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Some characterizations of strongly $\sigma$-short Boolean Algebras

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Abstract. We give some characterizations of strongly $\sigma$-short Boolean algebras.

In this report, we give some characterizations of strongly $\sigma$-short Boolean algebras. In [2], we introduced $\sigma$-short Boolean algebras and strongly $\sigma$-short Boolean algebras. We say that a subset $D$ of a Boolean algebra $\mathcal{B}$ is dense, in symbol $D \subset_d \mathcal{B}$, if for every positive element $b \in \mathcal{B}$ there exists $d \in D$ such that $0 < d \leq b$, $\sigma$-short if every strictly descending sequence of length $\omega$ in $D$ does not have a nonzero lower bound in $\mathcal{B}$, $\land$-closed if for every $d_1, d_2 \in D$ with $d_1 \land d_2 > 0$, $d_1 \land d_2 \in D$. $\mathcal{B}$ is said to be $\sigma$-short if it has a $\sigma$-short dense subset and strongly $\sigma$-short if it has a $\sigma$-short $\land$-closed dense subset. We denote by $d(\mathcal{B})$ the density of $\mathcal{B}$. We assume that Boolean algebras are infinite and atomless in this report.

In [2], it was open whether measure algebras are strongly $\sigma$-short. Jörg Brendle showed the following theorem (see [1]).

Theorem A (Brendle). Let $\mathcal{B}_\kappa$ be the algebra for adding $\kappa$ many random reals.

1. $\mathcal{B}_\omega$ is not strongly $\sigma$-short.
2. Suppose that $d(\mathcal{B}_\kappa) = \kappa$. Then $\mathcal{B}_\kappa$ is strongly $\sigma$-short.

We say that a Boolean algebra $\mathcal{B}$ has $(\kappa, \omega)$-caliber if for any uncountable subset $T \subseteq \mathcal{B}$ of size $\kappa$, there is countable $F \subseteq T$ such that $F$ has a non-zero lower bound in $\mathcal{B}$. It is well-known that the random algebra has $(\omega_1, \omega)$-caliber.

Y. Yoshinobu and I extended the first result above more general as follows (see [1]).

Theorem B (Takahashi-Yoshinobu). Suppose that $\mathcal{B}$ satisfies $(\kappa, \omega)$-caliber and $d(\mathcal{B}) \geq \kappa$. Then $\mathcal{B}$ is not strongly $\sigma$-short.

In this report, we extend these theorems and give some characterizations of strongly $\sigma$-short Boolean algebras.

For $X \subseteq \mathcal{B}$, let $\bigwedge X = \{x_1 \land \cdots \land x_n > 0 \mid x_1, \cdots, x_n \in X, n \in \omega\}$ and $[X]^{\omega} = \{Y \subseteq X \mid |Y| = \omega\}$, where $|Y|$ is the cardinality of $Y$. 

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Theorem. The following are equivalent.

(1) $B$ is strongly $\sigma$-short.

(2) There exists $X \subset B$ such that $\bigwedge X \subset_d B$ and $\bigwedge Y = 0$ for every $Y \in [X]^{\omega}$.

(3) There exists $X \subset_d B$ such that $\bigwedge Y = 0$ for every $Y \in [X]^{\omega}$.

(4) There exist $X \subset B, D \subset_d B$ and $f : D \rightarrow X$ such that $\bigwedge Y = 0$ for every $Y \in [X]^{\omega}$ and $d \wedge f(d) > 0$ for every $d \in D$.

(5) There exists $X \subset_d B$ such that $\{y \in X | y \geq x\}$ is finite for every $x \in X$.

(6) There exists a sequence $\{X_n\}_{n \in \omega}$ of subsets of $B$ which satisfies the following conditions:

(a) $X_n$ is a pairwise incomparable subset of $B$.
(b) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\leq x$.
(c) $\{y \in X_m | y \geq x\}$ is finite for every $m < n$ and $x \in X_n$.
(d) $X := \bigcup_{n \in \omega} X_n \subset_d B$

Proof of theorem. (1)$\Rightarrow$(2): Suppose that $B$ is strongly $\sigma$-short. Let $D$ be a $\sigma$-short, $\wedge$-closed and dense subset of $B$. Without loss of generality, we assume that $|D| = d(B)$.

