(\kappa, \omega_{1})} \neg\phi$ and $\dot{b}[\delta] \in S_{\delta}$ and so $\{b_{f} \mid f \in \omega 2\} \subseteq S_{\delta}$. Since $\{b_{f} \mid f \in \omega 2\} = 2^\omega$ and $S_{\delta}$ is countable, this is a contradiction.

\hfill \Box

Now back to the proof of theorem, we proceed by contradiction. Suppose $\langle b_{\beta} \mid \beta < \kappa \rangle$ and $\langle S_{\alpha} \mid \alpha < \omega_{1} \rangle$ satisfy club-wKH in $V[\text{Lv}(\kappa, \omega_{1})]$. Since $\text{Lv}(\kappa, \omega_{1})$ has the $\kappa$-c.c., we may assume $\langle S_{\alpha} \mid \alpha < \omega_{1} \rangle \subseteq V$. Then by claim, we know that $b_{\beta} \in V$ for all $\beta < \kappa$. Hence $2^{<\kappa} \geq \kappa$. But $\kappa$ is a strongly inaccessible cardinal. This is a contradiction.

\hfill \Box

The following is a later half of the exercise (J6) on p.300 in [K].

\textbf{4.3 Corollary.} $\neg\diamondsuit^{*}$ holds in $V[\text{Lv}(\kappa, \omega_{1})]$.

\textbf{Proof.} $\Diamond^{*}$ iff $\text{CH} + \Phi(\text{club})$. It in turn implies $\text{wKH} + \overline{\Phi}(\text{club})$. And so $\Diamond^{*}$ implies club-wKH.
§5. Not KH + Club-wKH

5.1 Theorem. Con(There exists a strongly inaccessible cardinal) implies Con(¬KH + club-wKH).

Proof. We first outline. Then provide some details.

(Out-line) Let κ be a strongly inaccessible cardinal in the ground model V. We first Levy collapse κ over V so that κ becomes new ω2, while ω1 remains the same. In this generic extension V[Lv(κ, ω1)], we have ¬KH due to Silver. We prepare some {bβ | β < κ} and {Sa | α < ω1} in V[Lv(κ, ω1)] such that

- bβ ∈ ω2 for all β < κ,
- Sα ⊆ α and Sα are countable for all α < ω1.
- If we denote Eβ = {α < ω1 | bβ[α ∈ Sα] and E = {X ∈ [κ]ω | ∀β ∈ X X ∩ ω1 ∈ Eβ}, then the Eβ are stationary in ω1 and so is E in [κ]ω.

We next side-by-side force over V[Lv(κ, ω1)] so that clubs Cβ are added with Cβ ⊆ Eβ for all β < κ. Let us denote this notion of forcing by R ∈ V[Lv(κ, ω1)]. We show that R has the κ-c.c. and is E-complete in the sense of [S] whose meaning explained later. In particular, R is σ-Baire and so preserves both ω1 and ω2. Hence club-wKH holds in the final extension V[Lv(κ, ω1)][R].

We claim ¬KH is preserved into V[Lv(κ, ω1)][R]. To this end, fix any possible Kurepa tree T in V[Lv(κ, ω1)][R]. We clarify the following among others.

- We factor V[Lv(κ, ω1)][R] into

V[Lv(κ, ω1)][R(β∗)][R(β∗, κ)]

so that T ∈ V[Lv(κ, ω1)][R(β∗)] for some β∗ < κ.

According to [J-S],

- ¬KH gets preserved over V[Lv(κ, ω1)] by any notion of forcing which is σ-Baire and of size at most ω1.

Hence T has at most ω1-many cofinal branches in the intermediate V[Lv(κ, ω1)][R(β∗)].

- We show no new cofinal branches are added through T over V[Lv(κ, ω1)][R(β∗)].

To this, we observe the quotient R(β∗, κ) is E-complete in V[Lv(κ, ω1)][R(β∗)]. We then modify Silver's construction for σ-closed notion of forcing to obverse the last item. Therefore T fails to be a Kurepa tree in V[Lv(κ, ω1)][R].

Some details follow.
(Step 1) Let $\kappa$ be a strongly inaccessible cardinal. We force with the Levy collapse $Lv(\kappa, \omega_1)$ over the ground model $V$. To save symbols, let us write $V[Lv(\kappa, \omega_1)]$ for the generic extensions.

