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A square principle in the context of $\mathcal{P}_\kappa \lambda$ (Forcing Method and Large Cardinal Axioms)

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A square principle in the context of $\mathcal{P}_\kappa \lambda$

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Abstract

We introduce a combinatorial principle for $\mathcal{P}_\kappa \lambda$ based upon $\square_\kappa$. Although we cannot transfer one of the clauses of $\square_\kappa$ to this context, we can replicate some of the desired consequences of that clause. We discuss this situation and its implications along with proving the relative consistency of some $\mathcal{P}_\kappa \lambda$ versions of $\square_\kappa$.

1 Introduction

In this paper, we discuss the problem of generalising the square principle to the context of $\mathcal{P}_\kappa \lambda$. The research presented below is discussed in the author's thesis, [9]. (In fact, the principles presented there are slight variations on those defined below.) This combinatorial research follows a well-established tradition and is guided by the idea of transferring interesting notions from the theory of the combinatorics of ordinal numbers. For example, Jensen's diamond principle (see [5]) has been usefully generalised to this context (originally by Jech in [4], but also by Matet in [8] and by Džamonja in [3]).

The square principle cannot be directly transferred to the context of $\mathcal{P}_\kappa \lambda$ for various reasons, as discussed below. The general approach that we follow is to establish a basic nontrivial square principle for $\mathcal{P}_\kappa \lambda$ then explicitly add

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further properties of square in more complex forcings. In this paper, following the basic principle, we define a second principle in which the square principle's non-reflection property is added.

Throughout this paper, $\kappa$ is a regular infinite cardinal and $\lambda$ is a cardinal with $\kappa \leq \lambda$. We now give some basic definitions and clarify the notation used in this paper.

Let $\mathcal{P}_\kappa \lambda = \{x \subseteq \lambda : |x| < \kappa\}$. $\mathcal{P}_\kappa \lambda$ is typically ordered by $\subseteq$ and will be throughout this paper. Combinatorial ideas such as clubs and stationarity can be defined in this context, as described in [6]. Note that $\kappa$ and $\lambda$ are arguments and may be replaced by specified cardinals or sets respectively. In this paper, we will frequently consider $\mathcal{P}_{|x|}(x)$ where $x$ is a set. Note that $\mathcal{P}_\kappa \lambda$ is also commonly written as $[\lambda]^{<\kappa}$.

The notation used in this paper is mostly standard. By $x \subseteq y$ we mean $(x \subseteq y$ and $x \neq y)$. We write $\lim(\alpha)$ as an abbreviation for “$\alpha$ is a limit ordinal”; $\text{otp}(X)$ denotes the ordertype of a wellordered set $X$ and for an ordinal $\alpha$, $\text{cf}(\alpha)$ denotes the cofinality. For a function $f$, $\text{dom}(f)$ denotes the domain of $f$ and $\text{im}(f)$ denotes image of $f$, while $f[X]$ denotes the restriction of $f$ to $X$ where $X \subseteq \text{dom}(f)$. In forcing proofs, we follow the convention that for two conditions $p, q$, $p \leq q$ means $p$ is a weaker condition than $q$. Lastly, we use $\text{reg}(X)$ to denote the set of elements of $X$ of regular cardinality.

Before we develop a version of the square principle in the context of $\mathcal{P}_\kappa \lambda$, we introduce the standard principle. This principle, denoted $\square_\kappa$, was developed by Jensen and has proved a useful tool in various areas of mathematical logic. It is defined as follows (although it should be noted that other equivalent formulations exist).

**Definition 1.1** $\square_\kappa$ is the statement that there is a sequence $\langle C_\alpha : \alpha \in \kappa^+; \lim(\alpha) \rangle$ with the following properties:

(i) $C_\alpha$ is a club subset of $\alpha$

(ii) if $\text{cf}(\alpha) < \kappa$ then $\text{otp}(C_\alpha) < \kappa$

(iii) (Coherence:) if $\beta \in C_\alpha$ and $\lim(\beta)$ then $C_\beta = C_\alpha \cap \beta$.

Forcing can be used to produce a model of set theory in which $\square_\kappa$ holds. This approach uses a partial order whose elements are initial segments of potential square sequences. It is also known that $\square_\kappa$ holds in $L$, the universe
of constructible sets. The best-known proof uses fine structure theory and is
due to Jensen; a good account of this proof is given in [2].

The square principle encapsulates various interesting properties. Coherence
and anticoherence are discussed in the next section. Another property, non-
reflection, is discussed further in the final section of this paper.

2 A square principle in the context of $\mathcal{P}_{\kappa}\lambda$

We will define a square-like principle that asserts the existence of a coherent
set of subsets of $\mathcal{P}_{\kappa}\lambda$ indexed by the elements of $\mathcal{P}_{\kappa}\lambda$. Note that in considering
$C_x \subseteq \mathcal{P}_{|x|}(x)$ for $x \in \mathcal{P}_{\kappa}\lambda$ we require $|x|$ to be regular and hence no club of
$\mathcal{P}_{|x|}(x)$ will have cardinality $<|x|$. Thus, the cardinalities of the clubs cannot
be limited as they are for those corresponding to singular ordinals in $\square_\kappa$. It
is necessary, therefore, to introduce alternative non-triviality conditions that
add some of the basic properties of $\square_\kappa$. Also, note that if $\kappa$ is a successor
 cardinal, coherence is trivial for a club of $\mathcal{P}_{\kappa}\lambda$, that is for the elements of
$[\lambda]^\kappa \setminus \kappa$. Thus, while $\square_\kappa$ actually asserts a property of $\kappa^+$, the principle defined
below does not “look ahead” at $\mathcal{P}_{\kappa^+}\lambda$.

For the reasons mentioned above, we must assume that $\kappa$ is a regular limit
cardinal. In fact, since we require stationary-many regular cardinals below
$\kappa$, for the remainder of this paper we assume that $\kappa$ is a Mahlo cardinal.

**Definition 2.1** Suppose $\kappa$ is a Mahlo cardinal and $\lambda$ is an infinite cardinal
with $\kappa \leq \lambda$. Suppose also that $S$ is a stationary subset of $\mathcal{P}_{\kappa}\lambda$. Then $\square_{\mathcal{P}_{\kappa}\lambda}(S)$
is the statement that there is a family of sets $\{C_x : x \in S\}$ with the following
properties:

(i) $C_x$ is a club subset of $\mathcal{P}_{|x|}(x)$ for all $x \in S$

(ii) (Coherence:) if $x \in S$ and $y \in C_x \cap S$ then $C_y = C_x \cap \mathcal{P}_{|y|}(y)$

(iii) (Anticoherence:) the set $\{x \in S : \text{there is a cofinal set of } y \in S \cap \mathcal{P}_{|x|}(x)
such that C_y \neq C_x \cap \mathcal{P}_{|y|}(y)\}$ is stationary in $\mathcal{P}_{\kappa}\lambda$.