Put $\kappa = d(B)$. Let $\{d_\alpha | \alpha < \kappa\}$ be an enumeration of elements of $D$. We shall find $D^\alpha \subset B$ and $\Lambda^\alpha \subset \Lambda$ for $\alpha < \kappa$ such that

(i) $\forall \alpha < \kappa [\Lambda^\alpha \neq \Lambda^{\alpha+1} \Rightarrow \Lambda^{\alpha+1} = \Lambda^\alpha \cup \{\alpha\}]$,
(ii) $\forall \alpha < \kappa \exists x \in D^{\alpha+1}[x \leq d_\alpha]$,
(iii) $D^\alpha = \bigwedge \{d_\beta | \beta \in \Lambda^\alpha\}$, and
(iv) $\forall \alpha < \kappa [\alpha \in \Lambda^{\alpha+1} \iff \forall d \in D^\alpha[d \not\leq d_\alpha]]$.

Assuming such $D^\alpha$ and $\Lambda^\alpha$ may be found, let

$\Lambda := \bigcup_{\alpha < \kappa} \Lambda^\alpha$ and $D' := \bigcup_{\alpha < \kappa} D^\alpha$.

By (ii), $D'$ is a dense subset of $B$. Put $X = \{d_\alpha | \alpha \in \Lambda\}$. By (iii), $D' = \bigwedge X$, so $\bigwedge X$ is a dense subset of $B$. Let $Y$ be a countable subset of $X$ and $\{d_{\alpha_n}\}_{n \in \omega}$ be its enumeration such that $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$. We show that $\bigwedge Y = 0$. Without loss of generality, we may assume that for any finite subset $Y_0$ of $Y$, $\bigwedge Y_0 > 0$. Put $e_n := d_{\alpha_n} \wedge \ldots \wedge d_{\alpha_n}$ for every $n \in \omega$. Since $\alpha_n \in \Lambda$, by (i), (iii) and (iv), we have $\alpha_n \in \Lambda^{\alpha_n+1}$, so that $d_{\alpha_n} \in D^{\alpha_n+1}$ and for every $d \in D^{\alpha_n}, d \not\leq d_{\alpha_n}$. Since
$e_{n-1} \in D_{n} \subset D_{n+1}$, $e_{n-1} \not\in d_{n}$. So we have $e_{0} > e_{1} > e_{2} > \cdots$. Hence \{e_{n}\}_{n\in\omega}$ is a strict decreasing sequence in $D$. Therefore $\bigwedge_{n\in\omega} Y = \bigwedge_{n\in\omega} e_{n} = 0$.

We define $D^{\alpha}$ and $\Lambda^{\alpha}$ by induction. Suppose that $D^{\beta}, \Lambda^{\beta} \ (\beta < \alpha)$ are defined. If $\alpha$ is limit, then

$$D^{\alpha} := \bigcup_{\beta < \alpha} D^{\beta} \quad \text{and} \quad \Lambda^{\alpha} := \bigcup_{\beta < \alpha} \Lambda^{\beta}.$$ 

If $\alpha$ is successor (say $\alpha = \alpha_{0} + 1$), then we define $D^{\alpha}, \Lambda^{\alpha}$ as follows.

If $\exists d \in D^{\alpha_{0}}[d \leq d_{\alpha_{0}}]$, then put

$$D^{\alpha} := D^{\alpha_{0}} \quad \text{and} \quad \Lambda^{\alpha} := \Lambda^{\alpha_{0}}.$$ 

If $\forall d \in D^{\alpha_{0}}[d \not\leq d_{\alpha_{0}}]$, then put

$$D^{\alpha} := \bigwedge(D^{\alpha_{0}} \cup \{d_{\alpha_{0}}\}) \quad \text{and} \quad \Lambda^{\alpha} := \Lambda^{\alpha_{0}} \cup \{\alpha_{0}\}.$$ 

It is easy to show that (i), (ii) and (iv) hold. We show (iii) by induction. Suppose that (iii) holds for every $\beta < \alpha$. If $\alpha$ is limit, then

$$D^{\alpha} = \bigcup_{\beta < \alpha} D^{\beta} = \bigcup_{\beta < \alpha} \bigwedge\{d_{\gamma} \mid \gamma \in \Lambda^{\beta}\} = \bigwedge\{d_{\beta} \mid \beta \in \Lambda^{\alpha}\}.$$ 

Suppose that $\alpha = \alpha_{0} + 1$. If $\exists d \in D^{\alpha_{0}}[d \leq d_{\alpha_{0}}]$, then it is clear that (iii) holds for $\alpha$. If $\forall d \in D^{\alpha_{0}}[d \not\leq d_{\alpha_{0}}]$, then

$$D^{\alpha} = \bigwedge(D^{\alpha_{0}} \cup \{d_{\alpha_{0}}\}) = \bigwedge(\bigwedge\{d_{\beta} \mid \beta \in \Lambda^{\alpha_{0}}\} \cup \{d_{\alpha_{0}}\}) = \bigwedge\{d_{\beta} \mid \beta \in \Lambda^{\alpha}\}.$$ 

(2)$\Rightarrow$(3): Easy.