Argue in $V[Lv(\kappa, \omega_1)]$. For each $\beta < \kappa$, Let us write $g_\beta : \omega_1 \rightarrow \beta$ for the $\beta$-th generic function added via $Lv(\kappa, \omega_1)$.

We prepare $(b_\beta | \beta < \kappa)$ and $(S_\alpha | \alpha < \omega_1)$. To define $b_\beta : \omega_1 \rightarrow 2$, we make use of $g_{\omega_1 + \beta}$. To define $S_\alpha$, say, for limit $\alpha$, we make use of $g_{\alpha} | \omega$ ($\alpha \leq i < \alpha + \alpha$). More precisely,

$$b_\beta(\alpha) = 1 \text{ iff } g_{\omega_1 + \beta}(\alpha) \text{ is odd.}$$

$$S_\alpha = \{ \sigma_\alpha^n | n < \omega \}, \quad \sigma_\alpha^n : \alpha \rightarrow 2.$$  

$$\sigma_\alpha^n(i) = 1 \text{ iff } g_{\alpha + i}(n) \text{ is odd.}$$

We know how to construct conditions via generic sequences with respect $Lv(\kappa, \omega_1)$ upon fixing countable elementary substructures. In such constructions, we know which parts of what $g_\beta$ are decided and what $g_\beta$ are left open. Hence it is not hard to show that $E = \{ X \in [\kappa]^\omega | \forall \beta \in X \cap \omega_1 \in E_\beta \}$ is stationary in $[\kappa]^\omega$. It then follows that each $E_\beta = \{ \alpha < \omega_1 | b_\beta(\alpha) \in S_\alpha \}$ must be stationary in $\omega_1$.

For an explicit proof, we show $E$ is stationary in $[\kappa]^\omega$. Suppose $p \Vdash_{Lv(\kappa, \omega_1)} \phi : \omega^\omega \rightarrow \kappa$”. We want to find $q^* \leq p$ and $X \in [\kappa]^\omega$ such that $q^* \Vdash_{Lv(\kappa, \omega_1)} X \in \bar{E}$ and $X$ is $\phi$-closed”, where $\bar{E}$ denotes the canonical name of $E$. To this end let $\theta$ be a sufficiently large regular cardinal and $N$ be a countable elementary substructure of $H_\theta$ with $p, \phi \in N$. Let $\delta = N \cap \omega_1$ and $X = N \cap \kappa$. Take a $Lv(\kappa, \omega_1)$-generic sequence $\langle p_n | n < \omega \rangle$ with $p_0 = p$. Let $q = \{ \beta_\delta \| \bar{E} \}$. Then $q \in Lv(\kappa, \omega_1)$ is $Lv(\kappa, \omega_1)$-generic and $\text{dom}(q) = N \cap (\kappa \times \omega_1) = X \times \delta$. Hence $q$ decides $g_{\omega_1 + \beta}[\delta]$ for all $\beta \in X$ and $q \Vdash_{Lv(\kappa, \omega_1)} X = N[\bar{G}] \cap \kappa$ is $\phi$-closed”.

We may place the countable set $\{ g_{\omega_1 + \beta}[\delta] | \beta \in X \}$ on $[\delta, \delta + \delta) \times \omega$. Namely, we may extend $g$ to $q^*$ so that $q^* \Vdash_{Lv(\kappa, \omega_1)} g_{\beta}[\delta] \in S_\delta$ for all $\beta \in X$. Hence $q^* \Vdash_{Lv(\kappa, \omega_1)} X \in \bar{E}$.

(Step 2) We side-by-side force clubs for all $E_\beta$ over $V[Lv(\kappa, \omega_1)]$. Let $X \subseteq \kappa$. Define $p \in R(X)$, if $p = \langle C^p_\beta | \beta \in X^p \rangle$ such that

- $X^p \in [X]^\omega$,
- $C^p_\beta$ is a countable closed subset of $E_\beta$ for all $\beta \in X^p$.

For $p, q \in R(X)$, set $q \leq R(X) p$, if

- $X^q \supseteq X^p$,
- $C^q_\beta$ end-extends $C^p_\beta$ at each $\beta \in X^p$.

Notice that we do not require $\max C^p_\beta = \max C^p_\beta$ for $\beta_1, \beta_2 \in X^p$.