We write $\square_{\mathcal{P}_{\kappa}\lambda}$ to mean that there is a stationary $S \subseteq \mathcal{P}_{\kappa}\lambda$ such that $\square_{\mathcal{P}_{\kappa}\lambda}(S)$
holds.

Note that the restriction to a stationary subset is not as serious a restriction
as it may appear since in this context we can at best have $C_x$ defined for all $x \in \text{reg}(\mathcal{P}_\kappa \lambda)$, which is stationary if Mahlo is $\kappa$, but cannot be club. Note that stronger forms of stationarity could be substituted with appropriate adjustments to the forcing below.

The anticoherence property is implicit in the definition of $\Box_\kappa$ but must be explicitly required for $\Box_{\mathcal{P}_\kappa \lambda}$. This ensures that the principle cannot be satisfied trivially, e.g. by setting $C_x = \mathcal{P}_{|x|}(x)$ for all $x \in \text{reg}(\mathcal{P}_\kappa \lambda)$.

The $\Box_{\mathcal{P}_\kappa \lambda}$ principle is consistent with $\text{ZFC} + \text{"$\kappa$ is Mahlo"}$, as we assert in the following theorem. Important questions remain unanswered, however. Noteably, it is not known whether the principle holds in $\text{ZFC} + \text{"V=L"}$ or even in $\text{ZFC}$.

Before we proceed with the consistency proof, note that we could also develop a principle based on clubs of $\mathcal{P}_{\kappa_x}(x)$ for each $x \in S$. Recall that $\kappa_x = x \cap \kappa$ if this is an ordinal and is undefined otherwise. Here, we would insist that $S$ contains only elements $x$ for which $\kappa_x$ is a regular cardinal. Assuming that $\kappa$ is Mahlo, the consistency of such a principle can be proved with a forcing analogous to the one for $\Box_{\mathcal{P}_\kappa \lambda}$.

**Theorem 2.2** Suppose $M$ is a countable model of a sufficiently rich fragment of $\text{ZFC}$ in which $\kappa$ is Mahlo and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which $\kappa$ is Mahlo and $\Box_{\mathcal{P}_\kappa \lambda}$ holds.

This theorem is proved by forcing with the partial order defined below. Essentially, the partial order is composed of fragments of possible witnesses to $\Box_{\mathcal{P}_\kappa \lambda}$.

**Definition 2.3** Let $P$ be a set whose elements $p$ are characterised as follows:

(i) $p$ is a function with $\text{dom}(p) \in \mathcal{P}_\kappa(\text{reg}(\mathcal{P}_\kappa \lambda))$

(ii) for all $x \in \text{dom}(p)$, $p(x)$ is either club in $\mathcal{P}_{|x|}(x)$ or the empty set

(iii) if $x \in \text{dom}(p)$ and $y \in p(x) \cap \text{reg}(\mathcal{P}_\kappa \lambda)$ then $y \in \text{dom}(p)$ and either $p(y) = p(x) \cap \mathcal{P}_{|y|}(y)$ or $p(y) = \emptyset$

(iv) if $\text{dom}(p) \cap \mathcal{P}_{|y|}(y)$ is stationary in $\mathcal{P}_{|y|}(y)$ then $y \in \text{dom}(p)$.

For $p,q \in P$, $p \leq q$ (meaning $q$ is stronger than $p$) iff $p \subseteq q$. We will also use the symbols $<_1$, $\geq_1$ and $>$ in the natural way.
Note that if we let $\emptyset$ be the function with empty domain then $\emptyset$ is the unique minimal element of $P$. Clearly, $P$ is non-empty. We must now establish various properties of $(P, \leq)$ to show that a suitable generic object exists and that the resulting forcing preserves cofinalities and cardinalities.

**Lemma 2.4** $(P, \leq)$ is separative.

**Proof.** Let $p \in P$ and let $x \in \text{reg}(\mathcal{P}_{\kappa})$ be such that $\bigcup \text{dom}(p) \in \mathcal{P}_{|x|}(x)$, which is possible because $|\bigcup \text{dom}(p)| < \kappa$. We now define $q \in P$ such that $p \leq q$ and $x \in \text{dom}(q)$ and $q(x) \neq \emptyset$. For $y \in \text{dom}(p)$, let $q(y) = p(y)$. Let $q(x)$ be any club of $\mathcal{P}_{|x|}(x)$ that does not intersect $\text{dom}(p)$. Such a club exists because $p$ satisfies (iv) of Definition 2.3. It is straightforward to check that $q \in P$, by checking against conditions (i)-(iv).

Now let $r \geq p$ be defined as follows. Let $r(y) = q(y)$ if $y \neq x$, let $r(x) = \emptyset$ and let $r(y)$ be undefined otherwise. Then $r \in P$ and $q, r$ are clearly incompatible extensions of $p$.

$\Box$.

Since $P$ is separative, there is a generic object $G$ in $M[G]$ that is not in the ground model, $M$. We will see that this generic provides an example of a $\square_{\mathcal{P}_{\kappa}, \lambda}$ set. First, however, we must show that the forcing preserves cofinalities and cardinalities.

**Lemma 2.5** $P$ satisfies the $\kappa^+$-chain condition.

**Proof.** Suppose $X \subseteq P$ and $|X| = \kappa^+$. We show that $X$ is not an antichain. Let $A = \{\text{dom}(p) : p \in X\}$. By a $\Delta$-system argument, using the fact that $\kappa$ is strongly inaccessible, we can find $B \subseteq A$ such that $|B| = \kappa^+$ and $B$ is a $\Delta$-system with root $R$. That is, for all $X, Y \in B$, $X \cap Y = R$.

