数理解析研究所講究録 1424 巻 2005 年 128-141

Borel Summability of Divergent Solutions for Singular 1st Order Linear PDEs of Nilpotent Type

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1 Introduction and Main Result.

In this paper we are concerned with the following first order linear partial differential equation:

$$A(x,y)D_xu(x,y)+B(x,y)D_yu(x,y)+C(x,y)u(x,y)=F(x,y), \tag{1.1}$$

where x, $y\in\mathbf{C}$, $D_x=\partial/\partial x$, $D_y=\partial/\partial y$. A, B, C and F are holomorphic at $(x,y)=(0,0)\in\mathbf{C}^2$.

First of all we give the following four fundamental assumptions:

$$A(x,0)\equiv 0, \tag{1.2}$$

$$\frac{\partial A}{\partial y}(0,0)\neq 0, \tag{1.3}$$

$$B(x,0)\equiv\frac{\partial B}{\partial y}(x,0)\equiv 0, \tag{1.4}$$

$$C(0,0)\neq 0. \tag{1.5}$$

In the following we always assume (1.2) $\sim$ (1.5). In §1.2 we will give one more important assumption (cf. (1.11)).

Remark 1.1 The assumptions (1.2) and (1.4) imply $A(0,0)=B(0,0)=0$, which means that the equation (1.1) is *singular at the origin.* Moreover it follows from (1.2), (1.3) and (1.4) that the Jacobi matrix $\partial(A,B)/\partial(x,y)|_{(x,y)=(0,0)}$ is a nilpotent matrix

$$\begin{pmatrix} 0 & (\partial A/\partial y)(0,0) \\ 0 & 0 \end{pmatrix}. \tag{1.6}$$

In this sence our equation is called of *nilpotent type.*

By assumptions we see that the equation (1.1) has a unique formal power series solution $u(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n$ ($u_n(x)$ are holomorphic in a common neighborhood of $x=0$), but it diverges in general and the rate of divergence is characterized in terms of the Gevrey index (cf. Definidion 1.1, (3) and Theorem 1.1). So we are concerned with the existence of Gevrey asymptotic solutions, and especially we are interested in the Borel summability of such divergent solutions (cf. Definition 1.1, (5)). Our main purpose is to obtain the conditions under which such divergent solutions are Borel summable. The main result in this paper will be given in Theorem 1.2.

1.1 Definition and Fundamental Result.

Firstly, in order to state our problem precisely, let us introduce the notation.

Definition 1.1 (1) $\mathcal{O}[R]$ denotes the ring of holomorphic functions on the closed ball $B(R) = \{x \in \mathbf{C};\ |x| \le R\}$, where R is a positive number.

(2) The ring of formal power series in y ($\in \mathbf{C}$) over the ring $\mathcal{O}[R]$ is denoted as $\mathcal{O}[R][[y]]$:

$$\mathcal{O}[R][[y]] = \left\{ u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n;\ u_n(x) \in \mathcal{O}[R] \right\}. \tag{1.7}$$

(3) We say that $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n$ ($\in \mathcal{O}[R][[y]]$) *belongs to* $\mathcal{O}[R][[y]]_2$ if there exist some positive constants C and K such that

$$\max_{|x|\le R} |u_n(x)| \le CK^n n! \tag{1.8}$$

for all $n = 0, 1, 2, \dots$. Therefore elements of $\mathcal{O}[R][[y]]_2$ diverge in general.

(4) For $\theta \in \mathbf{R}$ and $T > 0$, we define the region $O(\theta, T)$ by

$$O(\theta,T) = \{y;\ |y - Te^{i\theta}| < T\}. \tag{1.9}$$

(5) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$. We say that $u(x,y)$ *is Borel summable in a direction* θ if there exists a holomorphic function $U(x,y)$ on $B(r) \times O(\theta,T)$ for some $0 < r \le R$ and $T > 0$ which satisfies the following asymptotic estimates: There exist some positive constants C and K such that

$$\max_{|x|\le r} \left| U(x,y) - \sum_{n=0}^{N-1} u_n(x)y^n \right| \le CK^N N! |y|^N, \quad y \in O(\theta,T); \quad N = 1,2,\dots. \tag{1.10}$$

In general a given divergent power series $u(x,y) \in \mathcal{O}[R][[y]]_2$ is not necessarily Borel summable. However, if $u(x,y)$ is Borel summable in a direction θ, we see that the above holomorphic function $U(x,y)$ is unique (cf. Balser[1][2], Lutz-Miyake-Schäfke[5] and Malgrange[6]). So we call this $U(x,y)$ the *Borel sum of* $u(x,y)$ *in a direction* θ.

The following theorem is fundamental in the argument below.

Theorem 1.1 (cf. Hibino[4]) *Let us assume* (1.2) $\sim$ (1.5). *Then the equation* (1.1) *has a unique formal power series solution* $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ *for some* $R > 0$.

On the basis of Theorem 1.1, let us study the Borel summability of the formal solution.

1.2 Main Result.

In the following we study the Borel summability of the formal solution under the following condition:

$$\frac{\partial^2 B}{\partial y^2}(x,0) \equiv 0. \tag{1.11}$$

Now, before stating the main theorem, let us rewrite the equation (1.1).

By the condition (1.5), we see that $C(x,y) \neq 0$ in the neighborhood of $(x,y) = (0,0)$. Therefore by dividing the both sides of (1.1) by $C(x,y)$, we may assume that $C(x,y) \equiv 1$. Then it follows from (1.2), (1.3), (1.4) and (1.11) that the equation (1.1) is rewritten in the following form:

$$\{\alpha(x) + \beta(x,y)\}yD_x u(x,y) + \gamma(x,y)y^2 D_y u(x,y) + u(x,y) = f(x,y), \tag{1.12}$$

where α, β, γ and f are holomorphic at the origin. Moreover they satisfy

$$\alpha(0) \neq 0, \tag{1.13}$$

$$\beta(x,0) \equiv \gamma(x,0) \equiv 0. \tag{1.14}$$

Furthermore in this paper we assume for simplicity that $\alpha(x)$ is the constant. Precisely, we consider the Borel summability of the formal solution for the following equation:

$$\{\alpha + \beta(x,y)\}yD_x u(x,y) + \gamma(x,y)y^2 D_y u(x,y) + u(x,y) = f(x,y), \tag{1.15}$$

where α is the constant satisfying $\alpha \neq 0$. Our purpose in this paper is to give the conditions under which the formal solution $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ of the equation (1.15) is Borel summable in a given direction θ.

