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On the exact WKB analysis for the fourth Painlevé hierarchy

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1 Introduction

In the exact WKB analytic study of Painlevé equations, [AKT], [KT1], [KT2] and [T] gave the explicit description of the connection formulas of Painlevé transcendents. In their analysis some properties on the relation which holds between the Stokes geometry of the Painlevé equations and that of their Lax pairs play the key role. After their success, [KKNT] showed that these properties of the Stokes geometry also hold for several Painlevé hierarchies (P_{I}, $P_{\mathrm{II}-1}$ and $P_{\mathrm{II}-2}$ in the notation of [KKNT]).

In this article we discuss the fourth Painlevé hierarchy which was introduced by Gordoa-Joshi-Pickering [GJP], and report that such basic properties of the Stokes geometry also hold for this hierarchy. Here we give the outline of our results; the detailed explanation of our results including numerical results of the Stokes geometry and of *Nishikawa Phenomena* will be published somewhere.

2 The fourth Painlevé hierarchy and their Lax pairs with a large parameter

The fourth Painlevé hierarchy we treat here was introduced in [GJP]. The following is essentially same with that given in [GJP, p.336], but we introduce a large parameter η to employ the exact WKB analysis:

Definition 2.1. *We call*

$$(P_{IV})_m : \begin{cases} \eta^{-1}\partial_t X_m = 2Y_m + uX_m + g - 2\alpha, \\ \eta^{-1}X_m\partial_t Y_m = -vX_m^2 + (Y_m + \frac{1}{2}g - \alpha)^2 - \frac{1}{4}\beta^2. \end{cases} \tag{2.1}$$

the fourth Painlevé hierarchy for $m = 1, 2, \cdots$. Here α, β and g are arbitrary constants, and

$$X_m = K_m + gt, \quad Y_m = L_m,$$

where $\{K_j\}$ and $\{L_j\}$ are polynomial of unknown functions u, v and their derivatives recursively defined by the following relation:

(2.2)

$$\eta^{-1}\partial_t \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \eta^{-1}u' + u\eta^{-1}\partial_t - \eta^{-2}\partial_t^2 & 2\eta^{-1}\partial_t \\ 2\eta^{-1}v\partial_t + \eta^{-1}v_t & u\eta^{-1}\partial_t + \eta^{-2}\partial_t^2 \end{pmatrix}\begin{pmatrix} K_j \\ L_j \end{pmatrix}$$

($j \geq 0$) with $K_0 = 2$ and $L_0 = 0$. Here, and in what follows, $'$ denotes the differentiation with respect to t.

Remark 2.1. *Precisely speaking, this recursion relation* (2.2) *does not define $\{K_j\}$ and $\{L_j\}$ uniquely; there remain ambiguities of integral constants for each step. In the following we take these integral constants as zero at each step. See* [N1], [N2] *for the precise definition of $\{K_j\}$ and $\{L_j\}$, and for the proof that this recursion relation* (2.2) *really defines $\{K_j\}$ and $\{L_j\}$ as polynomials of u and v and their derivatives. Here we list up first few members of $\{K_j\}$ and $\{L_j\}$ in a vector notation:*

$$\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.3}$$

$$\begin{pmatrix} K_2 \\ L_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u^2 + 2v - \eta^{-1}u' \\ 2uv + \eta^{-1}v' \end{pmatrix}, \tag{2.4}$$

$$\begin{pmatrix} K_3 \\ L_3 \end{pmatrix} = \left(\frac{1}{2}\right)^2 \begin{pmatrix} u^3 + 6uv - 3\eta^{-1}uu' + \eta^{-2}u'' \\ 3u^2v + 3v^2 + 3\eta^{-1}uv' + \eta^{-2}v'' \end{pmatrix}. \tag{2.5}$$

Remark 2.2. $(P_{\mathrm{IV}})_1$ *becomes*

$$(2.6)\qquad \begin{cases} \eta^{-1}u' &= 2v + u(u+gt) + g - 2\alpha, \\ \eta^{-1}(u+gt)v &= -v(u+gt)^2 + (v + \frac{1}{2}\alpha - g)^2 - \frac{1}{4}\beta^2. \end{cases}$$

We can solve the first equation with respect to v. By substituting this v into the second equation, we obtain

$$(2.7)\qquad y'' = \frac{(y')^2}{2y} + \eta^2 \left\{ \frac{3}{2}y^3 - 2gty^2 + (\frac{g^2}{2}t^2 + g - 2\alpha - \eta^{-1}g)y - \frac{\beta^2}{2y} \right\},$$

where $y = u + gt$. This is the traditional fourth Painlevé equation (the standard form of the fourth Painlevé equation is obtained by setting $g = -2$ and $\eta = 1$ in (2.7)). This is the reason whey we call (2.1) the fourth Painlevé hierarchy.