Consider the numbers of functions with domain $R$ such that for each function $f$ and each $x \in R$, $f(x) \subseteq \mathcal{P}_{|x|}(x)$. Clearly, if we impose no further conditions on the value of $f(x)$, the number of distinct functions is equal to $\Pi_{x \in R} |\mathcal{P}(\mathcal{P}_{|x|}(x))|$. Now for all $x \in R$, $|\mathcal{P}_{|x|}(x)| < \kappa$ and since $\kappa^{< \kappa} = \kappa$, it follows that $|\mathcal{P}(\mathcal{P}_{|x|}(x))| \leq \kappa$. Furthermore, since $|R| < \kappa$, it follows that $\Pi_{x \in R} |\mathcal{P}(\mathcal{P}_{|x|}(x))| \leq \kappa$. In other words there are only $\kappa$-many suitable functions defined on $R$. But $B = \kappa^+$ so by the pigeonhole principle there must
be some function \( f \) defined on \( R \) such that \( p[R] = f \) for \( \kappa^+ \) many \( p \in X \) with \( \text{dom}(p) \in B \).

Now let \( Y = \{ p \in X : \text{dom}(p) \in B \text{ and } p[R] = f \} \). For \( p, q \in P \), if \( p(x) = q(x) \) for all \( x \in \text{dom}(p) \cap \text{dom}(q) \), it is easily proved that \( p \cup q \) is a common extension of \( p, q \) and hence that \( p, q \) are compatible. Thus, the elements of \( Y \) are pairwise compatible because they agree on \( R \), which is the intersection of their domains, by the definition of \( B \). Hence, \( X \) is not an antichain.

We can now conclude that the forcing preserves cofinalities and cardinalities > \( \kappa \). We now prove that \( P \) is < \( \kappa \)-directed closed. It will then follow that the forcing preserves cofinalities and cardinalities ≤ \( \kappa \).

**Lemma 2.6** \( P \) is < \( \kappa \)-directed closed.

**Proof.** Suppose \( \mu < \kappa \) and \( \{ p_\alpha : \alpha < \mu \} \) is a set of pairwise compatible conditions from \( P \). We define \( p_\mu^* = \bigcup_{\alpha<\mu} p_\alpha \). This is a function since the conditions are pairwise compatible. It is easily checked that \( p_\mu^* \) satisfies (i)-(iii) of Definition 2.3. However, there may be \( x \in \text{reg}(\mathcal{P}_\kappa \lambda) \setminus \text{dom}(p_\mu^*) \) such that \( \text{dom}(p_\mu^*) \) is stationary in \( \mathcal{P}_{\|x\|}(x) \) so condition (iv) may not hold. We now make a small adjustment to \( p_\mu^* \) to obtain \( p_\mu \) still satisfying (i)-(iii) but also satisfying (iv). Let \( p_\mu(x) = p_\mu^*(x) \) for all \( x \in \text{dom}(p_\mu^*) \). For \( x \in (\mathcal{P}(\cup \text{dom}(p_\mu^*)) \setminus \text{dom}(p_\mu^*)) \), let \( p_\mu(x) = \emptyset \). Then \( p_\mu \) is as required since (i)-(iv) hold and for all \( \alpha < \mu \), \( p_\alpha < p_\mu \).

Since the forcing is < \( \kappa \)-closed, no new sets of ordinals of size < \( \kappa \) are introduced. Hence, \( (\mathcal{P}_\kappa \lambda)^{M[G]} = (\mathcal{P}_\kappa \lambda)^M \) and we can write \( \mathcal{P}_\kappa \lambda \) for the name \( \mathcal{P}_\kappa \lambda \).

We must now ensure that for any generic \( G \) of \( P \), the set \( \{ x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset) \} \) is stationary in \( \mathcal{P}_\kappa \lambda \) in the generic extension. Before we do this we give a lemma that will be needed several times in the proof.

**Lemma 2.7** Suppose \( p \in P \) and \(|- (\mathcal{C} \text{ is a club of } \mathcal{P}_\kappa \lambda) \text{ and suppose } y \in \mathcal{P}_\kappa \lambda \). Then there is \( x \in \mathcal{P}_\kappa \lambda \) and \( q \in P \) such that \( q \geq p \) and \( q \mid - (y \subset x \text{ and } x \in \mathcal{C}) \).
Proof. Let $p$ be a name such that $p \models (y \subseteq x \text{ and } x \in \mathcal{C})$. This is possible because $p \models (\exists \mu < \kappa)(|x| < \mu)$ because $\kappa$ is a limit cardinal.

Let $p_0 \geq p$ and $\nu$ a cardinal such that $p_0 \models (\mu = \nu)$. So $p_0 \models (|x| < \nu$ and $\nu < \kappa)$. Thus we can find a condition $p_1 \geq p_0$ and a name for an enumeration of $x$ in an ordertype $< \alpha$ so that: $p_1 \models (i^* < \nu$ and $x = \{\gamma_i : i < i^*\})$ and we can extend again to obtain $\beta$ and $p_2$ such that $p_2 \models (x = \{\gamma_i : i < \beta\})$.

Now let $q_0 \geq p_2$ be such that $q_0 \models (\gamma_0 = \delta_0)$. That is, $q_0$ identifies the value of the name $\gamma_i$. By induction on $i < \beta$ we construct an increasing sequence $\langle q_0 : \alpha < \beta \rangle$ and a sequence $\langle \delta_\alpha : \alpha < \beta \rangle$ such that $q_\alpha \models (\forall \xi < \alpha)(\gamma_\xi = \delta_\alpha(i))$. This is possible because $P$ is $< \kappa$-closed.

Again, by the $< \kappa$-closure of $P$, it is possible to find $q \in P$ that identifies all the elements of $x$. That is, there is $z \in \mathcal{P}_\kappa \lambda$ and $q \geq p$ such that $q \models (y \subseteq z$ and $z \in \mathcal{C})$, as required.

$\therefore$

Lemma 2.8 Let $G$ be a generic of $P$. Then $M[G] \models \{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p)$ and $p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa \lambda$.

Proof. Let $S$ be a name of the set $\{x \in \mathcal{P}_\kappa \lambda : (\exists p \in G)(x \in \text{dom}(p)$ and $p(x) \neq \emptyset)\}$.

Suppose $p_0 \in G$ is such that $p_0 \models (\mathcal{C}$ is club in $\mathcal{P}_\kappa \lambda$ and $\mathcal{C} \cap S = \emptyset$ and $x_0 \in \mathcal{C} \cap \mathcal{P}_\kappa \lambda$). Note that we use the previous lemma to obtain $p_0 \models (x_0 \in \mathcal{C} \cap \mathcal{P}_\kappa \lambda)$. We derive a contradiction by finding $p \geq p_0$ such that $p \models (\mathcal{C} \cap S \neq \emptyset)$. The strategy is to fix a chain of elements of $\mathcal{C}$ and $S$ up to a regular limit where the two chains intersect.

