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<td>Author(s)</td>
<td>Shimomura, Shun</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1424: 177-183</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47227">http://hdl.handle.net/2433/47227</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
ON SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH THE QUASI-PAINLEVÉ PROPERTY II

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1. Introduction
We say that a nonlinear differential equation

$$y'' = R(x, y, y'), \quad R(x, y, z) \in \mathbb{C}(x, y, z)$$

has the quasi-Painlevé property if, for each solution, every movable singular point is an algebraic branch point. In the previous paper [3], we have shown that

$$y'' = \frac{10}{9} y^4 + x$$

admits the quasi-Painlevé property. In this paper, we treat a more general nonlinear equation of the form

(E) $$y'' = \frac{10}{9} y^4 + P(x),$$

where $P(x)$ is a polynomial with complex coefficients satisfying $\deg P \geq 1$.

Our main results are stated as follows:

**Theorem 1.** Let $y(x)$ be an arbitrary solution of (E). Suppose that $x = x_0$ is an algebraic branch point of $y(x)$. Then, around it,

$$y(x) = \xi^{-2/3} - \frac{9}{22} P(x_0) \xi^2 + c \xi^{8/3} + \frac{9 P'(x_0)}{14} \xi^3 + \sum_{j \geq 10} c_j \xi^{j/3},$$

where $\xi = x - x_0$, $c$ is an integration constant, $c_j (j \geq 10)$ are uniquely determined polynomials of $(x_0, c)$, and $\xi^{1/3}$ is an arbitrary branch of $\phi$ such that $\phi^3 = \xi$.

**Theorem 2.** If $\deg P \notin 4\mathbb{N}$, then (E) admits no meromorphic solutions.
Theorem 3. If $\deg P \in 4\mathbb{N}$, then (E) admits no meromorphic solutions except at most four polynomial solutions.

For an arbitrary polynomial $p(x)$, equation (E) with $P(x) = p''(x) - 10p(x)^4/9$ admits the polynomial solution $y = p(x)$. In particular, if $P(x) = -10(\alpha x + \beta)^4/9$, then there exist four solutions $\pm(\alpha x + \beta), \pm i(\alpha x + \beta)$.

Theorem 4. Equation (E) has the quasi-Painlevé property. For each solution, around every movable singularity $x = x_0$, it is expressible by a Puiseux series expansion of the form (1).

2. Lemmas

We review some lemmas which will be used in the proofs of our results.

Suppose that $F(x, u, v)$ and $G(x, u, v)$ are analytic functions in a neighbourhood of $(a_0, b_0, c_0) \in \mathbb{C}^3$. For a system of differential equations

\[ u' = F(x, u, v), \quad v' = G(x, u, v), \]

we have the following lemma due to Painlevé.

Lemma 5. Let $\Gamma (\subset \mathbb{C})$ be a curve with finite length terminating in $x = a_0$. Suppose that a solution $(u, v) = (\varphi(x), \psi(x))$ of (2) has the properties below:

(i) for every point $\xi \in \Gamma \setminus \{a_0\}$, $\varphi(x)$ and $\psi(x)$ are analytic at $x = \xi$;

(ii) there exists a sequence $\{a_n\} \subset \Gamma \setminus \{a_0\}$, $a_n \to a_0$ ($n \to \infty$) such that $(\varphi(a_n), \psi(a_n)) \to (b_0, c_0) \in \mathbb{C}^2$.

Then, $\varphi(x)$ and $\psi(x)$ are analytic at $x = a_0$.

