数理解析研究所講究録 1423 巻 2005 年 128-138

♣ and destructible gaps

依岡 輝幸*† (Teruyuki Yorioka)
神戸大学大学院自然科学研究科
(Graduate School of Science and Technology, Kobe University)

Abstract

We show that (1) ♣ plus $\mathsf{cof}(\mathcal{M}) = \aleph_1$ implies the existence of a destructible gap and (2) $\mathbf{Coll}(\omega, \omega_1)$ adds a destructible gap.

1 Introduction

In this paper, we deal with a pregap in the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$. A pregap in $\mathcal{P}(\omega)/\mathrm{fin}$ is a pair $(\mathcal{A}, \mathcal{B})$ of subsets of $\mathcal{P}(\omega)$ such that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the set $a \cap b$ is finite. For subsets a and b of ω, we say that a is almost contained in b (and denote $a \subseteq^* b$) if the set $a \setminus l$ is a subset of b for some $l \in \omega$. For a pregap $(\mathcal{A}, \mathcal{B})$, both ordered sets $\langle \mathcal{A}, \subseteq^* \rangle$ and $\langle \mathcal{B}, \subseteq^* \rangle$ are well ordered and these order types are κ and λ respectively, then we say that a pregap $(\mathcal{A}, \mathcal{B})$ has the type (κ, λ) or is a (κ, λ)-pregap. Moreover if $\kappa = \lambda$, we say that the pregap is symmetric. For a pregap $(\mathcal{A}, \mathcal{B})$, we say that $(\mathcal{A}, \mathcal{B})$ is separated if for some $c \in \mathcal{P}(\omega)$, $a \subseteq^* c$ and the set $c \cap b$ is finite for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. If a pregap is not separated, we say that it is a gap. Moreover if a gap has the type (κ, λ), it is called a (κ, λ)-gap.

We note that being a pregap is absolute in any model having the pregap, but being a gap is not. In [13], Kunen has investigated an (ω_1, ω_1)-gap and has given a characterization of being a gap in the forcing extension and in [23, Chapter 9], Todorčević has introduced a notion of an open coloring and has given Ramsey theoretic characterization of being a gap in the forcing extension (Theorem 1.1). From their characterizations, we note that an (ω_1, ω_1)-gap constructed by Hausdorff is still a gap in any extension preserving cardinals. We say that such a gap is indestructible. If an (ω_1, ω_1)-gap is not indestructible, that is, it is not a gap in some forcing extension not collapsing cardinals, it is called destructible. (We note that every gap not having the type (ω_1, ω_1), it can be separated by a ccc-forcing extension.) Kunen has proved that under Martin's Axiom for $\aleph_1$ many dense sets of ccc-forcing notions, all (ω_1, ω_1)-gap are indestructible. In [14], Laver has implied that a destructible gap consistently exists. Therefore it is not decided from ZFC that there exists a destructible gap.

A notion of a destructible gap can be an analogy of one of a Suslin tree ([1]). A Suslin tree is an ω_1-tree having no uncountable chains and antichains. A destructible gap is considered as a similar notion. For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ with the set $a_\alpha \cap b_\alpha$ empty for every $\alpha \in \omega_1$, we say here that α and β in ω_1 are compatible if

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset.$$

*Supported by JSPS Research Fellowships for Young Scientists.
†Supported by Grants-in-Aid for JSPS Fellow, No. 16·3977, Ministry of Education, Culture, Sports, Science and Technology.

Then by the characterization due to Kunen and Todorčević, we notice that an (ω_1, ω_1)-pregap is a destructible gap iff it has no uncountable pairwise compatible and incompatible subsets of ω_1.

Jensen has proved that if $\mathbf{V} = \mathbf{L}$, then there exists a Suslin tree. After that, he has introduced a combinatorial principle $\diamondsuit$ and has constructed a Suslin tree from $\diamondsuit$. In [19], Shelah has proved that adding a Cohen real adds a Suslin tree. The same results for a destructible gap are also true and proved by Todorčević ([5, Proposition 2.5] and [23, Theorem 9.3]). (Random reals effect the existence of a Suslin tree and a destructible gap quite different. [15, 9], [8, 10, 11]) (We must notice that from results of Farah and Hirschorn [6, 8], the existence of a destructible gap is independent with the existence of a Suslin tree.)

In [24], Velleman has modified a construction of a Suslin tree due to Shelah using a morass, and after that Miyamoto has modified a Velleman's construction using a connections of two models. The first version of Miyamoto's theorem also have a morass as a condition to build a Suslin tree, but in [3, §7], Brendle has modified again that situation and consequently, he constructed a Suslin tree from $\dagger$ plus the covering number $\mathsf{cov}(\mathcal{M})$ of the meager ideal is larger than $\aleph_1$. $\dagger$ is a combinatorial principle on ω_1, introduced in the paper [2], as follow: there is a sequence $\langle A_\alpha; \alpha \in \omega_1 \rangle$ of countable subsets of ω_1 such that for any uncountable subset B of ω_1 there is $\alpha \in \omega_1$ so that $A_\alpha \subseteq B$. A destructible gap can be constructed under the same situation, that is, $\dagger$ plus $\mathsf{cov}(\mathcal{M}) > \aleph_1$ implies the existence of a destructible gap.