Let $y_0 \in \text{reg}({\mathcal{P}_\kappa \lambda})$ be such that $(\bigcup \text{dom}(p_0) \cup x_0) \in \mathcal{P}_{|y_0|}(y_0)$. We now identify $p_0^* \geq p_0$ such that $y_0 \in \text{dom}(p_0^*)$.

Let $D_0$ be a linearly ordered club of $\mathcal{P}_{|y_0|}(y_0)$ that does not intersect $\text{dom}(p_0)$. Such a club exists by definition of $P$ (in particular, clause (iv) of Definition 2.3). Note that having $D_0$ linearly ordered is convenient but not strictly necessary; it is possible because $|y_0|$ is regular.
Let \( p_0^*(u) = \begin{cases} p_0(u) & \text{if } u \in \text{dom}(p_0) \\ D_0 & \text{if } u = y_0 \\ \emptyset & \text{if } u \in \text{reg}(\mathcal{P}(y_0) \setminus (\text{dom}(p_0) \cup \{y_0\})) \\ \text{undefined} & \text{otherwise} \end{cases} \)

Then by checking against Definition 2.3, it is apparent that \( p_0^* \in P \). Note also that \( p_0^* \geq p_0 \).

Now using the preceding lemma, let \( p_1 \geq p_0^* \) be such that for some \( x_1 \in \mathcal{P}_\lambda \), \( p_1 \models (x_1 \in \mathcal{C} \cap \mathcal{P}_\lambda \text{ and } y_0 \subseteq x_1) \).

We now proceed inductively to define \( p_\alpha, x_\alpha, y_\alpha, p_\alpha^* \) so that for all \( \beta < \alpha \), \( y_\beta \in p_\alpha(y_\alpha) \) and \( p_\beta \leq p_\alpha \leq p_\alpha^* \). In the case when \( \alpha \) is a limit ordinal, we describe the condition under which the induction will stop. We will then observe that this condition will be met at some stage \( \alpha < \kappa \).

**Case 1: \( \alpha = \beta + 1 \)**

By the inductive definition, \( p_\alpha \) and \( x_\alpha \) are already defined. We now define \( p_\alpha^* \) and \( y_\alpha \) then also define \( p_{\alpha+1} \) and \( x_{\alpha+1} \). Let \( y_\alpha \in \text{reg}(\mathcal{P}_\lambda) \) be such that \( \bigcup \text{dom}(p_\beta) \cup x_\alpha \in \mathcal{P}_{\text{val}}(y_\alpha) \). We now identify \( p_\alpha^* \geq p_\alpha \) such that \( y_\alpha \in \text{dom}(p_\alpha^*) \).

Unlike in the case \( \alpha = 0 \), we will define \( p_\alpha^*(y_\alpha) \) so that it has non-trivial coherence. In particular, for all \( \beta < \alpha \), we will have \( y_\beta \in p_\alpha^*(y_\alpha) \).

The inductive hypothesis implies that \( y_\beta \in \text{dom}(p_\alpha) \) so we can find a linearly ordered club \( D_\alpha \) of \( \mathcal{P}_{\text{val}}(y_\alpha) \) that does not intersect \( \text{dom}(p_\alpha) \) and satisfies \( u \in D_\alpha \Rightarrow y_\beta \subseteq u \). Such a club exists by (iv) of Definition 2.3 and by intersecting with the club \( \{ u \in \mathcal{P}_\lambda : y_\beta \subseteq u \} \). Now let \( D_\alpha^* = p_\alpha(y_\beta) \cup \{ y_\beta \} \cup D_\alpha \).

Let \( p_\alpha^*(u) = \begin{cases} p_\alpha(u) & \text{if } u \in \text{dom}(p_\alpha) \\ D_\alpha^* & \text{if } u = y_\alpha \\ \emptyset & \text{if } u \in \text{reg}(\mathcal{P}(y_\alpha) \setminus (\text{dom}(p_\alpha) \cup \{y_\alpha\})) \\ \text{undefined} & \text{otherwise} \end{cases} \)

It is easily checked that \( p_\alpha^* \) satisfies (i) to (iv) of Definition 2.3 and that \( p_\alpha^* \geq p_\alpha \). Note also that \( y_\beta \in p_\alpha^*(y_\alpha) \).

Now using the previous lemma, let \( p_{\alpha+1} \geq p_\alpha^* \) be such that for some \( x_{\alpha+1} \in \mathcal{P}_\lambda \), \( p_\alpha \models (x_\alpha \in \mathcal{C} \cap \mathcal{P}_\lambda \text{ and } y_\alpha \subseteq x_\alpha) \).

**Case 2: \( \alpha \) is a limit ordinal < \( \kappa \)**

Note that \( x_\alpha \) and \( p_\alpha \) are not yet defined. Let \( p_\alpha \in P \) be such that \( p_\alpha \geq p_\beta \) for all \( \beta < \alpha \). This is possible because \( P \) is < \( \kappa \)-closed. Let \( s_\alpha = \bigcup \{ y_\beta : \beta < \alpha \} \).
If \(|s_\alpha|\) is regular then this will be the final stage of the induction. We then proceed to define \(y\) and \(p\) as described below. So suppose now that \(|s_\alpha|\) is singular. Note in particular that \(s_\alpha \notin \text{reg}(\mathcal{P}_\kappa \lambda)\) so \(s_\alpha \notin \text{dom}(p_\alpha^*)\).

By the inductive definitions of \(y_\beta\), \(s_\alpha = \bigcup \{\text{dom}(p_\beta) : \beta < \alpha\}\), that is \(s_\alpha\) is the set of ordinals that are in at least one element of the domain of at least one \(p_\beta\). Let \(y_\alpha \in \text{reg}(\mathcal{P}_\kappa \lambda)\) be such that \(s_\alpha = \mathcal{P}_{|y_\alpha|}(y_\alpha)\). Thus, for any \(\beta < \alpha\), if \(u \in \text{dom}(p_\beta)\) then \(u \in \mathcal{P}_{|y_\alpha|}(y_\alpha)\).

Let \(D_\alpha\) be a linearly ordered club of \(\mathcal{P}_{|y_\alpha|}(y_\alpha)\) that does not intersect \(\text{dom}(p_\alpha)\) and such that if \(u \in D_\alpha\) then \(s_\alpha \subseteq u\). Let \(D_\alpha^* = \cup \{p_\beta(y_\beta) : \beta < \alpha\} \cup \{s_\alpha\} \cup D_\alpha\).