Now let us give the conditions which the coefficients should satisfy.

Assumptions.

First we define the region $E_+(\theta,\kappa)$ $(\kappa > 0)$ by

$$E_+(\theta,\kappa) = \{\xi;\ \mathrm{dis}(\xi, \mathbf{R}_+ e^{i\theta}) \equiv \inf\{|\xi - \zeta|;\ \zeta \in \mathbf{R}_+ e^{i\theta}\} \leq \kappa\}, \tag{1.16}$$

where $\mathbf{R}_+ = [0, +\infty)$ and $\mathbf{R}_+ e^{i\theta} = \{re^{i\theta};\ r \in \mathbf{R}_+\}$. We assume the following (A1) and (A2).

(A1) $\beta(x,y)$, $\gamma(x,y)$ and $f(x,y)$ are continued analytically to $E_+(\theta + \pi + \arg(\alpha), \kappa) \times \{y \in \mathbf{C};\ |y| \leq c\}$ for some $\kappa > 0$ and $c > 0$.

(A2) $\beta(x,y)$, $\gamma(x,y)$ and $f(x,y)$ have the following estimates on $E_+(\theta + \pi + \arg(\alpha), \kappa) \times \{y \in \mathbf{C};\ |y| \leq c\}$:

$$\sup_{x \in E_+(\theta+\pi+\arg(\alpha),\kappa),\ |y| \leq c} |\beta(x,y)| < \infty; \tag{1.17}$$

$$\max_{|y| \leq c} |\gamma(x,y)| \leq \frac{K}{\{1 + |x|\}^q}, \quad x \in E_+(\theta + \pi + \arg(\alpha), \kappa) \tag{1.18}$$

for some positive constants $K > 0$ and $q > 1$;

$$\max_{|y| \leq c} |f(x,y)| \leq Ce^{\delta|x|}, \quad x \in E_+(\theta + \pi + \arg(\alpha), \kappa) \tag{1.19}$$

for some positive constants $C > 0$ and $\delta > 0$.

Then we obtain the following main result in this paper.

Theorem 1.2 *Under the assumptions* (A1) *and* (A2) *the formal solution* $u(x,y)$ *of the equation* (1.15) *is Borel summable in the direction* θ.

Remark 1.2 When the formal solution $u(x,y)$ of (1.15) is Borel summable, we see that its Borel sum is a holomorphic solution of (1.15). This is an immediate consequence of the uniqueness of the Borel sum.

2 Formal Borel Transform of Equations.

Before proving Theorem 1.2, we give some preliminaries.

Definition 2.1 For $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$, we define a convergent power series $\mathcal{B}(u)(x,\eta)$ in a neighborhood of $(x,\eta) = (0,0)$ by

$$\mathcal{B}(u)(x,\eta) = \sum_{n=0}^{\infty} u_n(x)\frac{\eta^n}{n!}. \tag{2.1}$$

We call $\mathcal{B}(u)(x,\eta)$ the *formal Borel transform of* $u(x,y)$.

When we want to check the Borel summability of a given formal power series $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$, the following theorem plays a fundamental role in general.

Theorem 2.1 ([5] and [6]) *For* $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$, *let us put* $v(x,\eta) = \mathcal{B}(u)(x,\eta)$. *Then the following two conditions* (i) *and* (ii) *are equivalent:*

(i) $u(x,y)$ *is Borel summable in a direction* θ.

(ii) $v(x,\eta)$ *can be continued analytically to* $B(r_0) \times E_+(\theta,\kappa_0)$ *for some* $r_0 > 0$ *and* $\kappa_0 > 0$, *and has the following exponential growth estimate for some positive constants* C *and* δ:

$$\max_{|x|\le r_0} |v(x,\eta)| \le Ce^{\delta|\eta|}, \quad \eta \in E_+(\theta,\kappa_0). \tag{2.2}$$

When the condition (i) *or* (ii) *(therefore both) is satisfied, the Borel sum* $U(x,y)$ *of* $u(x,y)$ *in the direction* θ *is given by*

$$U(x,y) = \frac{1}{y}\int_{\mathbf{R}_+e^{i\theta}} e^{-\eta/y} v(x,\eta)d\eta. \tag{2.3}$$

Therefore in order to prove Theorem 1.2, it is sufficient to prove that the formal Borel transform $v(x,\eta) = \mathcal{B}(u)(x,\eta)$ of the formal solution $u(x,y)$ satisfies the above condition (ii) under the assumptions (A1) and (A2). In order to do that, firstly let us write down the equation which $\mathcal{B}(u)(x,\eta)$ should satisfy. By operating the formal Borel transform to (1.15), we see that $\mathcal{B}(u)(x,\eta)$ is a solution of the following equation:

$$\begin{aligned} &\alpha\int_0^\eta D_x v(x,t)dt + \int_0^\eta \mathcal{B}(\beta)(x,\eta-t)D_x v(x,t)dt \\ &\quad + \int_0^\eta \mathcal{B}(\gamma)_\eta(x,\eta-t)\cdot tv(x,t)dt - \int_0^\eta \mathcal{B}(\gamma)(x,\eta-t)v(x,t)dt + v(x,\eta) \\ &= \mathcal{B}(f)(x,\eta), \end{aligned} \tag{2.4}$$

where $\mathcal{B}(\beta)(x,\eta)$, $\mathcal{B}(\gamma)(x,\eta)$ and $\mathcal{B}(f)(x,\eta)$ are the formal Borel transforms of $\beta(x,y) = \sum_{n=1}^{\infty} \beta_n(x)y^n$, $\gamma(x,y) = \sum_{n=1}^{\infty} \gamma_n(x)y^n$ and $f(x,y) = \sum_{n=0}^{\infty} f_n(x)y^n$, respectively, that is,

$$\mathcal{B}(\beta)(x,\eta) = \sum_{n=1}^{\infty} \beta_n(x)\frac{\eta^n}{n!}, \quad \mathcal{B}(\gamma)(x,\eta) = \sum_{n=1}^{\infty} \gamma_n(x)\frac{\eta^n}{n!} \text{ and } \mathcal{B}(f)(x,\eta) = \sum_{n=0}^{\infty} f_n(x)\frac{\eta^n}{n!}.$$