(3)$\Rightarrow$(4): Put $D := X$ and $f := I|D$. 

(4)$\Rightarrow$(1): Put $D_{0} := \{d \wedge f(d) \mid d \in D\}$ and $D_{1} := \bigwedge D_{0}$. Since $D_{0}$ is dense in $B$, $D_{1}$ is also dense in $B$ and $\wedge$-close. To see that $D_{1}$ is $\sigma$-short, it is enough to show that $\bigwedge_{n\in\omega} Y = 0$ for every $Y \in [D_{1}]^{\omega}$. Let $Y := \{d_{n} \wedge f(d_{n}) \mid n \in \omega\}$. Then we have $\bigwedge_{n\in\omega} Y = \bigwedge_{n\in\omega} d_{n} \wedge \bigwedge_{n\in\omega} f(d_{n})$. Since $f$ is one-to-one, $f(d_{n}) \neq f(d_{m})$ for $n \neq m$. Hence $\{f(d_{n}) \mid n \in \omega\} \subseteq [X]^{\omega}$. Therefore $\bigwedge_{n\in\omega} Y \leq \bigwedge_{n\in\omega} f(d_{n}) = 0$.

(5)$\Leftrightarrow$(3): Easy.

(5)$\Rightarrow$(6): Let $X$ be a dense subset of $B$ such that $\{y \in X \mid y \geq x\}$ is finite for every $x \in X$. Put $X_{n} := \{d \in X \mid \{x \in X \mid x \geq d\} = n\}$ for every $n \in \omega$. Then it is easy to show that $X_{n}$ satisfies conditions (a)–(d).

(6)$\Rightarrow$(5): Put $X := \bigcup_{n\in\omega} X_{n}$. Then $X$ is dense in $B$ by (d). For every $x \in X$, there exists $n \in \omega$ such that $x \in X_{n}$ and $\{y \in X \mid y \geq x\} = \bigcup_{m<n} \{y \in X_{m} \mid y \geq x\}$ by (a), (b) and (c). Hence $\{y \in X \mid y \geq x\}$ is finite. \qed
Theorem B (Takahashi-Yoshinobu). Suppose that $B$ satisfies $(\kappa, \omega)$-caliber and $d(B) \geq \kappa$. Then $B$ is not strongly $\sigma$-short.

Proof of Theorem B: Suppose that $B$ is strongly $\sigma$-short. Then by virtue of main theorem, there exists $X \subset_d B$ such that $\bigwedge Y = 0$ for every $Y \in [X]^{\omega}$. Since $|T| \geq d(B) \geq \kappa$, there is countable $F \subseteq T$ such that $F$ has a non-zero lower bound in $B$. This contradicts that $B$ satisfies $(\kappa, \omega)$-caliber. $\Box$

Theorem A (Brendle). Let $B_{\kappa}$ be the algebra for adding $\kappa$ many random reals.

1. $B_{\omega}$ is not strongly $\sigma$-short.

2. Suppose that $d(B_{\kappa}) = \kappa$. Then $B_{\kappa}$ is strongly $\sigma$-short.

Proof: (1): Since $B_{\omega}$ satisfies $(\omega_1, \omega)$-caliber, $B_{\omega}$ is not strongly $\sigma$-short by virtue of Theorem B.

(2): Let $D \subseteq B_{\kappa}$ be dense, $|D| = \kappa$. Say $D = \{b_{\alpha}; \alpha < \kappa\}$. For each $\alpha$ choose $\gamma_{\alpha} \notin \text{supp}(b_{\alpha})$ in such a way that the $\gamma_{\alpha}$ are distinct for distinct $\alpha$. Let $f(b_{\alpha}) := [\{(\gamma_{\alpha}, 0), 0\}]$. Here $\{\{(\gamma_{\alpha}, 0), 0\}\}$ denotes the partial function $p : \kappa \times \omega \rightarrow 2$ with domain the singleton $\{\gamma_{\alpha}, 0\}$ and $p(\gamma_{\alpha}, 0) = 0$. $[p]$ is the open set defined by $p$. Then $f$ satisfies the assumption of (4) of the main theorem. Hence $B_{\kappa}$ is strongly $\sigma$-short. $\Box$

Open Problems

1. Are perfect tree forcings, Hechler forcing $\sigma$-short?

2. For every $\sigma$-short $B$, does there exist a sequence $\{X_n\}_{n\in\omega}$ of subsets of $B$ which satisfies the following conditions:
   
   (a) $X_n$ is a pairwise incomparable subset of $B$.

   (b) If $x \in X_m, y \in X_m$ and $n < m$, then $y \not\geq x$.

   (c) $\bigcup_{n\in\omega} X_n \subset_d B$

References