$\clubsuit$ is a combinatorial principle on ω_1 introduced by Ostaszewski ([17]. See also [20, I.§7]): There exists a sequence $\langle A_\alpha; \alpha \in \omega_1 \rangle$ of subsets of ω_1 such that for all $\alpha \in \omega_1$, $A_\alpha \subseteq \alpha$ and for every uncountable subset A of ω_1, the set $\{\alpha \in \omega_1; A_\alpha \subseteq A\}$ is stationary. We note that $\diamondsuit$ implies $\clubsuit$ and $\clubsuit$ plus the Continuum Hypothesis implies $\diamondsuit$ ([20]). From the result of Baumgartner [12, Theorem IV. 4] (or the result [16, Corollary 6.14]), it is consistent with ZFC that $\clubsuit$, the cofinality $\mathsf{cof}(\mathcal{M})$ of the meager ideal on the real line is equal to $\aleph_1$ and the continuum is larger than $\aleph_1$, hence in this model, $\diamondsuit$ does not hold. Brendle has proved that that a Suslin tree exists in the model satisfying $\clubsuit$ plus $\mathsf{cof}(\mathcal{M}) = \aleph_1$ ([3, Theorem 6]). As same as a Suslin tree, we can show that $\clubsuit$ plus $\mathsf{cof}(\mathcal{M}) = \aleph_1$ implies the existence of a destructible gap (Theorem 1).

The consistency of $\clubsuit$ plus $\neg\,\mathsf{CH}$ was an well known open problem. The first discovery of this consistency was due to Shelah. After that, this problem has been investigated by several set theorists. As far as I know, we have the following five types of models satisfying $\clubsuit$ and $\neg\,\mathsf{CH}$. (Here, κ is an uncountable regular cardinal.)

1. **Shelah** [20]. $(\clubsuit_{\omega_2} + 2^{\aleph_1} = \aleph_3)^{\mathbf{Coll}(\omega,\omega_1)} \models$" $\clubsuit + 2^{\aleph_0} = \aleph_2$".
2. **Fuchino-Shelah-Soukup** [7]. $\diamondsuit^{\text{pseudo-product}(\mathbb{C},\kappa)} \models$" $\clubsuit + 2^{\aleph_0} = \mathsf{cov}(\mathcal{M}) = \kappa$".
3. **Brendle** [3]. $\diamondsuit^{\text{pseudo-product}(\mathbb{B},\kappa)} \models$" $\clubsuit + 2^{\aleph_0} = \mathsf{cov}(\mathcal{N}) = \kappa$".
4. **Baumgartner** [12]. $\diamondsuit^{\text{csp}(\mathbb{S},\kappa)} \models$" $\clubsuit + \mathsf{cof}(\mathcal{M}) = \aleph_1 + 2^{\aleph_0} = \kappa$".
5. **Moore-Hrušák-Džamonja** [16]. $\mathbf{V}^{\text{csi}(\mathbb{S},\omega_2)} \models$" $\clubsuit + \mathsf{cof}(\mathcal{M}) = \aleph_1 + 2^{\aleph_0} = \aleph_2$".

From above results, we have known that the models 2, 4 and 5 have both a Suslin tree and a destructible gap. I will prove that $\mathbf{Coll}(\omega, \omega_1)$ adds a destructible gap (Theorem 3.1), hence it follows that the model 1 has a destructible gap. (I conjecture that **Coll**(ω, ω_1) adds a Suslin tree, so the model 1 also has a Suslin tree.) Well, it is not still known wethere the model 3 has a Suslin tree, or destructible gap.

Throughout this paper, we always deal with a symmetric pregap. For an ordinal α, if we say that $\langle a_\xi, b_\xi; \xi \in \alpha\rangle$ is a pregap, we always assume that

- if $\xi < \eta$ in α, $a_\xi \subseteq^* a_\eta$ and $b_\xi \subseteq^* b_\eta$, and
- for every $\xi \in \alpha$, the set $a_\xi \cap b_\xi$ is empty.

We have the following characterizations of being a gap and indestructibility.

Theorem 1.1 (E.g. [4, 13, 18, 22]). *Let $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ be an (ω_1, ω_1)-pregap.*

1. *The following statements are equivalent:*

   (i) *$(\mathcal{A}, \mathcal{B})$ forms a gap.*

   (ii) $\forall X \in [\omega_1]^{\omega_1}\ \exists \alpha \neq \beta \in X\ ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset)$.

2. *The following statements are equivalent:*

   (i) *$(\mathcal{A}, \mathcal{B})$ is destructible (may not be a gap).*

   (ii) $\forall X \in [\omega_1]^{\omega_1}\ \exists \alpha \neq \beta \in X\ ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset)$.

2 ♣ plus cof$(\mathcal{M}) = \aleph_1$ implies the existence of a destructible gap

In [5, Proposition 2.5], a destructible gap is constructed from $\diamondsuit$. This proof uses the CH to show the pregap constructed by recursion is really a gap. The following proof (and the proof in [25]) says that we do not need the CH to construct a destructible gap from $\diamondsuit$ also.

