Some characterizations of strongly $\sigma$-short Boolean Algebras (Forcing Method and Large Cardinal Axioms)

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Some characterizations of strongly $\sigma$-short Boolean Algebras

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Abstract. We give some characterizations of strongly $\sigma$-short Boolean algebras.

In this report, we give some characterizations of strongly $\sigma$-short Boolean algebras. In [2], we introduced $\sigma$-short Boolean algebras and strongly $\sigma$-short Boolean algebras. We say that a subset $D$ of a Boolean algebra $B$ is dense, in symbol $D \subset B$, if for every positive element $b \in B$ there exists $d \in D$ such that $0 < d \leq b$, $\sigma$-short if every strictly descending sequence of length $\omega$ in $D$ does not have a nonzero lower bound in $B$, $\wedge$-closed if for every $d_1, d_2 \in D$ with $d_1 \wedge d_2 > 0$, $d_1 \wedge d_2 \in D$. $B$ is said to be $\sigma$-short if it has a $\sigma$-short dense subset and strongly $\sigma$-short if it has a $\sigma$-short $\wedge$-closed dense subset. We denote by $d(B)$ the density of $B$. We assume that Boolean algebras are infinite and atomless in this report.

In [2], it was open whether measure algebras are strongly $\sigma$-short. Jörg Brendle showed the following theorem (see [1]).

Theorem A (Brendle). Let $B_\kappa$ be the algebra for adding $\kappa$ many random reals.

1. $B_\omega$ is not strongly $\sigma$-short.

2. Suppose that $d(B_\kappa) = \kappa$. Then $B_\kappa$ is strongly $\sigma$-short.

We say that a Boolean algebra $B$ has $(\kappa, \omega)$-caliber if for any uncountable subset $T \subseteq B$ of size $\kappa$, there is countable $F \subseteq T$ such that $F$ has a non-zero lower bound in $B$. It is well-known that the random algebra has $(\omega_1, \omega)$-caliber.

Y. Yoshinobu and I extended the first result above more general as follows (see [1]).

Theorem B (Takahashi-Yoshinobu). Suppose that $B$ satisfies $(\kappa, \omega)$-caliber and $d(B) \geq \kappa$. Then $B$ is not strongly $\sigma$-short.

In this report, we extend these theorems and give some characterizations of strongly $\sigma$-short Boolean algebras.

For $X \subseteq B$, let $\bigwedge X = \{x_1 \wedge \cdots \wedge x_n > 0 \mid x_1, \cdots, x_n \in X, n \in \omega\}$ and $[X]^{\omega} = \{Y \subseteq X \mid |Y| = \omega\}$, where $|Y|$ is the cardinality of $Y$.

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Theorem. The following are equivalent.

(1) \( \mathcal{B} \) is strongly \( \sigma \)-short.

(2) There exists \( X \subset \mathcal{B} \) such that \( \bigwedge X \subset_d \mathcal{B} \) and \( \bigwedge Y = 0 \) for every \( Y \in [X]^\omega \).

(3) There exists \( X \subset_d \mathcal{B} \) such that \( \bigwedge Y = 0 \) for every \( Y \in [X]^\omega \).

(4) There exist \( X \subset \mathcal{B}, D \subset_d \mathcal{B} \) and \( f : D \xrightarrow{1-1} X \) such that \( \bigwedge Y = 0 \) for every \( Y \in [X]^\omega \) and \( d \wedge f(d) > 0 \) for every \( d \in D \).

(5) There exists \( X \subset_d \mathcal{B} \) such that \( \{y \in X | y \geq x\} \) is finite for every \( x \in X \).

(6) There exists a sequence \( \{X_n\}_{n \in \omega} \) of subsets of \( \mathcal{B} \) which satisfies the following conditions:

- \( a \). \( X_n \) is a pairwise incomparable subset of \( \mathcal{B} \).
- \( b \). If \( x \in X_n, y \in X_m \) and \( n < m \), then \( y \nsubseteq x \).
- \( c \). \( \{y \in X_m | y \geq x\} \) is finite for every \( m < n \) and \( x \in X_n \).
- \( d \). \( X := \bigcup_{n \in \omega} X_n \subset_d \mathcal{B} \)

Proof of theorem. (1)\( \Rightarrow \) (2): Suppose that \( \mathcal{B} \) is strongly \( \sigma \)-short. Let \( D \) be a \( \sigma \)-short, \( \wedge \)-closed and dense subset of \( \mathcal{B} \). Without loss of generality, we assume that \( |D| = d(\mathcal{B}) \).

Put \( \kappa = d(\mathcal{B}) \). Let \( \{d_\alpha | \alpha < \kappa\} \) be an enumeration of elements of \( D \). We shall find \( D^\alpha \subset \mathcal{B} \) and \( \Lambda^\alpha \subset \mathcal{B} \) for \( \alpha < \kappa \) such that

- \( i \). \( \forall \alpha < \kappa \Lambda^\alpha \neq \Lambda^{\alpha+1} \Rightarrow \Lambda^{\alpha+1} = \Lambda^\alpha \cup \{\alpha\} \),
- \( ii \). \( \forall \alpha < \kappa \exists x \in D^{\alpha+1}[x \leq d_\alpha] \),
- \( iii \). \( D^\alpha = \bigwedge \{d_\beta | \beta \in \Lambda^\alpha \} \), and
- \( iv \). \( \forall \alpha < \kappa \alpha \in \Lambda^{\alpha+1} \Leftrightarrow \forall d \in D^\alpha[d \nsubseteq d_\alpha] \).