Furthermore by operating D_η to the equation (2.4) from the left, we see that $\mathcal{B}(u)(x,\eta)$ is a solution of the following initial value problem:

(2.5)

$$\begin{cases} \{D_\eta + \alpha D_x\}v(x,\eta) &= -\int_0^\eta \mathcal{B}(\beta)_\eta(x,\eta-t)D_x v(x,t)dt - \mathcal{B}(\gamma)_\eta(x,0)\cdot \eta v(x,\eta) \\ & \quad - \int_0^\eta \mathcal{B}(\gamma)_{\eta\eta}(x,\eta-t)\cdot tv(x,t)dt \\ & \quad + \int_0^\eta \mathcal{B}(\gamma)_\eta(x,\eta-t)v(x,t)dt + g(x,\eta), \\ v(x,0) = f(x,0), & \end{cases}$$

where $g(x,\eta) = \mathcal{B}(f)_\eta(x,\eta)$.

It is easy to prove that $\mathcal{B}(u)(x,\eta)$ is the unique locally holomorphic solution of (2.5). Hence Theorem 1.2 will be proved by showing that under the assumptions (A1) and (A2) the solution $v(x,\eta)$ of the equation (2.5) satisfies the condition (ii) in Theorem 2.1.

3 Proof of Theorem 1.2.

Let us prove that the solution $v(x,\eta)$ of the equation (2.5) satisfies the condition (ii) in Theorem 2.1. Firstly we remark that in general the solution $V(x,\eta)$ of the initial value problem of the following first order linear partial differential equation

$$\begin{cases} \{D_\eta + \alpha D_x\}V(x,\eta) = k(x,\eta), \\ V(x,0) = l(x) \end{cases} \tag{3.1}$$

is given by

$$V(x,\eta) = \int_0^\eta k(x-\alpha(\eta-t),t)dt + l(x-\alpha\eta). \tag{3.2}$$

Proof of Theorem 1.2. First, let us transform the equation (2.5) into the integral equation. It follows from (3.2) that the equation (2.5) is equivalent to the following equation:

$$v(x,\eta) = f(x-\alpha\eta,0) + \int_0^\eta g(x-\alpha(\eta-t),t)dt + Iv(x,\eta) + \sum_{i=5}^{7} I_i v(x,\eta),$$

where each operator I and I_i ($i = 5, 6, 7$) is given by

$$Iv(x,\eta) = -\int_0^\eta \int_0^t \mathcal{B}(\beta)_\eta(x-\alpha(\eta-t),t-s)v_x(x-\alpha(\eta-t),s)dsdt,$$

and

(3.3)

$$\begin{aligned} I_5 v(x,\eta) &= -\int_0^\eta \mathcal{B}(\gamma)_\eta(x-\alpha(\eta-t),0)\cdot tv(x-\alpha(\eta-t),t)dt, \\ I_6 v(x,\eta) &= -\int_0^\eta \int_0^t \mathcal{B}(\gamma)_{\eta\eta}(x-\alpha(\eta-t),t-s)\cdot sv(x-\alpha(\eta-t),s)dsdt, \\ I_7 v(x,\eta) &= \int_0^\eta \int_0^t \mathcal{B}(\gamma)_\eta(x-\alpha(\eta-t),t-s)v(x-\alpha(\eta-t),s)dsdt. \end{aligned}$$

Moreover, let us transform $Iv(x,\eta)$. By using Fubini's theorem, we write $\int_0^\eta \int_0^t \cdots dsdt = \int_0^\eta \int_s^\eta \cdots dtdt$. Here we remark that

$$\begin{aligned}&\int_s^\eta \mathcal{B}(\beta)_\eta(x-\alpha(\eta-t), t-s)v_x(x-\alpha(\eta-t), s)dt\\ =\ &\frac{1}{\alpha}\int_s^\eta \mathcal{B}(\beta)_\eta(x-\alpha(\eta-t), t-s)\frac{\partial}{\partial t}v(x-\alpha(\eta-t), s)dt.\end{aligned}$$

Therefore by an integration by parts and Fubini's theorem again we see that (2.5) is equivalent to the following equation:

$$v(x,\eta) = f(x-\alpha\eta, 0) + \int_0^\eta g(x-\alpha(\eta-t), t)dt + \sum_{i=1}^{7} I_i v(x,\eta), \tag{3.4}$$

where each operator I_i ($i = 1, 2, 3, 4$) is given by

$$\begin{aligned}I_1v(x,\eta) &= -\frac{1}{\alpha}\int_0^\eta \mathcal{B}(\beta)_\eta(x, \eta-t)v(x,t)dt,\\ I_2v(x,\eta) &= \frac{1}{\alpha}\int_0^\eta \mathcal{B}(\beta)_\eta(x-\alpha(\eta-t), 0)v(x-\alpha(\eta-t), t)dt,\\ I_3v(x,\eta) &= \frac{1}{\alpha}\int_0^\eta\int_0^t \mathcal{B}(\beta)_{\eta\eta}(x-\alpha(\eta-t), t-s)v(x-\alpha(\eta-t), s)dsdt,\\ I_4v(x,\eta) &= \frac{1}{\alpha}\int_0^\eta\int_0^t \mathcal{B}(\beta)_{x\eta}(x-\alpha(\eta-t), t-s)v(x-\alpha(\eta-t), s)dsdt,\end{aligned} \tag{3.5}$$

and I_5, I_6 and I_7 are same as (3.3).