Remark 2.3. *As is explained in* [GJP] *we can add more terms with arbitrary constants in $(P_{\mathrm{IV}})_m$. Our analysis in the following can be applied to such extended $(P_{\mathrm{IV}})_m$.*

The underlying Lax pair of $(P_{\mathrm{IV}})_m$ is also introduced in [GJP]:

$$(2.8)\qquad (L_{\mathrm{IV}})_m : \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\psi = 0, & (2.8.a) \\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\psi = 0, & (2.8.b) \end{cases}$$

where

$$(2.9)\qquad A = \frac{1}{gx}\begin{pmatrix} -\dfrac{1}{2}\eta^{-1}\dfrac{\partial S_m}{\partial x} & S_m \\ qS_m - \dfrac{1}{2}\eta^{-2}\dfrac{\partial^2 S_m}{\partial x^2} & \dfrac{1}{2}\eta^{-1}\dfrac{\partial S_m}{\partial x} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix},$$

and

$$(2.10)\qquad q = \left(x - \frac{1}{2}u\right)^2 - v + \frac{1}{2}\eta^{-1}u_t, \quad S_m = \frac{1}{2}\left(tg + \sum_{j=0}^{m} x^{m-j}K_j\right).$$

Actually the compatibility condition

$$(2.11)\qquad \frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta(AB - BA) = 0$$

of $(L_{\mathrm{IV}})_m$ is reduced to $(P_{\mathrm{IV}})_m$.

3 Relations of the Stokes geometry of the P_{IV} hierarchy and its underlying Lax pair

In this section we considered the Stokes geometry of $(P_{\mathrm{IV}})_m$ and $(L_{\mathrm{IV}})_m$ (we will fix m in the following). To this purpose we first construct a formal solution of $(P_{\mathrm{IV}})_m$ of the following form

$$\text{(3.1)} \qquad \begin{cases} u &= \hat{u}(t,\eta) &= u_0(t) + \eta^{-1}u_1(t) + \cdots, \\ v &= \hat{v}(t,\eta) &= v_0(t) + \eta^{-1}v_1(t) + \cdots. \end{cases}$$

By substituting this expansion into $(P_{\mathrm{IV}})_m$, we find that u_0 and v_0 satisfy

$$\text{(3.2)} \qquad \begin{cases} 2Y_m^{(0)} + uX_m^{(0)} + g - 2\alpha = 0, \\ -v(X_m^{(0)})^2 + (Y_m^{(0)} + \dfrac{1}{2}g - \alpha)^2 - \dfrac{1}{4}\beta^2 = 0, \end{cases}$$

or, equivalently,

$$\text{(3.3)} \qquad \begin{cases} 2L_m^{(0)} + uK_m^{(0)} + g - 2\alpha = 0, \\ -v(K_m^{(0)} + gt)^2 + (L_m^{(0)} + \dfrac{1}{2}g - \alpha)^2 - \dfrac{1}{4}\beta^2 = 0, \end{cases}$$

where $X_m^{(0)}$, $Y_m^{(0)}$, $K_m^{(0)}$ and $L_m^{(0)}$ are top order terms of X_m, Y_m, K_m and L_m with respect to the large parameter η, respectively. We can show that $L_m^{(0)}$ and $K_m^{(0)}$ are polynomials of u and v (See [N1], [N2].), and hence (3.3) is algebraic equation with respect to u and v. Once u_0 and v_0 are chosen and fixed so that they satisfy (3.3), higher order terms u_j and v_j $(j \geq 1)$ are determined uniquely and recursively. A formal solution (3.1) constructed in this way is called a zero parameter solution in the following.