Let \(p_\alpha^*(u) = \begin{cases} p_\alpha(u) & \text{if } u \in \text{dom}(p_\alpha) \\ 0 & \text{if } u = y_\alpha \\ \text{undefined} & \text{otherwise} \end{cases}\)

Then \(p_\alpha^* \in P\) and \((\forall \beta < \alpha)(p_\alpha^* \geq p_\beta)\).

As before, let \(p_{\alpha+1} \geq p_\alpha^*\) be such that for some \(x_{\alpha+1} \in \mathcal{P}_\kappa \lambda\), \(p_\alpha \models (x_\alpha \in \mathcal{C} \cap \mathcal{P}_\kappa \lambda \text{ and } y_\alpha \subseteq x_\alpha)\).

We repeat this procedure until we reach a limit ordinal \(\alpha = \mu < \kappa\) such that \(s_\alpha\) (as defined in Case 2) has inaccessible cardinality. There must be such a \(\mu\) because \(\kappa\) is Mahlo. Otherwise the set \(\{s_\alpha : \alpha < \kappa\}\) would be a club subset of \(\kappa\) that does not intersect the set of regular cardinals, contradicting the fact that \(\kappa\) is Mahlo. So suppose \(|s_\alpha|\) is inaccessible because the sequence \(\langle|y_\beta| : \beta < \alpha\rangle\) is strictly increasing by the inductive definitions of \(y_\beta\) for \(\beta < \alpha\).

Let \(y = s_\alpha\) and let \(E = \cup \{\text{dom}(p_\beta) : \beta < \alpha\}\). Now define \(p\) as follows.

Let \(p(u) = \begin{cases} p_\beta(u) & \text{if } (\exists \beta < \mu)(u \in \text{dom}(p_\beta)) \\ \bigcup \{p_\beta(y_\beta) : \beta < \mu\} & \text{if } u = y \\ \emptyset & \text{if } u \in \text{reg}(\mathcal{P}(y) \setminus (E \cup \{y\})) \\ \text{undefined} & \text{otherwise} \end{cases}\)

As before, by checking against (i)-(iv) of Definition 2.3, we see that \(p \in P\). We now show that \(p \models -(\mathcal{C} \cap \mathcal{S} \neq \emptyset)\).

Note that \(\cup_{\beta < \mu} y_\beta = y = \cup_{\beta < \mu} x_\beta\) because for any \(\beta < \mu\), \(x_\beta \subseteq y_\beta \subseteq x_{\beta+1} \subseteq y_{\beta+1}\). By the definition of \(p\), it is clear that \(p(y) \neq \emptyset\) and hence that \(p \models -y \in \mathcal{S}\). Also, since \(p \models -(\mathcal{C}\text{ is club in } \mathcal{P}_\kappa \lambda \text{ and } (\forall \beta < \mu)(x_\beta \in \mathcal{C}))\) it
follows that \( p \parallel -y \in C \). Hence, \( p \parallel -y \in C \cap S \), which is a contradiction because \( p \geq p_0 \) and \( p_0 \parallel -C \cap S = \emptyset \).

\[ \exists \]

We now establish that the proposed witness to \( \Box_{P,\lambda} \) satisfies the anticoherence condition.

**Lemma 2.9** Let \( G \) be a generic of \( P \). Then let
\[
S = \{ x \in P_{\kappa \lambda} : (\exists p \in G) (x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset) \}
\]
and let
\[
T = \{ x \in S : \text{there is a cofinal set of } y \in S \cap P_{|x|}(x) \text{ such that } (\exists p \in G) (\{x, y\} \subseteq \text{dom}(p) \text{ and } p(y) \neq p(x) \cap P_{|y|}(y)) \}.
\]

Then \( M[G] \models T \) is stationary in \( P_{\kappa \lambda} \).

**Proof (outline).** We proceed as in Lemma 2.8, forming the sequence of forcing conditions as before but at each stage, we interrupt the induction after setting \( p_\alpha^* \) but before setting \( x_\alpha+1 \). We set \( z_\alpha \supset y_\alpha \) and define \( q \geq p_\alpha^* \) such that \( z_\alpha \in \text{dom}(q) \) but \( q(z_\alpha) \cap q(y_\alpha) = \emptyset \). Now continue as before but defining \( x_\alpha+1 \) so that \( z_\alpha \subseteq x_\alpha+1 \) and with \( q \leq p \).

\[ \exists \]

Finally, we need to verify that \( \kappa \) is Mahlo in the generic extension \( M[G] \).

**Lemma 2.10** If \( G \) is a generic of \( P \) then \( M[G] \models \kappa \) is Mahlo.

**Proof.** Working in \( M[G] \), suppose \( C \) is a club in \( \kappa \). Then if \( C^* = \{ x \in P_{\kappa \lambda} : |x| \in C \} \), it follows that \( C^* \) is club in \( P_{\kappa \lambda} \). By Lemma 2.8, we can find \( y \) in \( C^* \cap \{ x \in P_{\kappa \lambda} : (\exists p \in G) (x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset) \} \). Then \( |y| \) is a regular cardinal in both \( M \) and \( M[G] \), by the preservation of cofinalities and cardinalities. Furthermore, \( |y| \in C \). Hence the set of regular cardinals is stationary in \( \kappa \). To see that \( \kappa \) remains a strong limit, note that for all \( \mu < \kappa \), \( (2^\mu)^{M[G]} = (2^\mu)^M \) by \( < \kappa \)-closure so \( \kappa \) remains a strong limit in the generic extension. Hence \( \kappa \) is Mahlo in \( M[G] \) as required.

\[ \exists \]

Given generic \( G \) of \( P \), let \( S = \{ x \in \text{reg}(P_{\kappa \lambda}) : (\exists p \in G) (p(x) \neq \emptyset) \} \) and for \( x \in S \), let \( C_x = p(x) \) where \( p \) is an element of \( G \) with \( x \in \text{dom}(p) \). The
preceding series of lemmas together prove that this $S$ and $\{C_x : x \in S\}$ provides a witness to $\square_{\mathcal{P}_{\kappa}\lambda}$ in $M[G]$. Thus, Theorem 2.2 is proved.

We proved in Lemma 2.10 that this forcing preserves the fact that $\kappa$ is Mahlo. In fact, we can do more than this and preserve supercompactness. Since forcing with $P$ is $\kappa$-directed closed, if $\kappa$ is supercompact in the ground model and we first force with a Laver preparation, then the supercompactness of $\kappa$ is preserved when we force with $P$.