In order to prove that the solution $v(x,\eta)$ of (3.4) satisfies the condition (ii) in Theorem 2.1 we employ the iteration method. Let us define $\{v_n(x,\eta)\}_{n=0}^\infty$ inductively as follows:

$$v_0(x,\eta) = f(x-\alpha\eta, 0) + \int_0^\eta g(x-\alpha(\eta-t), t)dt.$$

For $n \geq 0$,

$$v_{n+1}(x,\eta) = v_0(x,\eta) + \sum_{i=1}^{7} I_i v_n(x,\eta). \tag{3.6}$$

Next, we define $\{w_n(x,\eta)\}_{n=0}^\infty$ by $w_0(x,\eta) = v_0(x,\eta)$ and $w_n(x,\eta) = v_n(x,\eta) - v_{n-1}(x,\eta)$ ($n \geq 1$), and define $\{W_n(x,\eta,t)\}_{n=0}^\infty$ by

$$W_n(x,\eta,t) = w_n(x-\alpha(\eta-t), t). \tag{3.7}$$

Definition 3.1 (1) For $\lambda \geq 0$ and $\rho > 0$, $U_\rho[0,\lambda]$ denotes the ρ-neighborhood of $[0,\lambda]$ in $\mathbf{C}$. Precisely,

$$U_\rho[0,\lambda] = \{\tau \in \mathbf{C};\ \mathrm{dis}(\tau, [0,\lambda]) < \rho\}.$$

(2) For $\eta \in \mathbf{C}$ we define the function $G^\eta(\tau)$ by

$$G^\eta(\tau) = \tau e^{i\arg(\eta)}, \quad \tau \in \mathbf{C},$$

and define G^η and G^η_ρ as follows:

$$\begin{aligned} G^\eta &= \{G^\eta(R) \in \mathbf{C};\ 0 \le R \le |\eta|\}, \\ G^\eta_\rho &= \{G^\eta(\tau) \in \mathbf{C};\ \tau \in U_\rho[0, |\eta|]\}. \end{aligned}$$

We remark that G^η is the segment from 0 to η and that G^η_ρ is the ρ-neighborhood of G^η.

Now we can take $r_0 > 0$ and $\kappa_0 > 0$ such that

$$\{x - \alpha\zeta;\ |x| \le r_0,\ \zeta \in E_+(\theta, \kappa_0)\} \subset E_+(\theta + \pi + \arg(\alpha), \kappa), \tag{3.8}$$

where $\kappa > 0$ is the constant given in the assumption (A1). So let us define $\widetilde{\beta}(x, \zeta, y)$, $\widetilde{\gamma}(x, \zeta, y)$ as follows:

$$\widetilde{\beta}(x, \zeta, y) = \beta(x - \alpha\zeta, y), \tag{3.9}$$

$$\widetilde{\gamma}(x, \zeta, y) = \gamma(x - \alpha\zeta, y). \tag{3.10}$$

Then it follows from the assumptions and (3.8) that $\widetilde{\beta}(x, \zeta, y)$ and $\widetilde{\gamma}(x, \zeta, y)$ are holomorphic on $\{x \in \mathbf{C};\ |x| \le r_0\} \times E_+(\theta, \kappa_0) \times \{y \in \mathbf{C};\ |y| \le c\}$. Moreover it holds that

$$\sup_{|x| \le r_0,\, \zeta \in E_+(\theta, \kappa_0),\, |y| \le c} |\widetilde{\beta}(x, \zeta, y)| < \infty \tag{3.11}$$

and

$$\max_{|x| \le r_0,\, |y| \le c} |\widetilde{\gamma}(x, \zeta, y)| \le \frac{K_0}{(1 + |\zeta|)^q}, \quad \zeta \in E_+(\theta, \kappa_0), \tag{3.12}$$

for some positive constant K_0.

Next let us define $\mathcal{B}(\widetilde{\beta})(x, \zeta, \eta)$ and $\mathcal{B}(\widetilde{\gamma})(x, \zeta, \eta)$ by

$$\mathcal{B}(\widetilde{\beta})(x, \zeta, \eta) = \mathcal{B}(\beta)(x - \alpha\zeta, \eta) \quad \left(= \sum_{n=1}^{\infty} \beta_n(x - \alpha\zeta) \frac{\eta^n}{n!} \right) \tag{3.13}$$

and

$$\mathcal{B}(\widetilde{\gamma})(x, \zeta, \eta) = \mathcal{B}(\gamma)(x - \alpha\zeta, \eta) \quad \left(= \sum_{n=1}^{\infty} \gamma_n(x - \alpha\zeta) \frac{\eta^n}{n!} \right). \tag{3.14}$$

Then we see from (3.11), (3.12) and Cauchy's integral formula that $\mathcal{B}(\widetilde{\beta})(x, \zeta, \eta)$ and $\mathcal{B}(\widetilde{\gamma})(x, \zeta, \eta)$ are holomorphic on $\{x \in \mathbf{C};\ |x| \le r_0\} \times E_+(\theta, \kappa_0) \times \mathbf{C}$ and that there exist some positive constants M and δ_0 such that

(3.15)

$$\begin{cases} \displaystyle\sup_{|x| \le r_0,\, \zeta \in E_+(\theta, \kappa_0)} \left| \frac{1}{\alpha} \mathcal{B}(\widetilde{\beta})_\eta(x, \zeta, \eta) \right| \le M e^{\delta_0 |\eta|}, & \eta \in \mathbf{C}, \\ \displaystyle\sup_{|x| \le r_0,\, \zeta \in E_+(\theta, \kappa_0)} \left| \frac{1}{\alpha} \mathcal{B}(\widetilde{\beta})_{\eta\eta}(x, \zeta, \eta) \right| \le M e^{\delta_0 |\eta|}, & \eta \in \mathbf{C}, \\ \displaystyle\sup_{|x| \le r_0,\, \zeta \in E_+(\theta, \kappa_0')} \left| \frac{1}{\alpha} \frac{\partial}{\partial \zeta} \mathcal{B}(\widetilde{\beta})_\eta(x, \zeta, \eta) \right| \le M e^{\delta_0 |\eta|}, & \eta \in \mathbf{C}, \\ \displaystyle\max_{|x| \le r_0} |\mathcal{B}(\widetilde{\gamma})_\eta(x, \zeta, \eta)| \le \frac{M}{(1 + |\zeta|)^q} e^{\delta_0 |\eta|} \ (\le M e^{\delta_0 |\eta|}), & \zeta \in E_+(\theta, \kappa_0),\ \eta \in \mathbf{C}, \\ \displaystyle\max_{|x| \le r_0} |\mathcal{B}(\widetilde{\gamma})_{\eta\eta}(x, \zeta, \eta)| \le \frac{M}{(1 + |\zeta|)^q} e^{\delta_0 |\eta|}, & \zeta \in E_+(\theta, \kappa_0),\ \eta \in \mathbf{C}, \end{cases}$$

where $\kappa_0{}' = \kappa_0/2$. Under these preparations let us take a monotonically decreasing positive sequence $\{\rho_n\}_{n=0}^{\infty}$ satisfying

$$\widetilde{\kappa} = \kappa_0{}' - \sum_{n=0}^{\infty} \rho_n > 0. \tag{3.16}$$

Then we obtain the following lemma:

Lemma 3.1 *$W_n(x,\eta,t)$ is continued analytically to $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta, \kappa_0{}' - \sum_{j=0}^{n} \rho_j),\ t \in G_{\rho_n}^{\eta}\}$. Moreover on $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta, \kappa_0{}' - \sum_{j=0}^{n} \rho_j),\ t \in G^{\eta}\}$ we have the following estimate: For some positive constant C_1,*

$$\begin{aligned} &|W_n(x,\eta,G^{\eta}(R))| \\ &\le\ C_1 e^{\delta_1|\eta|}(9M)^n \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{R^l}{l!}, \quad 0 \le R \le |\eta|, \end{aligned} \tag{3.17}$$

where $\delta_1 = \max\{\delta, \delta_0\}$ (δ is the constant given in (1.19)).