Once a zero parameter solution $u = \hat{u}$, $v = \hat{v}$ is constructed, we substitute this zero parameter solution into our quantities A, B, K_m, L_m, S_m and q. As a result, we obtain expansions of A and B etc. with respect to the large parameter η like

$$\text{(3.4)} \qquad A_m\big|_{u=\hat{u},v=\hat{v}} = A_{m,0}(x,t) + \eta^{-1}A_{m,1}(x,t) + \cdots,$$

$$\text{(3.5)} \qquad B_m\big|_{u=\hat{u},v=\hat{v}} = B_{m,0}(x,t) + \eta^{-1}B_{m,1}(x,t) + \cdots.$$

Explicitly we obtain

$$\text{(3.6)} \qquad A_{m,0} = \frac{1}{gx}\begin{pmatrix} 0 & S_{m,0} \\ q_0 S_{m,0} & 0 \end{pmatrix}, \quad B_{m,0} = \begin{pmatrix} 0 & 1 \\ q_0 & 0 \end{pmatrix},$$

where

$$q_0 = (x - \frac{1}{2}u_0)^2 - v_0, \tag{3.7}$$

$$S_{m,0} = \frac{1}{2}\left(tg + \sum_{j=0}^{m} x^{m-i}K_{i,0}\right). \tag{3.8}$$

Now we consider the Stokes geometry of (2.8.a) which is defined with the characteristic polynomial

$$\det(\lambda - A_{m,0}(x,t)) = \lambda^2 + \det A_{m,0}(x,t). \tag{3.9}$$

First we find $A_{m,0} = (2S_{m,0}/gx)B_{m,0}$ by (3.6), which implies

$$\det A_{m,0} = \left(\frac{2S_{m,0}}{gx}\right)^2 \det B_{m,0}. \tag{3.10}$$

Hence we obtain the following proposition on turning points of (2.8.a):

Proposition 3.1. *(i) Eq (2.8.a) has m double turning points*

$$\{b_1(t), b_2(t), \ldots, b_m(t)\}$$

and each $b_j(t)$ is a root of $S_{m,0}(x,t) = 0$.
(ii) Eq (2.8.a) has two simple turning points $\{a_1(t), a_2(t)\}$ and each $a_j(t)$ is a root of

$$\det B_{m,0} = (x - \frac{1}{2}u_0)^2 - v_0 = 0.$$

We also obtain the relation between the eigenvalues of A_0 and that of B_0. The following proposition can be shown easily (See [KKNT]):

Proposition 3.2. *Let λ_+ and λ_- be eigenvalues of A_0, μ_+ and μ_- eigenvalues of B_0. Then we obtain*

$$\frac{\partial}{\partial t}\lambda_+ = \frac{\partial}{\partial x}\mu_+, \quad \frac{\partial}{\partial t}\lambda_- = \frac{\partial}{\partial x}\mu_-, \tag{3.11}$$

or

$$\frac{\partial}{\partial t}\lambda_+ = \frac{\partial}{\partial x}\mu_-, \quad \frac{\partial}{\partial t}\lambda_= = \frac{\partial}{\partial x}\mu_+. \tag{3.12}$$

Next we consider the Stokes geometry for the fourth Painlevé hierarchy. The Stokes geometry of $(P_{\mathrm{IV}})_m$ is defined as that of linearized equation of $(P_{\mathrm{IV}})_m$ at a zero parameter solutions $u = \hat{u}$, $v = \hat{v}$. Hence we first determine the (characteristic polynomial of) linearized equation. To this purpose we set $u = \hat{u} + \Delta u$ and $v = \hat{v} + \Delta v$ in $(P_{\mathrm{IV}})_m$, and consider the linear part in $(\Delta u, \Delta v)$. Let ΔX_m and ΔY_m be linear parts of X_m and Y_m in $(\Delta u, \Delta v)$, respectively. In terms of this ΔX_m and ΔY_m, the linearized equation of $(P_{\mathrm{IV}})_m$ can be written as

$$(3.13)\quad \begin{cases} \eta^{-1}\dfrac{d}{dt}\Delta X_m = 2\Delta Y_m + \hat{u}\Delta X_m + \hat{X}_m \Delta u, \\ \hat{X}_m \eta^{-1}\dfrac{d}{dt}\Delta Y_m = -\eta^{-1}\dfrac{d\hat{Y}_m}{dt}\Delta X_m - (\hat{X}_m)^2\Delta v - 2\hat{v}\hat{X}_m\Delta X_m \\ \qquad\qquad +2(\hat{Y}_m + \dfrac{1}{2}g - \alpha)\Delta Y_m, \end{cases}$$

where

$$(3.14)\qquad \hat{X}_m = X_m|_{u=\hat{u},v=\hat{v}}\,, \quad \hat{Y}_m = Y_m|_{u=\hat{u},v=\hat{v}}\,.$$