The following condition is a useful notion to construct a destructible gap. This is used in the proof of [5, Proposition 2.5]. (But we slightly modify the original one.)

Definition 2.1 ([25]). *We say that a pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ admits finite changes if for all $\alpha < \omega_1$, the set $a_\alpha \cap b_\alpha$ is empty and the set $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite, and for any $\beta < \alpha$ with $\beta = \eta + k$ for some $\eta \in \mathsf{Lim} \cap \alpha$ and $k \in \omega$, $H, J \in [\omega]^{<\omega}$ with $H \cap J = \emptyset$ and $i > \max(H \cup J)$ there exists $n \in \omega$ so that*

$$a_{\eta+n} \cap i = H,\ a_{\eta+n} \setminus i = a_\beta \setminus i,\ b_{\eta+n} \cap i = J,\ \text{and } b_{\eta+n} \setminus i = b_\beta \setminus i.$$

Theorem 1. *♣ and $\mathsf{cof}(\mathcal{M}) = \aleph_1$ implies the existence of a destructible gap.*

Proof. At first, we give some notation in the proof to avoid using many symbols in formulae.

For each $\alpha \in \omega_1$ and a pregap $\langle a_\xi, b_\xi; \xi < \alpha\rangle$, let $g \in 2^{\alpha \times \omega \times 2}$ be a function such that for all $\xi < \alpha$, $a_\xi = \{n \in \omega; g(\xi, n, 0) = 1\}$ and $b_\xi = \{n \in \omega; g(\xi, n, 1) = 1\}$, that is, g is a code of this pregap. Assume that α is a countable ordinal and g is a code of an (α, α)-pregap $\langle a_\xi, b_\xi; \xi \in \alpha\rangle$

which admits finite changes, and $a_\xi \cap b_\xi = \emptyset$ and $\omega \setminus (a_\xi \cup b_\xi)$ is infinite for all $\xi \in \alpha$. Then we define a subset $\mathcal{X}(g)$ of α^ω which is a collection of members x in α^ω such that

$$\bigcup_{\xi \in \mathrm{ran}(x)} a_\xi \cap \bigcup_{\xi \in \mathrm{ran}(x)} b_\xi = \emptyset.$$

We can identify $\mathcal{X}(g)$ as the Baire space ω^ω. (By the admission of finite changes of g, any node in $\mathcal{X}(g)$ has infinitely many successors.) For each $s \in \alpha^{<\omega}$, we let $[s] := \{x \in \mathcal{X}(g); s \subseteq x\}$ and denote $\mathcal{X}^{<\omega}(g)$ as the set of $s \in \alpha^{<\omega}$ such that $[s]$ is a basic open set in $\mathcal{X}(g)$, i.e.

$$\bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \cap \bigcup_{\xi \in \mathrm{ran}(s)} b_\xi = \emptyset.$$

Let O be a dense open subset of ω^ω. O is a union of countably many basic open sets, that is, O has a code as a countable sequence of members of $\omega^{<\omega}$. In this proof, we can consider O as a dense open subset of $\mathcal{X}(g)$ using its code. Moreover we define a space $\mathcal{Y}(g)$ such that

$$\begin{aligned}\mathcal{Y}(g) := \{y \in (\alpha \times \omega)^\omega; &\text{the sequence of the first coordinats of } y \text{ is in } \mathcal{X}(g)\\ &\text{and the second coordinats are strictly increasing }\}.\end{aligned}$$

$\mathcal{Y}(g)$ is also considered as the Baire space. For $y \in (\alpha \times \omega)^{\leq\omega}$ and $l < |y|$, we denote $y(l) = \langle y(l)(0), y(l)(1)\rangle$ and $\mathrm{ran}_0(y) := \{y(l)(0); l < |y|\}$. As in the definition of $\mathcal{X}^{<\omega}(g)$, we denote $\mathcal{Y}^{<\omega}(g)$ as the set of $t \in (\alpha \times \omega)^{<\omega}$ such that $[t]$ is a basic open set in $\mathcal{Y}(g)$.

Let $\langle A_\alpha; \alpha \in \omega_1\rangle$ be a $\clubsuit$-sequence. Since $\mathsf{cof}(\mathcal{M})$ is equal to the cofinality of the collection of closed nowhere dense sets (e.g. [21, Lemma 3.7]) and now $\mathsf{cof}(\mathcal{M}) = \aleph_1$, there exists a family $\mathcal{O}$ of open dense subsets of ω^ω of size $\aleph_1$ such that for any dense open subset O of ω^ω, there exists a member of $\mathcal{O}$ which is a subset of O. We write Lim as a class of limit ordinals. Let $\langle P_\beta; \beta \in \omega_1 \cap \mathsf{Lim}\rangle$ be a partition and f a function from ω_1 onto $\mathcal{O}$ such that for all $\beta \in \omega_1 \cap \mathsf{Lim}$,

- P_β is uncountable,
- the set $P_\beta \cap \beta$ is empty, and
- $f \restriction P_\beta$ is surjective.