Assuming such \( D^\alpha \) and \( \Lambda^\alpha \) may be found, let

\[
\Lambda := \bigcup_{\alpha < \kappa} \Lambda^\alpha \quad \text{and} \quad D' := \bigcup_{\alpha < \kappa} D^\alpha.
\]

By (ii), \( D' \) is a dense subset of \( \mathcal{B} \). Put \( X = \{d_\alpha | \alpha \in \Lambda\} \). By (iii), \( D' = \bigwedge X \), so \( \bigwedge X \) is a dense subset of \( \mathcal{B} \). Let \( Y \) be a countable subset of \( X \) and \( \{d_{\alpha_n}\}_{n \in \omega} \) be its enumeration such that \( \alpha_0 < \alpha_1 < \alpha_2 < \cdots \). We show that \( \bigwedge Y = 0 \). Without loss of generality, we may assume that for any finite subset \( Y_0 \) of \( Y \), \( \bigwedge Y_0 > 0 \). Put \( e_n := d_{\alpha_0} \wedge \cdots \wedge d_{\alpha_n} \) for every \( n \in \omega \). Since \( \alpha_n \in \Lambda \), by (i), (iii) and (iv), we have \( \alpha_n \in \Lambda^{\alpha_n+1} \), so that \( d_{\alpha_n} \in D^{\alpha_n+1} \) and for every \( d \in D^{\alpha_n}, d \nsubseteq d_{\alpha_n} \). Since
If \( n \leq d \) then we define \( D_n := D_{n+1} \) and \( A_n := A_{n+1} \).

(2) \( \Rightarrow \) (3): Easy.

(3) \( \Rightarrow \) (4): Put \( D := X \) and \( f := Id_D \).

(4) \( \Rightarrow \) (1): Put \( D_0 := \{ d \in D \mid d \leq d_{\alpha_0} \} \) and \( D_1 := \bigsqcup_{n \in \omega} D_n \). Since \( D_0 \) is dense in \( B \), \( D_1 \) is also dense in \( B \) and \( \wedge \)-close. To see that \( D_1 \) is \( \sigma \)-short, it is enough to show that \( \bigwedge_{n \in \omega} Y = 0 \) for every \( Y \in [D_1]^{\omega} \). Let \( Y := \{ d_n \wedge f(d_n) \mid n \in \omega \} \). Then we have \( \bigwedge_{n \in \omega} Y = \bigwedge_{n \in \omega} d_n \wedge \bigwedge_{n \in \omega} f(d_n) \). Since \( f \) is one-to-one, \( f(d_n) \neq f(d_m) \) for \( n \neq m \). Hence \( \{ f(d_n) \mid n \in \omega \} \in [X]^{\omega} \). Therefore \( \bigwedge_{n \in \omega} Y \leq \bigwedge_{n \in \omega} f(d_n) = 0 \).

(5) \( \Leftrightarrow \) (3): Easy.

(6) \( \Rightarrow \) (5): Put \( X := \bigcup_{n \in \omega} X_n \). Then \( X \) is dense in \( B \) by (d). For every \( x \in X \), there exists \( n \in \omega \) such that \( x \in X_n \). Then \( \{ y \in X \mid y \geq x \} = \bigcup_{m<n} \{ y \in X_m \mid y \geq x \} \) by (a), (b) and (c). Hence \( \{ y \in X \mid y \geq x \} \) is finite.

\( \square \)
Theorem B (Takahashi-Yoshinobu). Suppose that $B$ satisfies $(\kappa, \omega)$-caliber and $d(B) \geq \kappa$. Then $B$ is not strongly $\sigma$-short.

Proof of Theorem B: Suppose that $B$ is strongly $\sigma$-short. Then by virtue of main theorem, there exists $X \subset_d B$ such that $\bigwedge Y = 0$ for every $Y \in [X]^\omega$. Since $|T| \geq d(B) \geq \kappa$, there is countable $F \subseteq T$ such that $F$ has a non-zero lower bound in $B$. This contradicts that $B$ satisfies $(\kappa, \omega)$-caliber. \hfill \square

Theorem A (Brendle). Let $B_\kappa$ be the algebra for adding $\kappa$ many random reals.

1. $B_\omega$ is not strongly $\sigma$-short.

2. Suppose that $d(B_\kappa) = \kappa$. Then $B_\kappa$ is strongly $\sigma$-short.

Proof: (1): Since $B_\omega$ satisfies $(\omega_1, \omega)$-caliber, $B_\omega$ is not strongly $\sigma$-short by virtue of Theorem B.

(2): Let $D \subseteq B_\kappa$ be dense, $|D| = \kappa$. Say $D = \{b_\alpha; \alpha < \kappa\}$. For each $\alpha$ choose $\gamma_\alpha \notin \text{supp}(b_\alpha)$ in such a way that the $\gamma_\alpha$ are distinct for distinct $\alpha$. Let $f(b_\alpha) := \{(\langle \gamma_\alpha, 0 \rangle, 0)\}$. Here $\{(\langle \gamma_\alpha, 0 \rangle, 0)\}$ denotes the partial function $p : \kappa \times \omega \to 2$ with domain the singleton $\{\langle \gamma_\alpha, 0 \rangle\}$ and $p(\langle \gamma_\alpha, 0 \rangle) = 0$. $[p]$ is the open set defined by $p$. Then $f$ satisfies the assumption of (4) of the main theorem. Hence $B_\kappa$ is strongly $\sigma$-short. \hfill \square

Open Problems

1. Are perfect tree forcings, Hechler forcing $\sigma$-short?

2. For every $\sigma$-short $B$, does there exist a sequence $\{X_n\}_{n \in \omega}$ of subsets of $B$ which satisfies the following conditions:

   (a) $X_n$ is a pairwise incomparable subset of $B$.

   (b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \nleq x$.

   (c) $\bigcup_{n \in \omega} X_n \subset_d B$

References