Let $f(z)$ be a meromorphic function in the whole complex plane. For $r > 0$, consider the functions defined by

\[
\begin{align*}
m(r, f) &:= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})|d\phi, \\
N(r, f) &:= \int_0^r (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r, \\
T(r, f) &:= m(r, f) + N(r, f),
\end{align*}
\]

which are called, respectively, the proximity function, the counting function and the characteristic function ([1]). Here $\log^+ s := \max\{\log s, 0\}$ ($s > 0$), and $n(r, f)$ denotes the number of poles in the disc $|z| \leq r$, each counted according to its multiplicity. Clearly, if $f(x)$ is entire, then $T(r, f) = m(r, f)$. It is known that $T(r, f)$ is monotone increasing with respect to $r$, and is a convex function of $\log r$. Furthermore, $f(z)$ is a rational function, if and only if $T(r, f) = O(\log r)$ as $r \to \infty$; and $f(z)$ is a transcendental function, if and only if $\log r/T(r, f) = o(1)$ as $r \to \infty$. The following lemma is concerning the proximity function of a meromorphic solution of a differential equation ([1, Lemma 2.4.2]).
Lemma 6. Suppose that the differential equation $w^{p+1} = \Phi(z, w)$, $p \in \mathbb{N}$ admits a meromorphic solution $w = f(z)$, where $\Phi(z, w)$ is a polynomial of $(z, w, w', ..., w^{(q)})$. If the total degree of $\Phi(z, w)$ with respect to $w$ and its derivatives does not exceed $p$, then

$$m(r, f) = O(\log T(r, f) + \log r)$$

as $r \to \infty$, $r \not\in E$, where $E$ is an exceptional interval with finite length.

3. Proof of Theorem 1

For an algebraic branch point $x_0$ of $y(x)$, we have $|y(x_0)| = +\infty$. Indeed, if $|y(x_0)| < +\infty$, then, for some $x_1 \in \mathbb{C}$,

$$|y'(x)| \leq |y'(x_1)| + \left| \int_{x_1}^{x} \left( \frac{10}{9} y(t)^4 + P(t) \right) dt \right| < +\infty$$

along a curve tending to $x_0$. By Lemma 5, $y(x)$ is analytic at $x = x_0$, which is a contradiction. Putting

$$y(x) = c_0 \xi^\alpha (1 + o(1)), \quad \xi = x - x_0, \quad \alpha \in \mathbb{Q}, \quad \alpha < 0, \quad c_0 \neq 0,$$

and substituting into (E), we have $(10/9)c_0^3 \xi^{3\alpha} = \alpha(\alpha - 1)\xi^{-2}$. Hence

$$y(x) = \xi^{-2/3} + \sum_{j=1}^{\infty} c_j \xi^{j/(3p)}, \quad p \in \mathbb{N}, \quad l \in \mathbb{Z}, \quad l \geq -2p + 1,$$

where $\xi^{1/3}$ is an arbitrary branch of $\phi$ such that $\phi^3 = \xi$. Substituting this into (E) again, we have

$$\frac{10}{9} \xi^{-8/3} + \frac{l(l - 3p)}{9p^2} c_l \xi^{l/(3p) - 2} + \cdots = \frac{10}{9} (\xi^{-8/3} + 4c_l \xi^{l/(3p) - 2} + \cdots + P(x_0) + P'(x_0)\xi + \cdots).$$

From this it follows that $l = 6p$ and that $c_{6p} = -9P(x_0)/22$. In general, the coefficients $c_j$ are determined by

$$(3) \quad \left( \frac{j}{p} + 5 \right) \left( \frac{j}{p} - 8 \right) c_j = Q_j(x_0, c_k; k \leq j - 1), \quad c_{-2p} = 1,$$

where $Q_j$ are polynomials of $x_0$ and $c_k$. Suppose that $c_j \neq 0$ for some $j$ satisfying $j/p \notin \mathbb{Z}$, and let $j_0$ be the minimal one among such numbers. Then $Q_{j_0} = 0$, and hence $c_{j_0} = 0$, which is a contradiction. This implies $p = 1$. Using (3), we have

$$c_6 = -\frac{9}{22} P(x_0), \quad c_7 = 0, \quad c_8 = c, \quad c_9 = \frac{9}{14} P'(x_0),$$
where \( c \) is an arbitrary constant. Thus we obtain the desired series expansion.