We construct a pregap $\langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ with the following properties:

1. $a_0 = b_0 = \emptyset$, $a_\alpha \cap b_\alpha = \emptyset$ and the set $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite for all $\alpha < \omega_1$.
2. If $\beta \leq \alpha < \omega_1$, then both $a_\beta \subseteq^* a_\alpha$ and $b_\beta \subseteq^* b_\alpha$.
3. $\langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ admits finite changes.
4. For each $\alpha \in \omega_1 \cap \mathsf{Lim}$, if for any $\gamma, \delta \in A_\alpha$ with $\gamma < \delta$, there is $\beta > \gamma$ such that $\delta \in P_\beta$, then there exists a strictly increasing sequence $\langle j_k^\alpha; k \in \omega\rangle$ of natural numbers such that for each $\beta \in \alpha \cap \mathsf{Lim}$ and $\gamma \in P_\beta \cap A_\alpha$, there is an infinite subset S of ω so that for any $j \in \{j_k^\alpha; k \in S\}$ and $K \subseteq j$, there exists $s \in \mathcal{X}^{<\omega}(g_\beta)$ such that $[s]$ is a subset of the dense open subset $f(\gamma)$ in $\mathcal{X}(g_\beta)$, and

$$\bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \cap K = \emptyset, \quad \bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \setminus j \subseteq a_\alpha,$$

$$\bigcup_{\xi \in \mathrm{ran}(s)} b_\xi \cap j \subseteq K \quad \text{and} \quad \bigcup_{\xi \in \mathrm{ran}(s)} b_\xi \smallsetminus j \subseteq b_\alpha.$$

5. For each $\alpha \in \omega_1 \cap \mathsf{Lim}$, if for any $\gamma, \delta \in A_\alpha$ with $\gamma < \delta$, there is $\beta > \gamma$ such that $\delta \in P_\beta$, then there exists a strictly increasing sequence $\langle i_k^\alpha : k \in \omega \rangle$ of natural numbers such that for each $\beta \in \alpha \cap \mathsf{Lim}$ and $\gamma \in P_\beta \cap A_\alpha$, there is an infinite subset T of ω so that for any $i \in \{i_k^\alpha ; k \in T\}$, there exists $t \in \mathcal{Y}^{<\omega}(g_\beta)$ such that $t(0)(1) \geq i$, $[t]$ is a subset of the dense open subset $f(\gamma)$ in $\mathcal{Y}(g_\beta)$, and

$$\bigcup_{\xi \in \mathrm{ran}_0(t)} a_\xi \cap \Big[i,\ t(|t|-1)(1)\Big) \subseteq a_\alpha$$

and

$$\bigcup_{\xi \in \mathrm{ran}_0(t)} b_\xi \cap \Big[i,\ t(|t|-1)(1)\Big) \subseteq b_\alpha.$$

The construction at successor stages are trivial by the property 3.

Assume that α is a limit ordinal. We enumerate the set $\{\langle \beta, \gamma \rangle ; \beta \in \alpha \cap \mathsf{Lim} \text{ and } \gamma \in P_\beta \cap A_\alpha\}$ by $\{\langle \beta_k, \gamma_k \rangle ; k \in \omega\}$ such that each pair $\langle \beta, \gamma \rangle$ appears infinitely many often. (These sets may be empty. If so, we let all $\langle \beta_k, \gamma_k \rangle$ not be defined.) In order to construct a_α and b_α, we construct an increasing cofinal sequence $\langle \zeta_k ; k \in \omega \rangle$ of α and natural numbers $i_k^\alpha = i_k$, $j_k^\alpha = j_k$, with properties that

- $\langle \zeta_k ; k \in \omega \rangle \in \mathcal{X}(g_\alpha)$,
- $\beta_k < \zeta_{k-1}$ and $i_k < j_k < i_{k+1}$ for every $k \in \omega$, and
- $a_{\zeta_{k-1}} \cap j_{k-1} = a_{\zeta_k} \cap j_{k-1}$ and $b_{\zeta_{k-1}} \cap j_{k-1} = b_{\zeta_k} \cap j_{k-1}$ for every $k \in \omega$

as follows; then we define $a_\alpha := \bigcup_{k \in \omega} a_{\zeta_k}$ and $b_\alpha := \bigcup_{k \in \omega} b_{\zeta_k}$:

Assume that we have already constructed ζ_h, i_h and j_h, $h < k$, for some $k \in \omega$. (We put $i_{-1} = j_{-1} = 0$. If $\langle \beta_k, \gamma_k \rangle$'s are not defined, then we ignore the following construction and define a_α and b_α satisfying the properties 1 and 2 and for all $\mu \in \alpha$, both sets $a_\alpha \smallsetminus a_\mu$ and $b_\alpha \smallsetminus b_\mu$ are infinite.) Let $\{K_m ; m < 2^{j_{k-1}}\}$ enumerate $\mathcal{P}(j_{k-1})$. By the inductive hypothesis of the property 3, we pick $\eta_m \in \beta_k$ for each $m \leq 2^{j_{k-1}}$ and $s_m \in \mathcal{X}^{<\omega}(g_{\beta_k})$ for each $m < 2^{j_{k-1}}$ such that