5.1.1 Lemma. (1) $R(X)$ has the $\omega_2$-c.c.
(2) $R(X)$ is $E$-complete. I.e., for all sufficiently large regular cardinals $\theta$ and all countable elementary substructures $N$ of $H_\theta$ such that $R(X)\in N$ and $N\cap \kappa \in E$, if $(r_n \mid n < \omega)$ is a $(R(X), N)$-generic sequence, then there exists $r \in R(X)$ such that for all $n < \omega$, $r \leq_{R(X)} r_n$.

Proof. For (1): In $V[\text{Lv}(\kappa, \omega_1)]$, we have $\diamondsuit$ and so CH holds. By a standard $\Delta$-system lemma, we may conclude $R(X)$ has the $\omega_2$-c.c.

For (2): Let us fix any regular cardinal $\theta$ with $\theta > \kappa$. Let $N$ be any countable elementary substructure of $H_\theta$ such that $R(X) \in N$ and $N \cap \kappa \in E$. Hence we have

$$\forall \beta \in N \cap \kappa \ N \cap \omega_1 \in E_\beta.$$ 

Let $(r_n \mid n < \omega)$ be any $(R(X), N)$-generic sequence. Then by genericity, we have $N \cap X = \bigcup \{X^n \mid n < \omega\}$. For each $\beta \in N \cap X$, let $C_\beta = \bigcup \{C_\beta^n \mid \beta \in X^n, \ n < \omega\} \cup \{N \cap \omega_1\}$ and $r = (C_\beta \mid \beta \in N \cap X)$. Then $C_\beta \subset E_\beta$ are clubs. Hence $r \in R(X)$ such that for all $n < \omega$, we have $r \leq r_n$.

Let $R = R(\kappa)$. Since $R$ adds clubs $C_\beta$ with $C_\beta \subset E_\beta$ for all $\beta < \kappa$, we have club-$\text{wKH}$ in the extensions $V[\text{Lv}(\kappa, \omega_1)][R]$.

(Step 3) We want to show $V[\text{Lv}(\kappa, \omega_1)][R] \models \neg \text{KH}$. To this end let $T$ be a possible Kurepa tree in $V[\text{Lv}(\kappa, \omega_1)][R]$. Then by the $\kappa$-c.c. of $R$, we have $\beta^* < \kappa$ such that $T \in V[\text{Lv}(\kappa, \omega_1)][R(\beta^*)]$. Let $V_1 = V[\text{Lv}(\kappa, \omega_1)]$ for short. Then

- $R$ and $R(\beta^*) \times R([\beta^*, \kappa))$ are isomorphic in $V_1$.
- $V_1 \models "R(\beta^*)"$ is $E$-complete and so $\sigma$-Baire”.

Hence,

- $V_1[R(\beta^*)] \models "E \text{ remains stationary in } [\kappa]^\omega"$.
- Since $R(\beta^*)$ is $\sigma$-Baire and so by absoluteness,
- $V_1[R(\beta^*)] \models "R([\beta^*, \kappa))"$ is $E$-complete”.

Since $R(\beta^*)$ is of size $\omega_1$ in $V_1$, we have $\bar{\kappa} < \kappa$ such that

- $R(\beta^*) \in V[\text{Lv}(\bar{\kappa}, \omega_1)]$.

Since $R(\beta^*)$ is $\sigma$-Baire in $V[\text{Lv}(\bar{\kappa}, \omega_1)] \subset V[\text{Lv}(\kappa, \omega_1)]$, the p.o. set $\text{Lv}(\bar{\kappa}, \omega_1))$ has the same meaning in both $V[\text{Lv}(\bar{\kappa}, \omega_1)]$ and $V[\text{Lv}(\bar{\kappa}, \omega_1)][R(\beta^*)]$. Now we apply the Product Lemma in $V[\text{Lv}(\bar{\kappa}, \omega_1)]$ so that

- We have

$$V_1[R(\beta^*)] = V[\text{Lv}(\bar{\kappa}, \omega_1)][R(\beta^*)][\text{Lv}(\bar{\kappa}, \kappa, \omega_1)]$$

and so $V_1[R(\beta^*)] \models \neg \text{KH}$ holds.
Therefore $T$ has at most $\omega_1$-many cofinal branches in $V[R(\beta^*)]$. We know

$$V_1[R] = V_1[R(\beta^*)][R([\beta^*, \kappa)])$$

and $R([\beta^*, \kappa])$ is $E$-complete in $V[R(\beta^*)]$. Hence it suffices to show the following.