**Theorem 2.11** Suppose $M$ is a countable model of a sufficiently rich fragment of ZFC in which $\kappa$ is supercompact and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which $\kappa$ is supercompact and $\square_{\mathcal{P}_{\kappa}\lambda}$ holds.

*Proof.* This follows by forcing with a Laver preparation followed by forcing with $P$. We use the fact that $P$ is $\kappa$-directed closed.

\[ \dashv \]

## 3 A $\mathcal{P}_{\kappa}\lambda$ version of square with a non-reflection property

One of the useful properties encapsulated by the square sequence is that of stationary non-reflection. This is demonstrated in the theorem presented below, which makes use of Fodor's Lemma, which we present here without proof.

**Lemma 3.1 (Fodor's Lemma)** Suppose that $S$ is a stationary subset of a regular cardinal $\mu$. Suppose also that $f : S \to \mu$ is such that $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is a stationary subset $T \subseteq S$ such that $f$ is constant on $T$.

The following well-known theorem is presented here with proof to motivate the work towards a $\mathcal{P}_{\kappa}\lambda$ version of the theorem discussed in the remainder of this section.

**Theorem 3.2** If $\square_{\kappa}$ holds then $\kappa^+$ has a non-reflecting stationary subset.
Proof. Suppose $\langle C_\alpha : \alpha < \kappa^+ \text{ and } \lim(\alpha) \rangle$ is as specified in the definition of $\square_\kappa$. Let $T = \{\alpha < \kappa^+ : \text{cf}(\alpha) < \kappa < \alpha\}$. To see that this is stationary, let $C$ be an arbitrary club of $\kappa^+$ and let $C^* = C \setminus \kappa$. Then the $\omega$th element of $C^*$ is an element of $T$.

Now define $F : T \rightarrow \kappa$ by $F(\alpha) = \text{otp}(C_\alpha)$. By part (ii) of Definition 1.1 and the definition of $T$, $F(\alpha) < \kappa < \text{otp}(\alpha)$ for all $\alpha \in T$. Hence, by Fodor’s Lemma, we can select a stationary subset $R \subseteq T$ such that $F$ is constant on $R$.

Now suppose $R$ reflects in $\alpha$ for some $\alpha \in R$. Let $\beta, \gamma \in R \cap C_\alpha$ with $\beta < \gamma$. Then $C_\beta \cup \{\beta\} \subseteq C_\gamma$ as $\beta = \sup(C_\beta)$. Thus $F(\gamma) = \text{otp}(C_\gamma) \geq \text{otp}(C_\beta) + 1 > F(\beta)$. But this is a contradiction because $F$ is constant on $R$.

$\dashv$.

We now extend $\square_{\mathcal{P}_\kappa \lambda}$ to produce a square principle that has a non-reflection property explicitly built into the definition. We then give a non-reflection theorem using this new principle.

**Definition 3.3** $\square_{\mathcal{P}_\kappa \lambda}(S, f)$ holds if $f : S \rightarrow \kappa$ and $S$ is stationary and there is a witness $\{C_x : x \in S\}$ to $\square_{\mathcal{P}_\kappa \lambda}(S)$ such that in addition to (i)-(iii) from Definition 2.1 we have:

(iv) $f(x) \in x$

(v) if $y \in C_x$ then $f(x) \neq f(y)$.

We now prove the relative consistency of this principle by extending the partial order $P$ used in the proof of Theorem 2.2.

**Theorem 3.4** Suppose $M$ is a countable model of a sufficiently rich fragment of ZFC in which $\kappa$ is Mahlo and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which $\kappa$ is Mahlo and for some $f, S$, $\square_{\mathcal{P}_\kappa \lambda}(S, f)$ holds.

We force with the poset $Q$ defined below.

**Definition 3.5** $p, q \in Q$ iff $p \in P$ and $q$ is as follows:
(i) $q$ is a function with domain \{ $x \in \text{dom}(p) : p(x) \neq \emptyset$ \}

(ii) $q(x) \in x$ for all $x \in \text{dom}(q)$

(iii) if $x \in \text{dom}(p)$ and $y \in p(x) \cap \text{dom}(p)$ and $p(y) \neq \emptyset$ then $q(y) \neq q(x)$.

If $(p, q), (p', q') \in Q$ then $(p, q) \leq (p', q')$ iff $p \subseteq p'$ and $q \subseteq q'$.

We do not present all of the details of the forcing proof. Instead we describe how to upgrade the proof of Theorem 2.2 to include the new property.

Note that $(\emptyset, \emptyset) \in Q$ so $Q$ is non-empty and has a minimal element. We must now establish various properties of $(Q, \leq)$ to show that a suitable generic object exists and that the resulting forcing preserves cofinalities and cardinalities.

**Lemma 3.6** $(Q, \leq)$ is separative.

**Proof.** Let $(p, q) \in Q$ and let $x \in \text{reg}(\mathcal{P}_\kappa \lambda) \setminus \text{dom}(p)$ such that there is $\gamma \in x \setminus \text{im}(q)$. Let $(p_0, q_0) \geq (p, q)$ be such that $p_0(x)$ is a club in $\mathcal{P}_{||x||}(x)$ that does not intersect $\text{dom}(p)$ and let $q_0(x) = \gamma$. Such a $p_0$ can be found by Definition 2.3 (iv) and because $|\text{dom}(p)| < \kappa \leq |\text{reg}(\mathcal{P}_\kappa \lambda)|$ so there must be some $x \in \text{reg}(\mathcal{P}_\kappa \lambda) \setminus \text{dom}(p)$. Now let $(p_1, q_1) \geq (p, q)$ be such that $x \in \text{dom}(p_1)$ and $p_1(x) = \emptyset$ and hence $x \notin \text{dom}(q_1)$. Clearly $(p_0, q_0)$ and $(p_1, q_1)$ are incompatible extensions of $(p, q)$. Hence, $Q$ is separable.

\dashv

We now prove that forcing with $Q$ preserves cofinalities and cardinalities by showing that $Q$ has the $\kappa^+$-chain condition and is $< \kappa$-directed closed.

We now use the $\Delta$-System Lemma to show that $Q$ has the $\kappa^+$-chain condition.

**Lemma 3.7** $Q$ satisfies the $\kappa^+$-chain condition.

**Proof.** Let $A$ be a subset of $Q$ of size $\kappa^+$. Now let $A = \{ \text{dom}(p) : \exists q(p, q) \in A \}$. By the $\Delta$-System Lemma, using the fact that $\kappa$ is a strong limit, we can find $B \subseteq A$ such that $|B| = \kappa^+$ and $B$ is a $\Delta$-system with root $R$.