- $a_{\eta_m} \cap j_{k-1} = j_{k-1} \smallsetminus K_m$ and $b_{\eta_m} \cap j_{k-1} = K_m$,
- $\langle \eta_m \rangle \subseteq s_m$ (i.e. $s_m(0) = \eta_m$),
- $[s_m]$ is a subset of the dense open subset $f(\gamma_k)$ in $\mathcal{X}(g_{\beta_k})$,
- $\max(\eta_{m+1} \cap \mathsf{Lim}) = \max\{\max(\xi \cap \mathsf{Lim}) ; \xi \in \mathrm{ran}(s_m)\}$, and
- $\displaystyle\bigcup_{\xi \in \mathrm{ran}(s_m)} a_\xi \smallsetminus j_{k-1} = a_{\eta_{m+1}} \smallsetminus j_{k-1}$ and $\displaystyle\bigcup_{\xi \in \mathrm{ran}(s_m)} b_\xi \smallsetminus j_{k-1} = b_{\eta_{m+1}} \smallsetminus j_{k-1}$

(This can be done by the property 3.) Let $i_k > j_{k-1}$ be such that

$$a_{\eta_{2^{j_{k-1}}}} \smallsetminus i_k \subseteq a_{\zeta_{k-1}} \quad \text{and} \quad b_{\eta_{2^{j_{k-1}}}} \smallsetminus i_k \subseteq b_{\zeta_{k-1}},$$

and then we take $\zeta'_{k-1} \in \alpha$ (by the inductive hypothesis of the property 3) so that

$$a_{\zeta'_{k-1}} \cap j_{k-1} = a_{\zeta_{k-1}} \cap j_{k-1}, \quad a_{\zeta'_{k-1}} \cap \left[j_{k-1},\ i_k\right) = a_{\eta_{2^{j_{k-1}}}} \cap \left[j_{k-1},\ i_k\right)$$

$$a_{\zeta'_{k-1}} \setminus i_k = a_{\zeta_{k-1}} \setminus i_k, \quad b_{\zeta'_{k-1}} \cap j_{k-1} = b_{\zeta_{k-1}} \cap j_{k-1},$$

$$b_{\zeta'_{k-1}} \cap \left[j_{k-1},\ i_k\right) = b_{\eta_{2^{j_{k-1}}}} \cap \left[j_{k-1},\ i_k\right) \text{ and } b_{\zeta'_{k-1}} \setminus i_k = b_{\zeta_{k-1}} \setminus i_k.$$

The construction up to here is for the property 4. For the property 5, we pick $t \in \mathcal{Y}^{<\omega}(g_{\beta_k})$ such that $t(0)(1) \geq i_k$, $[t]$ is a subset of the dense open subset $f(\gamma_k)$ in $\mathcal{Y}(g_{\beta_k})$. (This can be done by the density of $f(\gamma_k)$. For the sequence $\langle\langle 0, i\rangle\rangle \in \mathcal{Y}(g_{\beta_k})^{<\omega}$, there is $t \in \mathcal{Y}(g_{\beta_k})^{<\omega}$ so that $\langle\langle 0, i\rangle\rangle \subseteq t$ and $[t]$ is a subset of $f(\gamma_k)$.) We let

$$\zeta''_{k-1} > \max\left(\mathrm{ran}_0(t) \cup \left\{\zeta'_{k-1}\right\}\right)$$

be a large enough ordinal less than α and $j_k > t(|t|-1)(1)(\geq i_k)$ be such that for all $\xi \in \mathrm{ran}_0(t) \cup \left\{\zeta'_{k-1}\right\}$,

$$a_\xi \setminus j_k \subseteq a_{\zeta''_{k-1}}, \quad b_\xi \setminus j_k \subseteq b_{\zeta''_{k-1}} \text{ and } \left|j_k \setminus \left(a_{\zeta''_{k-1}} \cup b_{\zeta''_{k-1}}\right)\right| \geq k$$

and find $\zeta_k < \alpha$ (by the inductive hypothesis of the property 3) so that

$$a_{\zeta_k} \cap i_k = a_{\zeta'_{k-1}} \cap i_k, \quad a_{\zeta_k} \cap \left[i_k,\ j_k\right) = \left(\bigcup_{\xi \in \mathrm{ran}_0(t)} a_\xi \cup a_{\zeta'_{k-1}}\right) \cap \left[i_k,\ j_k\right),$$

$$a_{\zeta_k} \setminus j_k = a_{\zeta''_{k-1}} \setminus j_k, \quad b_{\zeta_k} \cap i_k = b_{\zeta'_{k-1}} \cap i_k,$$

$$b_{\zeta_k} \cap \left[i_k,\ j_k\right) = \left(\bigcup_{\xi \in \mathrm{ran}_0(t)} b_\xi \cup b_{\zeta'_{k-1}}\right) \cap \left[i_k,\ j_k\right) \text{ and } b_{\zeta_k} \setminus j_k = b_{\zeta''_{k-1}} \setminus j_k,$$

which completes the construction.