5.1.2 Lemma. Let $P$ be a p.o. set which is $E$-complete for some stationary $E \subset [\kappa]^{\omega_1}$ and $T$ be a tree of height $\omega_1$ whose levels are all of size countable. Then $T$ gets now new cofinal branches in the generic extensions $V[P]$.

Proof. Suppose $\mathbf{p} \models \neg \Phi[b]$ is a cofinal branch through $T$ with $b \not\in V"$. We derive a contradiction. To this end, let $\theta$ be a sufficiently large regular cardinal and $N$ be a countable elementary substructure of $H_\theta$ with $\mathbf{p}, P, T, b \in N$ and $N \cap \kappa \in E$. This is possible, as $E$ is stationary. Denote $\delta = N \cap \omega_1$.

Construct $\langle (p_s, b_s) \mid s < \omega_2 \rangle$ by recursion on $|s|$ such that for each $s < \omega_2$,

- $p_s = \mathbf{p}$ and we may assume $\{b_s\} = T_0$.
- $p_s \in P \cap N$ and $b_s \in T \cap N$.
- $p_s \models \neg \Phi[b_s] \in \dot{b}$.
- $p_s^{\langle i \rangle} \leq p_s$, $b_s <_T b_s^{\langle i \rangle}$ for $i = 0, 1$ and $b_s^{\langle 0 \rangle}$, $b_s^{\langle 1 \rangle}$ are incomparable. I.e, $b_s^{\langle 0 \rangle} \notin T b_s^{\langle 1 \rangle}$ and $b_s^{\langle 1 \rangle} \notin T b_s^{\langle 0 \rangle}$.
- $\langle \mathbf{p} f_n \mid n < \omega \rangle$ is a $(P, N)$-generic sequence for all $f \in \omega_2$.

Since $P$ is $E$-complete, we may fix $p_f \in P$ such that $p_f \leq P p_f f_n$ for all $n < \omega$. We may assume, by extending $p_f$ further, there exists $b_f \in T_\delta$ such that $p_f \models \Phi[b_f] \in \dot{b}$. Since $|\{b_f \mid f \in \omega_2\}| = 2^\omega$ and $T_\delta$ is countable, this is a contradiction.

§6. $\clubsuit$ and $\Phi($stat$)$ are different

We separate $\Phi($stat$)$ and $\clubsuit$.

6.1 Theorem. $\text{Con}(\text{MA}_{\omega_1} (\text{Fn}(\omega_1, 2)) + \Phi($stat$))$.

6.2 Corollary. $\text{Con}(\neg \clubsuit + \Phi($stat$))$.

Proof. $\text{MA}_{\omega_1} (\text{Fn}(\omega_1, 2))$ implies $\neg \clubsuit$.

Proof of theorem. We first out-line. Then provide some details.

(Out-line) Since $\Phi($stat$)$ entails $\Phi(\text{cof})$, we must have $2^\omega < 2^{\omega_1}$. Suppose CH and $2^{\omega_1} = \omega_2$. Add $\omega_3$-many functions from $\omega_1$ into $\omega_1$. Then we have

- $\text{CH} + 2^{\omega_1} = \omega_3$. 

\[ \forall F : \mathcal{L}_{1} \to \omega \exists g : \omega \to \omega \forall b \in \mathcal{L}_{1} \{ \alpha < \omega_1 \mid F(b[\alpha]) = g(\alpha) \} \text{ is stationary.} \]

Next, we add \( \omega_2 \)-many subsets of \( \omega \). Since we can capture relevant names, we have

- \( 2^\omega = \omega_2 + \text{MA}_{\omega_1}(\text{Fn}(\omega_1, 2)) + 2^{\omega_1} = \omega_3 \).
- \( \forall F : \mathcal{L}_{1} \to \omega \exists g : \omega \to \omega \forall b \in \mathcal{L}_{1} \{ \alpha < \omega_1 \mid F(b[\alpha]) < g(\alpha) \} \) is stationary.