4. Proofs of Theorems 2 and 3

Suppose that \( \deg P \notin 4\mathbb{N} \), and that (E) admits a meromorphic solution \( Y(x) \). Then \( Y(x) \) is entire, because \( Y(x) \) has no poles. If \( Y(x) \) is a polynomial, then \( Y(x) = C_0 x^m + O(x^{m-1}) \), \( m \in \mathbb{N} \cup \{0\} \), \( C_0 \neq 0 \) around \( x = \infty \). Substitution of this into (E) yields that \( \deg P = 4m \), which is a contradiction. Hence \( Y(x) \) is transcendental and entire. Applying Lemma 6 to the equality \( 10Y(x)^4/9 = Y''(x) - P(x) \), we have \( T(r,Y) = m(r,Y) = O(\log T(r,Y) + \log r) \) as \( r \to \infty \), \( r \notin E_0 \), \( \mu(E_0) < \infty \). Hence \( T(r,Y) \leq K_0 \log r \) for \( r \notin E_0 \), where \( K_0 \) is some positive number. For each \( r > 0 \), there exists a number \( r'(r) \geq r \) such that \( r'(r) - r \leq 2\mu(E_0) \) and that \( r'(r) \notin E_0 \). Since \( T(r,Y) \) is monotone increasing,

\[
T(r,Y) \leq T(r'(r),Y) \leq K_0 \log r'(r) \leq K_0 \log(r + 2\mu(E_0)) = O(\log r),
\]

which contradicts the transcendency of \( Y(x) \). Thus Theorem 2 has been proved. Next suppose that \( \deg P = 4m, m \in \mathbb{N} \), namely \( P(x) = P_0 x^{4m} + \cdots + P_{4n} \), and that \( Y_0(x) = C_0 x^d + C_1 x^{d-1} + \cdots + C_0 \) is a polynomial solution of (E). Then \( 4d = 4m \) and \( 10C_0^4/9 + P_0 = 0 \). Hence the number of polynomial solutions does not exceed four. Thus we obtain Theorem 3.

5. Proof of Theorem 4

Let \( y(x) \) be an arbitrary solution of (E).

5.1. Equivalent system of equations. Suppose that \( x = x_0 \) is an algebraic branch point of \( y(x) \). Since \( P(x_0) = P(x) - P'(x_0)\xi + O(\xi^2), \xi = x - x_0 \), series (1) is written in the form

\[
y(x) = \xi^{-2/3} \left(1 - \frac{9}{22} P(x)\xi^{8/3} + 81 \frac{1}{77} P'(x)\xi^{11/3} + \cdots\right).
\]

Putting \( y = u^{-2} \), we have

\[
\xi^{1/3} = \pm u \left(1 - \frac{9}{44} P(x)u^8 + \frac{c}{2} u^{10} \pm \frac{81}{154} P'(x)u^6 + \cdots\right),
\]

and

\[
y'(x) = \frac{2}{3} u^{-5} \left(-\frac{2}{3} - \frac{3}{2} P(x)u^8 + \frac{13}{3} cu^{10} + \cdots\right) + \frac{9}{2} P'(x)u^6.
\]

Observing these facts, we define new unknown variables \( u, v \) by

\[
y = u^{-2},
\]

\[
y' = \frac{2}{3} u^{-5} \left(1 + \frac{9}{4} P(x)u^8 + u^{10}v\right) + \frac{9}{2} P'(x)u^6.
\]
Then, we have a system of equations with respect to $u$, $v$. Now regarding $x$, $v$ as unknown functions of $u$, we get

\[ \frac{dx}{du} = \pm 3u^2 U(x, u, v)^{-1}, \quad \frac{dv}{du} = 3u^3 V(x, u, v)U(x, u, v)^{-1}, \]

with

\[
U(x, u, v) = 1 + \frac{9}{4} P(x) u^8 + u^{10} v \mp \frac{27}{4} P'(x) u^{11},
\]

\[
V(x, u, v) = -\frac{5}{3} u^6 v^2 + \frac{3}{4} u^4 (-3P(x) \pm 33P'(x) u^3) v
\]

\[- \frac{81}{16} u^2 (P(x) \mp 3P'(x) u^3)(P(x) \mp 6P'(x) u^3) + \frac{27}{4} P''(x).\]