We check that $\langle a_\alpha, b_\alpha; \alpha \in \omega_1\rangle$ is a destructible gap, i.e. we will prove the following two statements.

(a) $\forall X \in [\omega_1]^{\omega_1}\ \exists \alpha \neq \beta \in X\ ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset)$.

(b) $\forall X \in [\omega_1]^{\omega_1}\ \exists \alpha \neq \beta \in X\ ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset)$.

(We recall that (a) means that the pregap is destructible, and (b) means that the pregap is a gap.)

For a proof of (a), assume that there exists an uncountable subset X of ω_1 such that for all $\gamma \neq \delta \in X$,

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset.$$

Without loss of generality, we may moreover assume that for all $\gamma \in \omega_1$, there exists $\delta \in X$ such that

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset.$$

We note that the set

$$C := \{\alpha \in \mathsf{Lim} \cap \omega_1; \forall \gamma \in \alpha\ \exists \delta \in X \cap \alpha\ ((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset)\}$$

is club on ω_1. We construct an uncountable subset A of ω_1 as follows. Assume that we have already constructed A up to δ for some countable ordinal δ. Then there is $\beta \in C \setminus (\delta+1)$. We notice that the set

$$D_\beta := \{x \in \mathcal{X}(g_\beta); \mathrm{ran}(x) \cap X \neq \emptyset\}$$

is dense open in $\mathcal{X}(g_\beta)$. So there exists $\gamma \in P_\beta$ such that $f(\gamma)$ is contained in D_β and let $A \cap (\gamma+1) := (A \cap \delta) \cup \{\gamma\}$ which completes the construction of A.

By the ♣-sequence, we can find $\alpha \in C$ such that $A_\alpha \subseteq A$. By the construction of A, A_α satisfies the first assumption of the property 4. We take any $\eta \in X \setminus \alpha$. Then there is a natural number m such that

$$a_\alpha \setminus m \subseteq a_\eta \text{ and } b_\alpha \setminus m \subseteq b_\eta.$$

We fix any $\gamma \in A_\alpha$. Then by the construction of A, for some $\beta \in \alpha$, $\gamma \in P_\beta$ and $f(\gamma)$ is a subset of D_β. Applying the property 4 for $\langle \alpha, \beta, \gamma \rangle$, we can find $j \geq m$ which satisfies the conclusion of the property 4. Then we can find $s \in \mathcal{X}^{<\omega}(g_\beta)$ such that $[s]$ is a subset of $f(\gamma)$ and

$$\bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \cap b_\eta \cap j = \emptyset, \quad \bigcup_{\xi \in \mathrm{ran}(s)} a_\xi \setminus j \subseteq a_\alpha,$$

$$\bigcup_{\xi \in \mathrm{ran}(s)} b_\xi \cap j \subseteq b_\eta \cap j \text{ and } \bigcup_{\xi \in \mathrm{ran}(s)} b_\xi \setminus j \subseteq b_\alpha.$$

By the definition of D_β, there exists $\xi \in \mathrm{ran}(s) \cap X$. (Because if $\mathrm{ran}(s) \cap X = \emptyset$, then let $\zeta \in \mathrm{ran}(s)$ and $x \in \beta^\omega$ such that $s \subseteq x$ and $x(i) = \zeta$ for all $i \geq |s|$, and then $x \in ([s] \cap \mathcal{X}(g_\beta)) \setminus D_\beta$, which contradicts an assumption of s. The point is that for any $s_0, s_1 \in \alpha^{<\omega}$, the intersection $[s_0] \cap [s_1]$ is empty if s_0 and s_1 are incomparable, otherwise $[s_0] \cap [s_1]$ is either $[s_0]$ or $[s_1]$.) But then

$$(a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi) = \emptyset$$

which is a contradiction and completes the proof of (a).

A proof of (b) is similar to one of (a), but we will use the property 5 instead of 4. We assume that there exists an uncountable subset Y of ω_1 such that for all $\gamma \neq \delta \in Y$,

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset.$$

Without loss of generality, we may moreover assume that for all $\gamma \in \omega_1$, there exists $\delta \in Y$ such that

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset.$$

We note again that the set

$$C' := \{\alpha \in \mathsf{Lim} \cap \omega_1; \forall \gamma \in \alpha\ \exists \delta \in Y \cap \alpha\ ((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset)\}$$

is club on ω_1. We construct an uncountable subset B of ω_1 as follows. Assume that we have already constructed B up to δ for some countable ordinal δ. Then there is $\beta \in C' \setminus (\delta+1)$. We define the subset E_β of $\mathcal{Y}(g_\beta)$ such that $y \in E_\beta$ if there exists $\xi \in Y$ so that for some $l \in \omega$, either

$$a_\xi \cap \left(\bigcup_{\zeta \in \mathrm{ran}_0(y)} b_\zeta \right) \cap \big[y(l), y(l+1)\big) \neq \emptyset$$

or

$$\left(\bigcup_{\zeta\in\mathrm{ran}_0(y)} a_\zeta\right)\cap b_\xi\cap\big[y(l),y(l+1)\big)\neq\emptyset.$$

We note that E_β is dense open in $\mathcal{Y}(g_\beta)$, hence there exists $\gamma\in P_\beta$ such that $f(\gamma)$ is contained in E_β and let

$$B\cap(\gamma+1):=(B\cap\delta)\cup\{\gamma\}$$

which completes the construction of B.