Here are some details:

(Step 1) Let \( P = \text{Fn}(\omega_3 \times \omega_1, \omega_1) \). Then \( P \) is \( \sigma \)-closed. By CH, \( P \) has the \( \omega_2 \)-c.c. Let \( \langle g_\xi \mid \xi < \omega_3 \rangle \) denote the canonical objects added by \( P \). In particular, \( g_\xi : \omega_1 \to \omega_1 \). By counting the number of \( P \)-names, we have

\[ V[[g_\xi \mid \xi < \omega_3]] = \text{"CH + } 2^{\omega_1} = \omega_3}. \]

Let \( F : \mathcal{L}_{1} \to \omega_2 \) in \( V[[g_\xi \mid \xi < \omega_3]] \). Since \( P \) has the \( \omega_2 \)-c.c, we have \( \xi^* < \omega_3 \) such that \( F \in V[[g_\xi \mid \xi < \xi^*]] \). Notice

\[ V[[g_\xi \mid \xi < \omega_3]] = V[[g_\xi \mid \xi < \xi^*]][g_{\xi^*}][g_\xi \mid \xi^* < \xi < \omega_3]]. \]

Let \( V_1 = V[[g_\xi \mid \xi < \xi^*]] \) and \( Q = \text{Fn}(\xi^*, \omega_3) \times \omega_1, \omega_1, \omega_1) \). Then the following suffices.

6.2.1 Claim. \( \forall \mathcal{V}_{\mathcal{Q}}, \forall b : \omega_1 \to \omega_2 \{ \alpha < \omega_1 \mid F(b[\alpha]) = g_\xi(\alpha) \} \) is stationary.

Proof. Argue in \( V_1 \). Suppose \( r = \mathcal{V}_Q \upharpoonright b : \omega_1 \to \omega_2 \) and \( \dot{C} \subseteq \omega_1 \) is a club*. Let \( \theta \) be a sufficiently large regular cardinal and \( N \) be a countable elementary substructure of \( H_\theta \) with \( r, Q, b, \dot{C} \in N \). Let \( \langle r_\eta \mid n < \omega \rangle \) be a \( (Q, \mathcal{N}) \)-generic sequence with \( r_0 = r \). Let \( r' = \bigcup \langle r_n \mid n < \omega \rangle \) and \( \delta = N \cap \omega_1 \). Then there is \( \sigma \in \xi_\omega \) such that \( r' \upharpoonright \mathcal{V}_{\mathcal{Q}}, \forall b[\delta] = \sigma \). Let \( r^* = r' \cup \{ (\xi^*, \delta, F(\sigma)) \} \). Then \( r^* \leq r' \) and \( r^* \upharpoonright \mathcal{V}_{\mathcal{Q}} F(b[\delta]) = g_\xi(\delta) \) and \( \delta \in \dot{C} \).

(Step 2) For notational simplicity, suppose the following in \( V \).

- \( \text{CH } + 2^{\omega_1} = \omega_3 \).

\[ \forall F : \mathcal{L}_{1} \to \omega \exists g : \omega \to \omega \forall b \in \mathcal{L}_{1} \{ \alpha < \omega_1 \mid F(b[\alpha]) = g(\alpha) \} \text{ is stationary.} \]

We force with \( Q = \text{Fn}(\omega_2 \times \omega, 2) \) over \( V \). Then in \( V[Q] \),

6.2.2 Claim. \( \forall F : \mathcal{L}_{1} \to \omega \exists g : \omega \to \omega \forall b \in \mathcal{L}_{1} \{ \alpha < \omega_1 \mid F(b[\alpha]) < g(\alpha) \} \) is stationary.

Proof. Let \( \mathcal{V}_Q \upharpoonright F : \mathcal{L}_{1} \to \omega_1 \). Let \( \mathcal{A} = \{ A \subset Q \mid A \text{ is an antichain of } Q \} \). Then \( |\mathcal{A}| = \omega_2 \). Define \( F_0 : \mathcal{L}_{1} \to \omega_1 \) so that for any \( \sigma \in \mathcal{A} \), we have \( \mathcal{V}_{\mathcal{Q}} F(s(\sigma)) < \)
$F_0(\sigma)$, where $s(\sigma)$ is a member of $\mathcal{A}$ naturally defined from $\sigma$ in $V[Q]$. This is possible, as $Q$ has the c.c.c.

Now by assumption, we have $g_0 : \omega_1 \to \omega_1$ such that

$$\forall b \in \omega_1. A \{ \alpha < \omega_1 \mid F_0(b, \alpha) = g_0(\alpha) \} \text{ is stationary.}$$

### 6.2.2.1 Sub claim. $\exists \forall b \in \omega_1. \{ \alpha < \omega_1 \mid \dot{F}(b, \alpha) < g_0(\alpha) \}$ is stationary”.