Consider the number of pairs of functions $(p, q)$ definable on $R$ such that for each function $(p, q)$ and each $x \in R$, $p(x) \in \mathcal{P}(\mathcal{P}_{||x||}(x))$ and $q(x) \in x$. By the
argument in the proof of Lemma 2.5, the number of possible values that \( p(x) \)
can take is \(< \kappa \). The number of possible values that \( q(x) \) can take is clearly
\(|x|\). Since \(|x| < \kappa\), the number of possible pairs \((p(x), q(x))\) is \(< \kappa \). But
\(|B| = \kappa^+\) so by the pigeonhole principle there must be some pair of functions
\((g, h)\) defined on \( R \) such that \( p[R = g \) and \( q[R = h \) for \( \kappa^+\) many \((p, q) \in X
with \( \text{dom}(p) \in B \).

Now let \( Y = \{(p, q) \in X : p[R = g \) and \( q[R = h \} \). For any \((p_0, q_0), (p_1, q_1) \in
Y\), using the fact that \( p_0, p_1 \) and \( q_0, q_1 \) agree \( R \), it is straightforward to verify
that \((p_0 \cup p_1, q_0 \cup q_1) \in Q \). Thus, \((p_0, q_0), (p_1, q_1) \) have a common extension
in \( Q \) and hence are compatible. Hence, \( A \) is not an antichain.

\(-\).

**Lemma 3.8** \( Q \) is \(< \kappa\)-directed closed.

**Proof.** Suppose \( \mu < \kappa \) and \( \{(p_\alpha, q_\alpha) : \alpha < \mu \} \) is a set of pairwise compatible
conditions from \( Q \). We define \( p_\mu^* = \bigcup_{\alpha < \mu} p_\alpha \) and \( q_\mu^* = \bigcup_{\alpha < \mu} q_\alpha \). Now extend
\( p_\mu^* \) to \( p_\mu \) as in the proof of the \( < \kappa\)-directed closure of \( P \). Note that we need
not add new elements to the domain of \( q_\mu^* \) since \( x \in \text{dom}(p_\mu) \backslash \text{dom}(p_\mu^*) \Rightarrow
p_\mu(x) = \emptyset \). That is, we may set \( q_\mu = q_\mu^* \). Now for any \( x, y \in \text{dom}(q_\alpha) \),
there is some \( \alpha < \mu \) such that \( x, y \in \text{dom}(q_\alpha) \). Since \((p_r, q_r) \in Q \) it follows
that \( x \in p_\mu(x) \Rightarrow q_\mu(x) \neq q_\mu(y) \) and vice versa as required. It follows that
\((p_\alpha, q_\alpha) \in Q \) and for all \( \beta < \mu, (p_\alpha, q_\alpha) \leq (p_\mu, q_\mu) \).

\(-\).

It follows from the preceding lemmas that forcing with \( Q \) preserves cofinalities
and cardinalities. As with \( P \), this forcing is \(< \kappa\)-closed so for a generic \( G \)
of \( Q \), \( (P_{\kappa \lambda})^{M[G]} = (P_{\kappa \lambda})^M \) and we can write \( P_{\kappa \lambda} \) for the name \( P_{\kappa \lambda} \) in
the following. We must now ensure that for any generic \( G \) of \( Q \), the set
\( \{x \in P_{\kappa \lambda} : \exists (p, q) \in G \}(x \in \text{dom}(p) \) and \( p(x) \neq \emptyset \} \) is stationary in \( P_{\kappa \lambda} \).
Note that the following variation on Lemma 2.7 holds. The proof is almost
identical to the proof of Lemma 2.7.

**Lemma 3.9** Suppose \( (p, q) \in Q \) and \( (p, q) \models (C \text{ is a club of } P_{\kappa \lambda}) \). Then
there is \( x \in P_{\kappa \lambda} \) and \( (p', q') \in Q \) such that \( (p', q') \geq (p, q) \) and \( (p', q') \models x \in
C \).

**Lemma 3.10** Let \( G \) be a generic of \( Q \). Then \( M[G] \models \{x \in P_{\kappa \lambda} : (\exists p \in
G)(x \in \text{dom}(p) \) and \( p(x) \neq \emptyset \} \) is stationary in \( P_{\kappa \lambda} \).
Proof. We proceed as in the proof of Lemma 2.8 but define \((p_\alpha, q_\alpha)\) and \((p^{*_\alpha}, q^{*_\alpha})\) at each stage. We now describe how to set \(q_\alpha\). Let \(\gamma \in y_0 \setminus \{q(y_0)\}\). We insist, without loss of generality, that for all \(\alpha\), \(\gamma\) is not in the image of \(q_\alpha\) or \(q^{*_\alpha}\). For all \(\alpha < \mu\) we set \(q(y_\alpha) = \gamma_\alpha \in y_\alpha \setminus \bigcup_{\beta < \alpha} y_\beta\). By definition of \(y_\alpha\), such a \(\gamma_\alpha\) will always exist. At the final stage, when defining \((p, q)\), we define \(p\) as before and set \(q(y) = \gamma\).

The last two lemmas that we need follow by arguments exactly analogous to the corresponding lemmas for \(P\).

**Lemma 3.11** Let \(G\) be a generic of \(Q\). Then let

\[
\mathcal{S} = \{x \in \mathcal{P}_\kappa \lambda : (\exists (p, q) \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}
\]

and let

\[
\mathcal{S} = \{x \in \mathcal{S} : \text{there is a cofinal set of } y \in \mathcal{S} \cap \mathcal{P}_{|x|}(x) \text{ such that } (\exists (p, q) \in G)((x, y) \subseteq \text{dom}(p) \text{ and } p(y) \neq p(x) \cap \mathcal{P}_{|y|}(y))\}.
\]

Then \(M[G] \models T\) is stationary in \(\mathcal{P}_\kappa \lambda\).

**Lemma 3.12** If \(G\) is a generic of \(Q\) then \(M[G] \models \kappa\) is Mahlo.

By forcing with the partial order \((Q, \leq)\), Theorem 3.4 is proved. We set

\[
\mathcal{S} = \{x \in \mathcal{P}_\kappa \lambda : (\exists (p, q) \in G)(x \in \text{dom}(p) \text{ and } p(x) \neq \emptyset)\}
\]

and set \(f = \bigcup\{q : \exists p((p, q) \in G)\}\). Then \(f\) and \(\{C_x : (\exists (p, q) \in G)(C_x = p(x) \neq \emptyset)\}\), together witness that \(\square_{\mathcal{P}_\kappa \lambda}(S, f)\) holds, as required.