We shall prove Lemma 3.1 in §4. Here let us admit it. Then Theorem 1.2 is proved as follows: It follows from Lemma 3.1 that $w_n(x,\eta)$ $(= W_n(x,\eta,\eta))$ is continued analytically to $B(r_0) \times E_+(\theta, \kappa_0{}' - \sum_{j=0}^{n} \rho_j)$ with the estimate

$$\begin{aligned} |w_n(x,\eta)| &= |W_n(x,\eta,G^{\eta}(|\eta|))| \\ &\le C_1 e^{\delta_1|\eta|}(9M)^n \sum_{k=0}^{n}\frac{1}{(q-1)^k}\binom{n}{k}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{|\eta|^l}{l!}. \end{aligned}$$

Hence on $B(r_0) \times E_+(\theta, \widetilde{\kappa})$ we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} |w_n(x,\eta)| &\le C_1 e^{\delta_1|\eta|}\sum_{n=0}^{\infty}(9M)^n \sum_{k=0}^{n}\frac{1}{(q-1)^k}\binom{n}{k}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{|\eta|^l}{l!} \\ &\le \widetilde{C}e^{\widetilde{\delta}|\eta|}, \end{aligned}$$

for some positive constants $\widetilde{C}$ and $\widetilde{\delta}$.

This shows that $v_n(x,\eta)$ $(= \sum_{k=0}^{n} w_k(x,\eta))$ converges to the solution $V(x,\eta)$ of (3.4) uniformly on $B(r_0) \times E_+(\theta,\widetilde{\kappa})$. Therefore $V(x,\eta)$ is the analytic continuation of $v(x,\eta)$ and it holds that

$$\max_{|x|\le r_0} |V(x,\eta)| \le \widetilde{C}e^{\widetilde{\delta}|\eta|}, \quad \eta \in E_+(\theta,\widetilde{\kappa}).$$

It follows from the above argument that $v(x,\eta)$ satisfies the condition (ii) in Theorem 2.1. This completes the proof of Theorem 1.2. ∎

4 Proof of Lemma 3.1.

Let us prove Lemma 3.1. It is proved by the induction with respect to n.

Proof of Lemma 3.1. The case $n = 0$ have been already proved in [3]. We assume that the claim of the lemma is proved up to n and prove it for $n+1$.

By (3.6) and (3.7) we have the following relation between W_n and W_{n+1}:

$$W_{n+1}(x,\eta,t)=\sum_{i=1}^{7}\mathcal{I}_i W_n(x,\eta,t), \tag{4.1}$$

where

$$\begin{aligned}
\mathcal{I}_1 W_n(x,\eta,t) &= I_1 w_n(x-\alpha(\eta-t),t)\\
&= -\frac{1}{\alpha}\int_0^t \mathcal{B}(\widetilde{\beta})_\eta(x,\eta-t,t-s)W_n(x,\eta-t+s,s)ds,\\
\mathcal{I}_2 W_n(x,\eta,t) &= I_2 w_n(x-\alpha(\eta-t),t)\\
&= \frac{1}{\alpha}\int_0^t \mathcal{B}(\widetilde{\beta})_\eta(x,\eta-s,0)W_n(x,\eta,s)ds,\\
\mathcal{I}_3 W_n(x,\eta,t) &= I_3 w_n(x-\alpha(\eta-t),t)\\
&= \frac{1}{\alpha}\int_0^t\int_0^s \mathcal{B}(\widetilde{\beta})_{\eta\eta}(x,\eta-s,s-z)W_n(x,\eta-s+z,z)dzds,\\
\mathcal{I}_4 W_n(x,\eta,t) &= I_4 w_n(x-\alpha(\eta-t),t)\\
&= -\frac{1}{\alpha}\int_0^t\int_0^s \frac{\partial}{\partial\zeta}\mathcal{B}(\widetilde{\beta})_\eta(x,\zeta,s-z)\Big|_{\zeta=\eta-s}\cdot W_n(x,\eta-s+z,z)dzds,\\
\mathcal{I}_5 W_n(x,\eta,t) &= I_5 w_n(x-\alpha(\eta-t),t)\\
&= -\int_0^t \mathcal{B}(\widetilde{\gamma})_\eta(x,\eta-s,0)\cdot sW_n(x,\eta,s)ds,\\
\mathcal{I}_6 W_n(x,\eta,t) &= I_6 w_n(x-\alpha(\eta-t),t)\\
&= -\int_0^t\int_0^s \mathcal{B}(\widetilde{\gamma})_{\eta\eta}(x,\eta-s,s-z)\cdot zW_n(x,\eta-s+z,z)dzds,\\
\mathcal{I}_7 W_n(x,\eta,t) &= I_7 w_n(x-\alpha(\eta-t),t)\\
&= \int_0^t\int_0^s \mathcal{B}(\widetilde{\gamma})_\eta(x,\eta-s,s-z)W_n(x,\eta-s+z,z)dzds.
\end{aligned}$$

Let us prove that each $\mathcal{I}_iW_n(x,\eta,t)$ $(i=1\sim 7)$ is well-defined on $\{(x,\eta,t);\ |x|\le r_0,\ \eta\in E_+(\theta,{\kappa_0}'-\sum_{j=0}^{n+1}\rho_j),\ t\in G^\eta_{\rho_{n+1}}\}$ by taking suitable paths of integrations. Let $|x|\le r_0$, $\eta\in E_+(\theta,{\kappa_0}'-\sum_{j=0}^{n+1}\rho_j)$, $t\in G^\eta_{\rho_{n+1}}$, and let us write $t\in G^\eta_{\rho_{n+1}}$ as $t=G^\eta(\tau)$ $(\tau\in U_{\rho_{n+1}}[0,|\eta|])$.