**Proof.** By the Maximal Principle of the $Q$-names, we may take $b : \omega_1 \to A$ such that for all $\alpha < \omega_1$, $\exists Q^{b, \alpha} s(\sigma) = s(b, \alpha')$. By the choice of $g_0$, we have

$$\{ \alpha < \omega_1 \mid F_0(b, \alpha) = g_0(\alpha) \} \text{ is stationary.}$$

Notice $F_0(b, \alpha) = g_0(\alpha)$ implies $\exists Q^{\dot{F}(b, \alpha) = \dot{F}(s(b, \alpha)) < F_0(b, \alpha) = g_0(\alpha)}$. Since the stationary subsets of $\omega_1$ remain stationary in $V[Q]$, we conclude

$$\{ \alpha < \omega_1 \mid \dot{F}(b, \alpha) < g_0(\alpha) \} \text{ is stationary.}$$

### 6.2.3 Claim. $\text{MA}_{\omega_1}(\text{Fn}(\omega_1, 2))$ holds in $V[Q]$.

**Proof.** Given $D = \{ D_i \mid i < \omega_1 \}$ dense subsets of $\text{Fn}(\omega_1, 2)$, there exists $\beta < \omega_2$ such that $D \in V[Q][\beta]$. Hence the next $\omega_1$-many coordinates provide a $D$-generic filter.

We may separate $\clubsuit$ and $\Phi(\text{stat})$ the other way round, too.

### 6.3 Theorem. $\text{Con}(\clubsuit + \neg \Phi(\text{stat}))$.

**Proof.** Since $2^{\omega} = 2^{\omega_1}$ negates $\Phi(\text{stat})$, we look for this property. We consider a model in $[S]$, where $\text{Con}(\clubsuit + \neg \text{CH})$ is shown.

Let $2^{\omega} = \omega_1$, $2^{\omega_1} = \omega_2$, $2^{\omega_2} = \omega_3$ and $\diamond(S_0^2)$ in $V$. First add $\omega_3$-many new subsets of $\omega_1$. Then collapse $\omega_1$ to countable. Let

$$V^* = V[\text{Fn}(\omega_3, 2, \omega_1)][\text{Fn}(\omega, \omega_1)].$$

Then we have $2^{\omega} = 2^{\omega_1} = \omega_2$ and $\clubsuit$ in $V^*$.

We record:

- $V[\text{Fn}(\omega_3, 2, \omega_1)] \models "2^{\omega} = \omega_1 + 2^{\omega_1} = 2^{\omega_2} = \omega_3 + \clubsuit(S_0^2)."
- $V^* \models "2^{\omega} = 2^{\omega_1} = 2^{\omega_2} = \omega_2 + \clubsuit."$
§7. A summary of implications, the chart

(A)

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(B)

$\Phi(\text{club})$ $\Rightarrow$ $\Phi(\text{stat})$ $\Rightarrow$ $\Phi(\text{cof})$

$\overline{\Phi}(\text{club})$ $\overline{\Phi}(\text{stat})$ $2^{\omega} < 2^{\omega_1} + \overline{\Phi}(\text{cof})$

(C)

$(<^*) \Rightarrow \overline{\Phi}(\text{coint}) \Rightarrow \overline{\Phi}(\text{club}) \Rightarrow \overline{\Phi}(\text{stat}) \Rightarrow \overline{\Phi}(\text{cof})$

$\downarrow$

$\neg \text{CB}$

(D)

False

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

CH + $\Phi(\text{coint})$ $\Rightarrow$ CH + $\Phi(\text{club})$ $\Rightarrow$ CH + $\Phi(\text{stat})$ $\Leftrightarrow$ CH + $\Phi(\exists \alpha \geq \omega)$

wKH + $\overline{\Phi}(\text{coint})$ $\Rightarrow$ wKH + $\overline{\Phi}(\text{club})$ $\Rightarrow$ wKH + $\overline{\Phi}(\text{stat})$ $\Rightarrow$ wKH + $\overline{\Phi}(\text{cof})$

(E)

CH + $2^{\omega_1} = \omega_3 + \text{GMA}_{\omega_3}$ $\Rightarrow$ CH + $(<^*)$ $\Rightarrow$ wKH + $\overline{\Phi}(\text{coint})$

7.1 Note. ([W]) Con(\text{NS}_{\omega_1} is $\omega_1$-dense and wKH).
References