Note that (5) is equivalent to (E), and that $(x(u), v(u))$ is a solution of (5) analytic at $u = 0$ and satisfying $x(0) = x_0, v(0) = -13c/2$.

5.2. Lyapunov function. The second equation of (4) is written in the form

\[
\left( y' - \frac{9}{2} P'(x) y^{-3} \right)^2 = \frac{4}{9} u^{-10} \left( 1 + \frac{9}{4} P(x) y^{-4} + y^{-5} v \right)^2,
\]

which implies that

\[ V = (y')^2 - 9P'(x)y^{-3}y' - \frac{4}{9} y^5 - 2P(x)y \]

satisfies

\[ V = -\frac{81}{4} P'(x)^2 y^{-6} + \frac{9}{4} P(x)^2 y^{-3} + \left( \frac{8}{9} + 2P(x) y^{-4} \right) v + \frac{4}{9} y^{-5} v^2. \]

Furthermore, $V(x)$ satisfies the first order equation

\[ V' - 27P'(x)y^{-4}V = 243P'(x)^2y^{-7}y' - 9P''(x)y^{-3}y' + 45P(x)P'(x)y^{-3}. \]

Using this equation, we have

**Lemma 7.** If $y(x)^{-1}$ is bounded along a path $\Gamma$ with finite length, then $V(x)$ is also bounded along $\Gamma$.

5.3. Derivation of Theorem 4. Suppose that $x = a$ is a singular point of $y(x)$. Let $\Gamma$ be a segment terminating in $a$ such that each $\xi \in \Gamma \setminus \{a\}$ is at most an algebraic branch point of $y(x)$. Modifying $\Gamma$ if necessary, we may suppose that $\Gamma$ is a curve terminating in $a$, and that $y(x)$ is analytic along $\Gamma \setminus \{a\}$. Put

\[ A = \liminf_{x \rightarrow a, x \in \Gamma} |y(x)|. \]

We divide into three cases (i) $0 < A < +\infty$, (ii) $A = +\infty$, (iii) $A = 0$. 
Case (i) $0 < A < +\infty$: By Lemma 7, $V(x)$ is bounded on $\Gamma$ near $a$. Then, by (6), there exists a sequence $\{a_n\} \subset \Gamma$ such that $y(a_n) \to y_0 (\neq 0, \infty)$, $y'(a_n) \to y_1 (\neq \infty)$, $a_n \to a$. This fact together with Lemma 5 implies that $y(x)$ is analytic at $x = a$.

Case (ii) $A = +\infty$: By supposition, $y(x) \to \infty$ as $x \to a$ along $\Gamma$, and $V(x)$ is bounded along $\Gamma$ near $a$. Then, regarding (7) as a quadratic equation with respect to $y$, we can choose a branch $v_-(x)$ of $v(x)$ which is bounded along $\Gamma$. Let $u_-(x)$ be the corresponding branch of $u(x)$ such that $u_-(x)^{-2} = y(x)$ (cf. (4)). Denote by $x = x(u)$ the inverse function of $u = u_-(x)$. Then, $x = x(u)$ and $v = v_-(x(u))$ are analytic functions of $u$ along $\Gamma^*: = u_-(\Gamma \setminus \{a\})$ satisfying

(a) $x(u) \to a$ as $u \to u_-(a) = 0$ along $\Gamma^*$;
(b) $v_-(x(u))$ is bounded along $\Gamma^*$.