On $\mathcal{I}_1W_n(x,\eta,G^\eta(\tau))$: Let us take a path of integration as

$$s(\sigma)=\sigma e^{i\arg(\eta)}\quad(\sigma\in[0,\tau]), \tag{4.2}$$

where $[0,\tau]$ is a segment from 0 to τ. Then we have $\eta-G^\eta(\tau)+s(\sigma)\in E_+(\theta,{\kappa_0}'-\sum_{j=0}^{n}\rho_j)$ and $s(\sigma)\in G^{\eta-G^\eta(\tau)+s(\sigma)}_{\rho_n}$. Hence $W_n(x,\eta-G^\eta(\tau)+s(\sigma),s(\sigma))$ is well-defined. It is clear that $\mathcal{B}(\widetilde{\beta})_\eta(x,\eta-G^\eta(\tau),G^\eta(\tau)-s(\sigma))$ is well-defined. Therefore $\mathcal{I}_1W_n(x,\eta,G^\eta(\tau))$ is well-defined.

On $\mathcal{I}_2W_n(x,\eta,G^\eta(\tau))$ and $\mathcal{I}_5W_n(x,\eta,G^\eta(\tau))$: Let us take a path of integration as (4.2). Then we obtain $\eta\in E_+(\theta,{\kappa_0}'-\sum_{j=0}^{n}\rho_j)$ and $s(\sigma)\in G^\eta_{\rho_n}$. Hence $W_n(x,\eta,s(\sigma))$ is well-defined. It is clear that $\mathcal{B}(\widetilde{\beta})_\eta(x,\eta-s(\sigma),0)$ and $\mathcal{B}(\widetilde{\gamma})_\eta(x,\eta-s(\sigma),0)$ is well-defined. Therefore $\mathcal{I}_2W_n(x,\eta,G^\eta(\tau))$ and $\mathcal{I}_5W_n(x,\eta,G^\eta(\tau))$ are well-defined.

On $\mathcal{I}_iW_n(x,\eta,G^\eta(\tau))$ $(i=3,\ 4,\ 6$ and $7)$: We only state paths of integrations. The suitable paths of integrations are

$$\text{(4.3)}\qquad \begin{cases} s(\sigma) = \sigma e^{i\arg(\eta)} & (\sigma \in [0,\tau]), \\ z(\lambda) = \lambda e^{i\arg(\eta)} & (\lambda \in [0,\sigma]). \end{cases}$$

By taking the above paths of integrations, we see that each $\mathcal{I}_i W_n(x,\eta,t)$ $(i = 1 \sim 7)$ is well-defined (therefore $W_{n+1}(x,\eta,t)$ is well-defined) on $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta, \kappa_0' - \sum_{j=0}^{n+1}\rho_j),\ t \in G^\eta_{\rho_{n+1}}\}$. Moreover on $\{(x,\eta,t);\ |x| \le r_0,\ \eta \in E_+(\theta, \kappa_0' - \sum_{j=0}^{n+1}\rho_j),\ t \in G^\eta\}$ we have the following representations:

$$\begin{aligned}
\mathcal{I}_1 W_n(x,\eta,G^\eta(R)) &= -\frac{1}{\alpha}\int_0^R \mathcal{B}(\widetilde{\beta})_\eta(x, (|\eta|-R)e^{i\arg(\eta)}, (R-R_1)e^{i\arg(\eta)}) \\
&\qquad\qquad \times \widetilde{W}_n(x,\eta,R,R_1)e^{i\arg(\eta)}dR_1, \\
\mathcal{I}_2 W_n(x,\eta,G^\eta(R)) &= \frac{1}{\alpha}\int_0^R \mathcal{B}(\widetilde{\beta})_\eta(x, (|\eta|-R_1)e^{i\arg(\eta)}, 0)\widetilde{W}_n(x,\eta,R_1,R_1)e^{i\arg(\eta)}dR_1, \\
\mathcal{I}_3 W_n(x,\eta,G^\eta(R)) &= \frac{1}{\alpha}\int_0^R\int_0^{R_1} \mathcal{B}(\widetilde{\beta})_{\eta\eta}(x, (|\eta|-R_1)e^{i\arg(\eta)}, (R_1-R_2)e^{i\arg(\eta)}) \\
&\qquad\qquad \times \widetilde{W}_n(x,\eta,R_1,R_2)\{e^{i\arg(\eta)}\}^2 dR_2 dR_1, \\
\mathcal{I}_4 W_n(x,\eta,G^\eta(R)) &= -\frac{1}{\alpha}\int_0^R\int_0^{R_1} \frac{\partial}{\partial\zeta}\mathcal{B}(\widetilde{\beta})_\eta(x,\zeta,(R_1-R_2)e^{i\arg(\eta)})\Big|_{\zeta=(|\eta|-R_1)e^{i\arg(\eta)}} \\
&\qquad\qquad \times \widetilde{W}_n(x,\eta,R_1,R_2)\{e^{i\arg(\eta)}\}^2 dR_2 dR_1, \\
\mathcal{I}_5 W_n(x,\eta,G^\eta(R)) &= -\int_0^R \mathcal{B}(\widetilde{\gamma})_\eta(x, (|\eta|-R_1)e^{i\arg(\eta)}, 0) \\
&\qquad\qquad \times R_1\widetilde{W}_n(x,\eta,R_1,R_1)\{e^{i\arg(\eta)}\}^2 dR_1, \\
\mathcal{I}_6 W_n(x,\eta,G^\eta(R)) &= -\int_0^R\int_0^{R_1} \mathcal{B}(\widetilde{\gamma})_{\eta\eta}(x, (|\eta|-R_1)e^{i\arg(\eta)}, (R_1-R_2)e^{i\arg(\eta)}) \\
&\qquad\qquad \times R_2\widetilde{W}_n(x,\eta,R_1,R_2)\{e^{i\arg(\eta)}\}^3 dR_2 dR_1, \\
\mathcal{I}_7 W_n(x,\eta,G^\eta(R)) &= \int_0^R\int_0^{R_1} \mathcal{B}(\widetilde{\gamma})_\eta(x, (|\eta|-R_1)e^{i\arg(\eta)}, (R_1-R_2)e^{i\arg(\eta)}) \\
&\qquad\qquad \times \widetilde{W}_n(x,\eta,R_1,R_2)\{e^{i\arg(\eta)}\}^2 dR_2 dR_1,
\end{aligned}$$

where

$$\text{(4.4)}\qquad \widetilde{W}_n(x,\eta,\mu,\nu) = W_n(x, (|\eta|-\mu+\nu)e^{i\arg(\eta)}, G^{(|\eta|-\mu+\nu)e^{i\arg(\eta)}}(\nu)).$$

Let us estimate each $\mathcal{I}_i W_n(x,\eta,G^\eta(R))$.