We now show how \(\square_{\mathcal{P}_\kappa \lambda}(S, f)\) gives non-reflection in \(\mathcal{P}_{|x|}(x)\) for stationary many \(x \in \mathcal{P}_\kappa \lambda\). We then state without proof some related results proved by Abe in [1] and by Koszmider in [7].

The following is proved by forcing and draws on Gitik’s method of shooting clubs in \(\mathcal{P}_\kappa \lambda\).

**Theorem 3.13 (Abe)** Let \(V \subset W\) be two models of ZFC with the same ordinals, \((\kappa^+)^V = (\kappa^+)^W\); let \(C\) be a club subset of \(\kappa\) of \(V\)-inaccessibles; let \(\kappa\) be an inaccessible cardinal in \(W\) and let \(T = \{x \in \mathcal{P}_\kappa \kappa^+ : V \models |x| \text{ is not inaccessible}\}\). Then there is a forcing notion in \(W\) that preserves cofinalities and cardinals and such that there is a stationary \(S \subset \mathcal{P}_\kappa \kappa^+\) such that \(S \cap \mathcal{P}_\kappa \kappa^+(x)\) is non-stationary for any \(x \in T\).
Koszmider in [7] gives a different kind of non-reflection result, considering reflection in $\mathcal{P}_\kappa(X)$ where $X \subseteq \lambda$.

**Theorem 3.14 (Koszmider)** It is consistent that there is a stationary set $S \subseteq \mathcal{P}_\kappa \lambda$ such that $S \cap \mathcal{P}_\kappa X$ is non-stationary in $\mathcal{P}_\kappa X$ for any $X \subseteq \lambda$ with $|X| \geq \kappa$ in the generic extension.

Finally we consider the following theorem of Abe which gives a form of non-reflection when $\kappa$ is supercompact.

**Theorem 3.15 (Abe)** If it is consistent that there is a supercompact cardinal then it is consistent that there is a supercompact $\kappa$, a cardinal $\lambda \geq \kappa$ and a stationary set $X \subseteq \mathcal{P}_\kappa \lambda$ such that $X \cap \mathcal{P}_\kappa \alpha$ is non-stationary in $\mathcal{P}_\kappa \alpha$ for any $\alpha < \lambda$.

The following definition presents the form of non-reflection that we examine with $\Box_{\mathcal{P}_\kappa \lambda}(S, f)$.

**Definition 3.16** A stationary set $S \subseteq \mathcal{P}_\kappa \lambda$ reflects in $\mathcal{P}_{|x|}(x)$ if $S \cap \mathcal{P}_{|x|}(x)$ is stationary in $\mathcal{P}_{|x|}(x)$.

The non-reflection theorem follows easily from the $\Box_{\mathcal{P}_\kappa \lambda}(S, f)$ principle. Note that the proof is closely analogously to the proof of non-reflection from $\Box_\kappa$ in the theory of cardinals. This theorem draws on the variation on Fodor's Lemma presented below. Lacking a suitable reference, we present a proof.

**Lemma 3.17** Suppose that $S$ is a stationary subset of $\mathcal{P}_\kappa \lambda$. Suppose also that $f : S \rightarrow \lambda$ is such that $f(x) \in x$ for all $x \in S$. Then there is a stationary subset $T \subseteq S$ such that $f$ is constant on $T$.

**Proof.** Suppose $f : S \rightarrow \lambda$ is a counterexample. For each $\alpha < \lambda$ choose $C_\alpha$ club in $\mathcal{P}_\kappa \lambda$ with $(f^{-1}(\alpha)) \cap C_\alpha = \emptyset$. Now let $D$ be the diagonal intersection of the $C_\alpha$, $D = \Delta(C_\alpha : \alpha < \lambda)$ and take $y \in S \cap D$, guaranteed to exist because $D$ is club. Then $f(y) \in y$ so since $y \in D$ we have $y \in C_{f(y)}$. Hence, $y \in f^{-1}(f(y)) \cap C_{f(y)}$, contradicting the choice of $C_{f(y)}$.

$\dashv$
Theorem 3.18 Suppose \( \kappa \) is Mahlo and \( \lambda \geq \kappa \). Then if \( \square_{\mathcal{P}_{\kappa}\lambda}(S, f) \) holds then there is a stationary set \( T \subseteq S \) such that \( T \) does not reflect in \( \mathcal{P}_{|x|}(x) \) for any \( x \in S \).

Proof. Let \( \{C_x : x \in S\} \) witness \( \square_{\mathcal{P}_{\kappa}\lambda}(S, f) \). Note that since \( f(x) \in x \), by the preceding lemma it follows that there is a stationary set \( T \subseteq S \) such that \( f(x) \) is constant on \( T \). Now suppose \( T \) reflects in \( \mathcal{P}_{|x|}(x) \) for some \( x \in S \). Let \( y \in T \cap C_x \). The set \( \{u \in \mathcal{P}_{|x|}(x) : y \subseteq u \text{ and } |y| < |u|\} \) is club in \( \mathcal{P}_{|x|}(x) \) so we can find \( z \in T \cap C_x \) such that \( y \in \mathcal{P}_{|z|}(z) \). By the definition of \( \square_{\mathcal{P}_{\kappa}\lambda}(S, f) \), we have that \( C_z = C_x \cap \mathcal{P}_{|z|}(z) \) so \( y \in C_z \). But then \( f(y) \neq f(z) \), contradicting the definition of \( T \). Thus \( T \) cannot reflect in \( \mathcal{P}_{|x|}(x) \).

\( \dashv \).

It should be noted that for some \( \kappa \), for example the first Mahlo cardinal, the conclusion of this theorem holds in ZFC. (Simply let \( S = T = \text{reg}(\mathcal{P}_{\kappa}\lambda) \).) The theorem becomes more relevant for cardinals higher in the Mahlo hierarchy (i.e. those that are \( \alpha - \text{Mahlo} \) for \( \alpha > 0 \)).

As with \( \square_{\mathcal{P}_{\kappa}\lambda}(S) \) we may use a Laver preparation to prove that \( \square_{\mathcal{P}_{\kappa}\lambda}(S, f) \) is consistent even for supercompact \( \kappa \). Thus, supercompactness of \( \kappa \) does not prevent this principle or the corresponding non-reflection theorem.

References