Take a sequence $\{b_n\} \subset \Gamma^*$ satisfying $b_n \to u_-(a) = 0$, $x(b_n) \to a$, $v_-(x(b_n)) \to v_0 \neq \infty$. Observe that $(x(u), v_-(x(u)))$ is a solution of (5). By Lemma 5, $x(u)$ is analytic at $u = 0$, implying that $x = a$ is at most an algebraic branch point of $y(x) = u(x)^{-2}$.

Case (iii) $A = 0$: For $y(x)$, we note the following lemma, which is obtained from [2, Lemma 2.2] with $R_0 = \Delta = 1/2$, $K = 1 + |a|$.

Lemma 8. Set $\theta_0 = (1 + |a|)^{-1}/42$. Let $c$ be a point such that $|c - a| < 1/4$, and suppose that $y(x)$ is analytic at $x = c$. If the inequalities $|y(c)| \leq \theta_0/6$, $|y'(c)| \geq 2$ hold, then $y(x)$ is analytic for $|y'(c)||x - c| < \theta_0$ and satisfies $|y(x)| \geq \theta_0/4$ on the circle $|y'(c)||x - c| = \theta_0/2$.

Let us consider the set $\Gamma_0 = \{x \in \Gamma \mid |y(x)| \leq \theta_0/6\}$. By the supposition $A = 0$, we have $\Gamma_0 \cap \{x \mid |x - a| < \varepsilon\} \neq \emptyset$ for every $\varepsilon > 0$. We may suppose that $|y'(x)| \geq 2$ for $x \in \Gamma_0$. Indeed, if this is not the case, then there exists a sequence $\{a_n\} \subset \Gamma_0$, $a_n \to a$ such that $y(a_n)$ and $y'(a_n)$ are bounded, and hence $y(x)$ is analytic at $x = a$. Now we proceed along $\Gamma$ toward $x = a$. Suppose that we meet the first point $c_1 \in \Gamma_0$. By Lemma 8, there exists a disc $D_1 : |x - c_1| \leq |y'(c_1)|^{-1}\theta_0/2$ such that $|y(x)| \geq \theta_0/4$ on the boundary $\partial D_1$. Note that $a \notin D_1$. Restarting from a point in $\Gamma \cap \partial D_1$, we proceed along $\Gamma$ until we meet the next point $c_2 \in \Gamma_0$. Take the disc $D_2 : |x - c_2| \leq |y'(c_2)|^{-1}\theta_0/2$, and repeat the procedure above. In this way, we get a sequence of discs $\{D_j\}$ such that $|y(x)| \geq \theta_0/4$ on the boundary $\partial D_j$. Then, $|y(x)| \geq \theta_0/6$ on the boundary of the set $\Gamma \cup \bigcup_{j=1}^\infty D_j$, which contains a curve $\gamma$ with the properties: (i) $\gamma$ terminates in $a$; (ii) $|y(x)| \geq \theta_0/6$; (iii) $y(x)$ is analytic along $\gamma \setminus \{a\}$. Hence this case is reduced to either (i) or (ii), which completes the proof of Theorem 4.

6. A remark

As was shown in [3], the equation

$$y'' = \frac{10}{9} y^4 + x$$
has the quasi-Painlevé property, and, the quadratic version of this is the first Painlevé equation

(I) \[ y'' = 6y^2 + x. \]

For (E), the corresponding version is

(8) \[ y'' = 6y^2 + P(x). \]

In general, this equation does not always admit the quasi-Painlevé property. In fact, equation (8) with \( P(x) = x^2 \) possesses the solution of the form

\[ y = \xi^{-2} - \frac{x_0^2}{10} \xi^2 - \frac{x_0}{3} \xi^3 + \left( c + \frac{1}{7} \log \xi \right) \xi^4 + \cdots, \quad \xi = x - x_0, \]

which means that \( x = x_0 \) is a logarithmic branch point.

References