On $\mathcal{I}_1 W_n(x,\eta,G^\eta(R))$: It follows from the assumption of the induction that

$$\text{(4.5)}\qquad \begin{aligned} &|\widetilde{W}_n(x,\eta,R,R_1)| \\ &\le C_1 e^{\delta_1|\eta|}e^{-\delta_1 R}e^{\delta_1 R_1}(9M)^n \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{{R_1}^l}{l!}. \end{aligned}$$

Hence (3.15) and $\delta_0 \le \delta_1$ imply that

$$\begin{aligned}
&|\mathcal{I}_1 W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n M \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\int_0^R \frac{{R_1}^l}{l!}dR_1 \\
=\ & C_1 e^{\delta_1|\eta|}(9M)^n M \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+1}^{2n+1}\binom{n}{l-1-n}\frac{R^l}{l!}.
\end{aligned}$$

On $\mathcal{I}_2 W_n(x,\eta,G^\eta(R))$: Let us consider R_1 instead of R in (4.5). Then we have

$$\begin{aligned}
&|\widetilde{W}_n(x,\eta,R_1,R_1)| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R_1)^{k(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{{R_1}^l}{l!} \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{{R_1}^l}{l!}.
\end{aligned}\tag{4.6}$$

Hence we see by (3.15) and $\delta_0 \le \delta_1$ that $\mathcal{I}_2 W_n(x,\eta,G^\eta(R))$ has the same estimate as that of $\mathcal{I}_1 W_n(x,\eta,G^\eta(R))$. Therefore it holds that

$$\begin{aligned}
&|\mathcal{I}_1 W_n(x,\eta,G^\eta(R))| + |\mathcal{I}_2 W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n (2M) \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+1}^{2n+1}\binom{n}{l-1-n}\frac{R^l}{l!}.
\end{aligned}\tag{4.7}$$

On $\mathcal{I}_3 W_n(x,\eta,G^\eta(R))$: It follows from the assumption of the induction that

$$\begin{aligned}
&|\widetilde{W}_n(x,\eta,R_1,R_2)| \\
\le\ & C_1 e^{\delta_1|\eta|}e^{-\delta_1 R_1}e^{\delta_1 R_2}(9M)^n \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R_1)^{k(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{{R_2}^l}{l!} \\
\le\ & C_1 e^{\delta_1|\eta|}e^{-\delta_1 R_1}e^{\delta_1 R_2}(9M)^n \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{{R_2}^l}{l!}.
\end{aligned}\tag{4.8}$$

Hence (3.15) and $\delta_0 \le \delta_1$ imply that

$$\begin{aligned}
&|\mathcal{I}_3 W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n M \\
&\times \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n}^{2n}\int_0^R\int_0^{R_1} \frac{{R_2}^l}{l!}dR_2 dR_1 \\
=\ & C_1 e^{\delta_1|\eta|}(9M)^n M \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+2}^{2(n+1)}\binom{n}{l-2-n}\frac{R^l}{l!}.
\end{aligned}$$

Similarly we can prove that $\mathcal{I}_4 W_n(x,\eta,G^\eta(R))$ and $\mathcal{I}_7 W_n(x,\eta,G^\eta(R))$ have the same estimates as that of $\mathcal{I}_3 W_n(x,\eta,G^\eta(R))$. Therefore it holds that

$$\begin{aligned}
&|\mathcal{I}_3 W_n(x,\eta,G^\eta(R))| + |\mathcal{I}_4 W_n(x,\eta,G^\eta(R))| + |\mathcal{I}_7 W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n (3M) \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+2}^{2(n+1)}\binom{n}{l-2-n}\frac{R^l}{l!}.
\end{aligned}\tag{4.9}$$

Moreover let us note that

$$\binom{n}{l-1-n}+\binom{n}{l-2-n}=\binom{n+1}{l-(n+1)}. \tag{4.10}$$

Then it follows from (4.7) and (4.9) that

$$\begin{aligned} &\sum_{i=1,2,3,4,7} |\mathcal{I}_i W_n(x,\eta,G^\eta(R))| \\ \leq\ & C_1 e^{\delta_1|\eta|}(9M)^n(3M)\sum_{k=0}^{n}\frac{1}{(q-1)^k}\binom{n}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+1}^{2(n+1)}\binom{n+1}{l-(n+1)}\frac{R^l}{l!}. \end{aligned} \tag{4.11}$$

On $\mathcal{I}_5 W_n(x,\eta,G^\eta(R))$: (3.15), (4.6) and $\delta_0\leq\delta_1$ imply that

$$\begin{aligned} &|\mathcal{I}_5 W_n(x,\eta,G^\eta(R))| \\ \leq\ & C_1 e^{\delta_1|\eta|}(9M)^n M\sum_{k=0}^{n}\frac{1}{(q-1)^k}\binom{n}{k}\int_0^R\frac{R_1}{(1+|\eta|-R_1)^{k(q-1)+q}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{{R_1}^l}{l!}dR_1 \\ \leq\ & C_1 e^{\delta_1|\eta|}(9M)^n M\sum_{k=0}^{n}\frac{1}{(q-1)^k}\binom{n}{k}\int_0^R\frac{1}{(1+|\eta|-R_1)^{k(q-1)+q}}dR_1\sum_{l=n}^{2n}\binom{n}{l-n}\frac{R^{l+1}}{l!}. \end{aligned}$$