By the $\clubsuit$-sequence, we can find $\alpha\in C'$ such that $A_\alpha\subseteq B$. By the construction of B, A_α satisfies the first assumption of the property 4. We take any $\eta\in Y\setminus\alpha$. Then there is a natural number m such that

$$a_\alpha\setminus m\subseteq a_\eta \text{ and } b_\alpha\setminus m\subseteq b_\eta.$$

We take any $\gamma\in A_\alpha$, then by the construction of B, for some $\beta\in\alpha$, $\gamma\in P_\beta$ and $f(\gamma)$ is a subset of E_β. Applying the property 5 for $\langle\alpha,\beta,\gamma\rangle$, we can find $i\geq m$ which satisfies the conclusion of the property 5. Then we can find $t\in\mathcal{Y}^{<\omega}(g_\beta)$ such that $t(0)(1)\geq i$, $[t]$ is a subset of $f(\gamma)$ and

$$\left(\bigcup_{\zeta\in\mathrm{ran}_0(t)} a_\zeta\right)\cap\Big[i,\ t(|t|-1)(1)\Big)\subseteq a_\alpha$$

and

$$\left(\bigcup_{\zeta\in\mathrm{ran}_0(t)} b_\zeta\right)\cap\Big[i,\ t(|t|-1)(1)\Big)\subseteq b_\alpha.$$

By the definition of E_β, there exists $\xi\in Y$ such that for some $l<|t|-1$, either

$$a_\xi\cap\left(\bigcup_{\zeta\in\mathrm{ran}_0(t)} b_\zeta\right)\cap\Big[t(l)(1),\ t(l+1)(1)\Big)\neq\emptyset$$

or

$$\left(\bigcup_{\zeta\in\mathrm{ran}_0(t)} a_\zeta\right)\cap b_\xi\cap\Big[t(l)(1),\ t(l+1)(1)\Big)\neq\emptyset.$$

But then, since $t(l)(1)\geq i$,

$$(a_\xi\cap b_\eta)\cup(a_\eta\cap b_\xi)\neq\emptyset$$

which is a contradiction and completes the proof of (b). □

3 Coll(ω,ω_1) adds a destructible gap

Coll(ω,ω_1) is a forcing notion collapsing $\aleph_1$ to $\aleph_0$ by finite approximations.

Adding a Cohen real always adds a destructible gap. Exactly, if $\langle a_\alpha,b_\alpha;\alpha\in\omega_1\rangle$ is an (ω_1,ω_1)-gap and c is Cohen (over the ground model), then $\langle a_\alpha\cap c,b_\alpha\cap c;\alpha\in\omega_1\rangle$ is a destructible gap (in the Cohen extension). The following proof is essentially the same proof of the case of Cohen forcing.

Theorem 3.1. **Coll**(ω,ω_1) *adds a destructible gap.*

Proof. We write $\mathbb{P} := \mathbf{Coll}(\omega, \omega_1)$. We note that in the extension with $\mathbf{Coll}(\omega, \omega_1)$,

$$\aleph_0{}^{\mathbf{V}} = \left|\aleph_1{}^{\mathbf{V}}\right| = \aleph_1 \text{ and } \aleph_2{}^{\mathbf{V}} = \aleph_1.$$

Hausdorff has proved that there exists an (ω_1, ω_1)-gap under ZFC. (Of course (?), we now assume the axiom of choice.) So we have a $\mathbf{Coll}(\omega, \omega_1)$-name $\left\langle \dot{a}_\alpha, \dot{b}_\alpha; \alpha \in \omega_2 \right\rangle$ such that

$$\Vdash_{\mathbf{Coll}(\omega,\omega_1)} \text{“ } \left\langle \dot{a}_\alpha, \dot{b}_\alpha; \alpha \in \check{\omega_2} \right\rangle \text{ is an } (\dot{\omega_1}, \dot{\omega_1})\text{-gap, (note that } \check{\omega_2} = \dot{\omega_1})$$

$$\text{and } \forall \alpha \in \check{\omega_2} \left(\dot{a}_\alpha \cap \dot{b}_\alpha = \emptyset \right) \text{ ”}.$$

We note that $\mathbb{P}$ is forcing-equivalent to the product $\mathbf{Coll}(\omega, \omega_1) \times \mathbb{C}$, where $\mathbb{C}$ is a partial order $\langle 2^{<\omega}, \supseteq \rangle$. In this proof, we identify a condition p in $\mathbb{C}$ with a finite subset $\{i \in |p| ; p(i) = 1\}$ of $|p|$. Letting $\dot{c}$ be a $\mathbb{C}$-name for a generic real, we now show that

$$\Vdash_{\mathbb{P}} \text{“ } \left\langle \dot{a}_\alpha \cap \dot{c}, \dot{b}_\alpha \cap \dot{c}; \alpha \in \check{\omega_2} \right\rangle \text{ is a destructible gap ”},$$

and this finishes the proof.