This linearized equation can be expressed in the following form:

$$(3.15)\quad \eta^{-1}\frac{d}{dt}\begin{pmatrix}\Delta X_m \\ \Delta Y_m\end{pmatrix} = \begin{pmatrix} \hat{u} & 2 \\ -2\hat{v} - \eta^{-1}\dfrac{1}{\hat{X}_m}\dfrac{d\hat{Y}_m}{dt} & -\hat{u}\end{pmatrix}\begin{pmatrix}\Delta X_m \\ \Delta Y_m\end{pmatrix} + \hat{X}_m\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix}$$

where we have used

$$(3.16)\qquad 2\hat{Y}_m + g - 2\alpha = -u\hat{X}_m.$$

We will now determine ΔX_m and ΔY_m. By its definition we easily find that $\Delta X_m = \Delta K_m$ and $\Delta Y_m = \Delta Y_m$. Since K_m and L_m are polynomials in u, v and their derivatives, there exits a 2×2 matrix valued linear differential operator

$$(3.17)\qquad P^{(m)}(t, \eta^{-1}\partial_t; \eta) = P_0^{(m)}(t, \eta^{-1}\partial_t) + \eta^{-1}P_1^{(m)}(t, \eta^{-1}\partial_t) + \cdots .$$

for which

$$(3.18)\qquad \begin{pmatrix}\Delta K_m \\ \Delta L_m\end{pmatrix} = P^{(m)}(t, \eta^{-1}\partial_t; \eta)\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix}$$

holds. In terms of this operator $P^{(m)}(t, \eta^{-1}\partial_t; \eta)$, the characteristic polynomial $C(t,\nu)$ (i.e., the top order part with respect to η of the symbol obtained by replacing $\eta^{-1}\partial_t$ by ν) of (3.15) is expressed as

(3.19)

$$\begin{aligned} C(t,\nu) &= \det\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} \nu - u_0 & -2 \\ 2v_0 & \nu + u_0 \end{pmatrix} P_0^{(m)}(t,\nu) + \hat{X}_{m,0}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\right] \\ &= \det\left[\begin{pmatrix} -\nu + u_0 & 2 \\ 2v_0 & \nu + u_0 \end{pmatrix} P_0^{(m)}(t,\nu) + \hat{X}_{m,0}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]. \end{aligned}$$

(Here we multiply a matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for the later convenience).

Proposition 3.3. *We have*

$$C(t,\nu) = (-1)^m 4 \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{\mu=\nu/2, x=b_j(t)}, \tag{3.20}$$

where $\{b_j(t)\}$ are double turning points defined in Proposition 3.1.

Proof. To determine $P_0^{(m)}(t,\nu)$ we use the recursion relation (2.2). Consider the linear part in $(\Delta u, \Delta v)$ of both side of (2.2), we find that $\{P_0^{(j)}(t,\nu)\}$ should satisfy the following recursion relation:

$$P_0^{(j+1)}(t,\nu) = \mathcal{R}P_0^{(j)}(t,\nu) + \frac{K_{j,0}}{2} I_2 \tag{3.21}$$

where

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} -\nu + u_0 & 2 \\ 2v_0 & \nu + u_0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.22}$$

By solving this recursion relation with the initial condition $P_0^{(1)}(t,\nu) = I_2$, we have

$$P_0^{(m)}(t,\nu) = \frac{1}{2}\sum_{j=0}^{m-1} \mathcal{R}^{m-j-1}\hat{K}_{j,0}. \tag{3.23}$$