Here it holds that

$$\begin{aligned} \int_0^R\frac{1}{(1+|\eta|-R_1)^{k(q-1)+q}}dR_1 &= \left[\frac{1}{k+1}\frac{1}{q-1}\frac{1}{(1+|\eta|-R_1)^{(k+1)(q-1)}}\right]_{R_1=0}^{R} \\ &\leq \frac{1}{k+1}\frac{1}{q-1}\frac{1}{(1+|\eta|-R)^{(k+1)(q-1)}} \end{aligned} \tag{4.12}$$

and that

$$\begin{aligned} \sum_{l=n}^{2n}\binom{n}{l-n}\frac{R^{l+1}}{l!} &= \sum_{l=n}^{2n}\binom{n}{l-n}(l+1)\frac{R^{l+1}}{(l+1)!} = \sum_{l=n+1}^{2n+1}\binom{n}{l-1-n}l\frac{R^l}{l!} \\ &\leq 2(n+1)\sum_{l=n+1}^{2n+1}\binom{n}{l-1-n}\frac{R^l}{l!}. \end{aligned}$$

Hence we have

$$\begin{aligned} &|\mathcal{I}_5 W_n(x,\eta,G^\eta(R))| \\ \leq\ & C_1 e^{\delta_1|\eta|}(9M)^n M \\ &\times\sum_{k=0}^{n}\frac{1}{(q-1)^{k+1}}\binom{n}{k}\frac{2(n+1)}{k+1}\frac{1}{(1+|\eta|-R)^{(k+1)(q-1)}}\sum_{l=n+1}^{2n+1}\binom{n}{l-1-n}\frac{R^l}{l!} \\ =\ & C_1 e^{\delta_1|\eta|}(9M)^n M \\ &\times\sum_{k=1}^{n+1}\frac{1}{(q-1)^{k}}\binom{n}{k-1}\frac{2(n+1)}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+1}^{2n+1}\binom{n}{l-1-n}\frac{R^l}{l!}. \end{aligned} \tag{4.13}$$

On $\mathcal{I}_6 W_n(x,\eta,G^\eta(R))$: (3.15), (4.8) and $\delta_0 \le \delta_1$ imply that

$$\begin{aligned}
&|\mathcal{I}_6 W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n M \\
&\times \sum_{k=0}^{n} \frac{1}{(q-1)^k}\binom{n}{k}\int_0^R \frac{1}{(1+|\eta|-R_1)^{k(q-1)+q}}\sum_{l=n}^{2n}\binom{n}{l-n}\int_0^{R_1}\frac{{R_2}^{l+1}}{l!}dR_2 dR_1.
\end{aligned}$$

Here let us estimate as

$$\int_0^{R_1}\frac{{R_2}^{l+1}}{l!}dR_2 = \frac{{R_1}^{l+2}}{l!(l+2)} \le 2(n+1)\frac{R^{l+2}}{(l+2)!}, \quad l = n, n+1, \dots, 2n. \tag{4.14}$$

Then (4.12) and (4.14) imply that

$$\begin{aligned}
&|\mathcal{I}_6 W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n M \\
&\times \sum_{k=0}^{n}\frac{1}{(q-1)^{k+1}}\binom{n}{k}\frac{2(n+1)}{k+1}\frac{1}{(1+|\eta|-R)^{(k+1)(q-1)}}\sum_{l=n}^{2n}\binom{n}{l-n}\frac{R^{l+2}}{(l+2)!} \\
=\ & C_1 e^{\delta_1|\eta|}(9M)^n M \\
&\times \sum_{k=1}^{n+1}\frac{1}{(q-1)^k}\binom{n}{k-1}\frac{2(n+1)}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+2}^{2(n+1)}\binom{n}{l-2-n}\frac{R^l}{l!}.
\end{aligned} \tag{4.15}$$

Therefore by (4.10), (4.13) and (4.15) we obtain

$$\begin{aligned}
&|\mathcal{I}_5 W_n(x,\eta,G^\eta(R))| + |\mathcal{I}_6 W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n M \\
&\times \sum_{k=1}^{n+1}\frac{1}{(q-1)^k}\binom{n}{k-1}\frac{2(n+1)}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+1}^{2(n+1)}\binom{n+1}{l-(n+1)}\frac{R^l}{l!}.
\end{aligned} \tag{4.16}$$

Finally let us combine (4.11) and (4.16). Then it holds that

$$\begin{aligned}
&|W_{n+1}(x,\eta,G^\eta(R))| \\
\le\ & \sum_{i=1}^{7}|\mathcal{I}_i W_n(x,\eta,G^\eta(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n(3M)\cdot\left\{1+\sum_{k=1}^{n}Z(k)+\frac{2}{(q-1)^{n+1}}\frac{1}{(1+|\eta|-R)^{(n+1)(q-1)}}\right\} \\
&\times \sum_{l=n+1}^{2(n+1)}\binom{n+1}{l-(n+1)}\frac{R^l}{l!},
\end{aligned}$$

where

$$Z(k) = \frac{1}{(q-1)^k}\left\{\binom{n}{k}+\binom{n}{k-1}\frac{2(n+1)}{k}\right\}\frac{1}{(1+|\eta|-R)^{k(q-1)}}, \quad k = 1, \dots, n.$$

Here let us note that

$$\binom{n}{k}+\binom{n}{k-1}\frac{2(n+1)}{k}=\binom{n}{k}+2\binom{n+1}{k}\le 3\binom{n+1}{k}.$$

Then we obtain that

$$\begin{aligned}
&|W_{n+1}(x,\eta,G^{\eta}(R))| \\
\le\ & C_1 e^{\delta_1|\eta|}(9M)^n(3M) \\
&\times 3\Bigg\{1+\sum_{k=1}^{n}\frac{1}{(q-1)^k}\binom{n+1}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}} \\
&\qquad +\frac{1}{(q-1)^{n+1}}\frac{1}{(1+|\eta|-R)^{(n+1)(q-1)}}\Bigg\}\sum_{l=n+1}^{2(n+1)}\binom{n+1}{l-(n+1)}\frac{R^l}{l!} \\
=\ & C_1 e^{\delta_1|\eta|}(9M)^{n+1}\sum_{k=0}^{n+1}\frac{1}{(q-1)^k}\binom{n+1}{k}\frac{1}{(1+|\eta|-R)^{k(q-1)}}\sum_{l=n+1}^{2(n+1)}\binom{n+1}{l-(n+1)}\frac{R^l}{l!},
\end{aligned}$$

which implies the lemma for $n+1$. The proof is completed. ∎

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