Assume that $\langle \dot{\alpha}_\xi ; \xi \in \omega_2 \rangle$ is $\mathbb{P}$-names for countable ordinals (i.e. less than ω_2 ordinals) such that

$$\Vdash_{\mathbb{P}} \text{“ } \dot{\alpha}_\xi < \dot{\alpha}_\eta < \check{\omega_2} \text{ ”}$$

if $\xi < \eta < \omega_2$. For each $\xi \in \omega_2$, we take a condition $\langle \sigma_\xi, s_\xi \rangle \in \mathbb{P}$ and $\beta_\xi \in \omega_2$ such that

$$\langle \sigma_\xi, s_\xi \rangle \Vdash_{\mathbb{P}} \text{“ } \dot{\alpha}_\xi = \check{\beta}_\xi \text{ ”}.$$

Check being a gap. Since $|\mathbb{P}| = \aleph_1$, without loss of generality, we may assume that all $\langle \sigma_\xi, s_\xi \rangle$ are the same condition $\langle \sigma, s \rangle$. We note that

$$\sigma \Vdash_{\mathbf{Coll}(\omega,\omega_1)} \text{“ } \left\langle \dot{a}_{\check{\beta}_\xi} \setminus |\check{s}|, \dot{b}_{\check{\beta}_\xi} \setminus |\check{s}| ; \alpha \in \check{\omega_2} \right\rangle \text{ is a gap,}$$

$$\text{(note that } \left\{ \check{\beta}_\xi ; \xi \in \check{\omega_2} \right\} \text{ is an uncountable set)”},$$

thus by the chracterization of being a gap, we can find $\sigma' \leq_{\mathbf{Coll}(\omega,\omega_1)} \sigma$, $\xi \neq \eta \in \omega_2$ and $k \in \omega$ so that

$$\sigma' \Vdash_{\mathbf{Coll}(\omega,\omega_1)} \text{“ } \left(\left((\dot{a}_{\check{\beta}_\xi} \cap \dot{b}_{\check{\beta}_\eta}) \cup (\dot{a}_{\check{\beta}_\eta} \cap \dot{b}_{\check{\beta}_\xi}) \right) \setminus |\check{s}| \right) \cap \check{k} \neq \emptyset \text{ ”}.$$

Let $s' := s^\frown \mathbf{1} \restriction [|s|, k)$, then

$$\langle \sigma', s' \rangle \Vdash_{\mathbb{P}} \text{“ } (\dot{a}_{\check{\beta}_\xi} \cap \dot{b}_{\check{\beta}_\eta} \cap \dot{c}) \cup (\dot{a}_{\check{\beta}_\eta} \cap \dot{b}_{\check{\beta}_\xi} \cap \dot{c}) \neq \emptyset \text{ ”}.$$

Therefore we have shown that

$$\Vdash_{\mathbb{P}} \text{“ } \forall X \in [\dot{\omega_1}]^{\dot{\omega_1}} \; \exists \alpha \neq \beta \in X \; \left((\dot{a}_\alpha \cap \dot{b}_\beta \cap \dot{c}) \cup (\dot{a}_\beta \cap \dot{b}_\alpha \cap \dot{c}) \neq \emptyset \right) \text{ ”}.$$

Check a destructibility. For each $\xi \in \omega_2$, without loss, there may exist $t_\xi, u_\xi \in 2^{|s_\xi|}$ so that

$$\sigma_\xi \Vdash_{\mathbf{Coll}(\omega,\omega_1)} \text{“ } \dot{a}_{\beta_\xi} \restriction |\check{s}_\xi| = \check{t}_\xi \text{ and } \dot{b}_{\beta_\xi} \restriction |\check{s}_\xi| = \check{u}_\xi \text{ ”}.$$

Without loss of generality, we may assume that all σ_ξ, s_ξ, t_ξ and u_ξ are some σ, s, t and u respectively. We must notice that, by our assumption, $t \cap u = \emptyset$. We fix any $\xi \neq \eta \in \omega_2$ with $\xi < \eta$. Then we can find σ' and $k \in \omega$ so that

$$\sigma' \Vdash_{\mathbf{Coll}(\omega,\omega_1)} \text{“ } \dot{a}_{\check{\beta}_\xi} \setminus \check{k} \subseteq \dot{a}_{\check{\beta}_\eta} \text{ and } \dot{b}_{\check{\beta}_\xi} \setminus \check{k} \subseteq \dot{b}_{\check{\beta}_\eta} \text{ ”}.$$

Let $s' := s^\frown \mathbf{0} \restriction [|s|, k)$, then

$$\langle \sigma', s' \rangle \Vdash_{\mathbb{P}} \text{“ } (\dot{a}_{\check{\beta}_\xi} \cap \dot{b}_{\check{\beta}_\eta} \cap \dot{c}) \cup (\dot{a}_{\check{\beta}_\eta} \cap \dot{b}_{\check{\beta}_\xi} \cap \dot{c}) = \emptyset \text{ ”}.$$

□

Acknowledgement. I would like to thank Jörg Brendle for letting me show his unpublished paper [3].

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