Hence we obtain

$$
\begin{aligned}
(3.24)\qquad C(t,\nu) &= \det\left[2\mathcal{R}P_0^{(m)}(t,\nu)+\hat{X}_{m,0}\right] \\
(3.25)\qquad &= \det\left[\sum_{j=0}^{m}\mathcal{R}^{m-j}K_{j,0}+gtI_2\right] \\
(3.26)\qquad &= 4\det\left[\frac{1}{2}\left(gtI_2+\sum_{j=0}^{m}\mathcal{R}^{m-j}K_{j,0}\right)\right].
\end{aligned}
$$

Here we note that $S_{m,0}$ can be factorized as

$$
(3.27)\qquad S_{m,0}=\frac{1}{2}\left(tg+\sum_{j=0}^{m}x^{m-j}K_{j,0}\right)=\prod_{j=1}^{m}(x-b_j(t))
$$

(See Proposition 3.1 (i)). Substituting $\mathcal{R}$ into x in this equation (3.27) we find

$$
(3.28)\qquad C(t,\nu)=4\prod_{j=1}^{m}(\mathcal{R}-b_jI_2).
$$

Hence we conclude that

$$
\begin{aligned}
(3.29)\qquad C(t,\nu) &= 4\det\prod_{j=1}^{m}(\mathcal{R}-b_j(t)I_2) \\
(3.30)\qquad &= 4\prod_{j=1}^{m}\left\{\frac{1}{4}\begin{pmatrix}-\nu+u_0-2b_j & 2\\ 2v_0 & \nu+u_0-2b_j\end{pmatrix}\right\} \\
(3.31)\qquad &= 4\prod_{j=1}^{m}\left\{-(\frac{1}{2}\nu)^2+(b_j-\frac{1}{2}u_0)^2-v_0\right\} \\
(3.32)\qquad &= (-1)^m4\prod_{j=1}^{m}\det(\mu-B_0)\Big|_{\mu=\nu/2,x=b_j(t)}
\end{aligned}
$$

This completes the proof of the proposition. □

As is explained already, Stokes geometry for $(P_{\mathrm{IV}})_m$ is defined in terms of $C(t,\nu)$. Note that it follows from the Proposition 3.3 that $C(t,\nu)$ has the form $f(\nu^2, t)$ with some polynomial f of degree m. This implies that there are two kind of turning points for $(P_{\rm IV})_m$: (i) A turning points where the degree 0 part of f vanishes ("a turning point of the first kind"), and (*ii*) a turning point where the discriminant of f vanishes ("a turning point of the second kind"). The following theorems are obtained by using Proposition 3.2 and 3.3:

Theorem 3.1. (i) *Let $t = \tau^{\rm I}$ be a turning point of the first kind of $(P_{IV})_m$. Assume* (a) *$t = \tau$ is not a turning point of second kind,* (b) *$\partial C/\partial t$ does not vanish at $t = \tau$, $\nu = 0$. Then at $t = \tau^{\rm I}$ a double turning point $x = b_j(t)$ merges with a simple turning point $x = a_k(t)$ in the Stokes geometry of* (2.8.a). *Consequently the two eigenvalues $\nu_{j,\pm}$ of C merge and vanish at $t = \tau^{\rm I}$. Furthermore the following relation holds:*

$$\frac{1}{2}\int_{\tau^{\rm I}}^{t}(\nu_{j,+} - \nu_{j,-})dt = \int_{a_k(t)}^{b_j(t)}(\lambda_+ - \lambda_-)dx. \tag{3.33}$$

(ii) *Let $t = \tau^{\rm II}$ be a turning point of the second kind of $(P_{\rm IV})_m$. Assume* (a) *$t = \tau$ is not a turning point of the fist kind,* (b) *$t = \tau$ is a simple zero of discriminant of $f(t, z)$. Then at $t = \tau^{\rm II}$ a double turning point $x = b_j(t)$ merges with another double turning point $x = b_{j'}(t)$. Consequently two eigenvalues $\nu_{j,+}$ and $\nu_{j',+}$ of C merge at $t = \tau^{\rm II}$, and so do $\nu_{j,-}$ and $\nu_{j',-}$. Furthermore the following relation holds:*

$$\int_{\tau^{\rm II}}^{t}(\nu_{j,+} - \nu_{j',+})dt = -\int_{\tau^{\rm II}}^{t}(\nu_{j,-} - \nu_{j',-})dt = \int_{b_{j'}(t)}^{b_j(t)}(\lambda_+ - \lambda_-)dx. \tag{3.34}$$

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