

# The general 1D Schrödinger equation as an exactly solvable problem

André Voros\*

CEA, Service de Physique Théorique de Saclay  
(CNRS URA 2306)  
F-91191 Gif-sur-Yvette CEDEX, France  
E-mail : voros@spht.saclay.cea.fr

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We review an exact WKB resolution method for the stationary 1D Schrödinger equation with a general polynomial potential. This contribution covers already published material: we supply a commented summary here, stressing a few aspects which were less highlighted before. As some of our earlier papers needed later corrections, we also recapitulate these here in footnotes in the bibliography. The latter, not meant to be exhaustive, focuses on narrowly relevant works plus some broader ones that do offer more extensive bibliographies.

## 1 General setting

We will treat a Schrödinger equation on the real line with a polynomial potential function  $V$  (real, normalized as  $V(q) = +q^N + v_1 q^{N-1} + \dots + v_{N-1} q$ ):

$$\left( -\frac{d^2}{dq^2} + [V(q) + \lambda] \right) \psi(q) = 0, \quad (1)$$

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\*Also at: Institut de Mathématiques de Jussieu-Chevaleret (CNRS UMR 7586), Université Paris 7, F-75251 Paris CEDEX 05, France.

(a general Sturm–Liouville problem). It is convenient to pose the problem on a *half-line*, initially  $q \in [0, +\infty)$ ; exact treatments however rely on analytical continuations to the *whole complex domain*:  $q \in \mathbb{C}$ ,  $(\vec{v}, \lambda) \in \mathbb{C}^N$  (where  $\vec{v} \equiv (v_1, \dots, v_{N-1})$ ).

## 1.1 Outline of results [47]–[50]

The traditional view about eq.(1) (in any dimension) is that

- $N = 2$  is exactly solvable: the harmonic oscillator, “the nice” case;
- no other  $N$  is solvable (generically):
  - $N = 1$  gives the **Airy** equation (more transcendental than the higher-degree  $N = 2$ , a paradox!)
  - $N \geq 3$  yield the **anharmonic oscillators**, the ground of choice for all approximation methods (perturbative, semiclassical ...).

We see eq.(1) instead as an **exactly solvable problem for any  $N$** , thanks to an *exact WKB* method with two key ingredients:

- a semiclassical analysis exploiting **zeta-regularization** [43, 29];
- exact quantization conditions that are Bohr–Sommerfeld-like but **selfconsistent**, and akin to the **Bethe Ansatz** equations that solve many exact models in 2D statistical mechanics [21, 40] or quantum field theory [7].

Moreover, degrees  $N$  and  $4/N$  behave similarly in this framework (a qualitative duality) [46]:

- $N = 1$  is transcendental like  $N = 4$ , clarifying the above paradox;
- $N = 2$  (selfdual) becomes completely explicitly solvable as a degenerate, rather pathological, case. A *countable family* of similarly solvable cases is actually obtained [50, Sec.4]: all the generalized eigenvalue problems  $[-\frac{d^2}{dq^2} + (q^N + \Lambda q^{\frac{N}{2}-1})] \psi(q) = 0$  (for even degrees  $N > 0$ ).

## 1.2 A few early sources

*Exact asymptotics* (which uses drastically divergent representations to perform fully exact calculations) is a paradoxical idea; its advent rests on various seminal developments which mainly took place in the '70s (restricting to works we felt as directly influential; a comprehensive bibliography would need much more space):

- Balian and Bloch’s representation of quantum mechanics, where the latter can be *exactly rebuilt in principle* just from the classical trajectories,

- provided these are taken in their full *complex* extension [5];
- the *exact geometrical* content of high-frequency asymptotics: singularities, wave front sets, and their propagation properties [26, 15, 23]; (reviewed in [27, chap. VIII])
  - the same in the *analytic* (*not*  $C^\infty$ ) category (a *key step* to exactness): ramified Cauchy problem [32], analytic pseudo-differential operators [11], hyper- and micro-functions [36], localized Fourier transforms [13]; (reviewed in [39])
  - *Borel resummations* of asymptotic series [20], large-order perturbation theory [9], tunneling (e.g., [25, 6, 33]), instantons and Zinn-Justin's conjectures [53];
  - seemingly unrelated direct monodromy calculations resulting in *exact functional relations* for Stokes multipliers [38].

The Balian–Bloch idea led to a first concrete exact WKB algorithm for 1D problems like eq.(1), based on (generalized) Borel resummations [42, 30, 17, 18, 12, 28], and best formalized in Écalle's framework of *resurgent functions* [24]; the explicit exact WKB results are indeed special instances of *bridge equations* in resurgence algebras (or resurgence equations). Still, in this approach, one basic resurgence theorem is not yet fully proved [18, thm 1.2.1 and its Comment], and a fairly explicit solution stage has only been reached for homogeneous polynomial potentials, rather tortuously [44]–[46]. Otherwise, many developments have been achieved in this and neighboring directions: proofs of Bender–Wu [30, 19] and Zinn-Justin conjectures [17, 18], hyperasymptotics (reviewed in [28, Part II]), results on Painlevé functions (reviewed in [28, Part III]).

This presentation is about a more direct exact WKB method [47]–[50] (surveyed in [51]), based upon functional relations *à la* Sibuya [38], plus crucial inputs of *zeta-regularization* both at the quantum and classical levels. At its core, it needs just one exact Wronskian identity plus a structure equation for a pair of spectral determinants, to generate the exact solution as a fixed point of an explicit system of nonlinear equations (“exact quantization conditions”, but also *Bethe-Ansatz* equations in an integrable-model interpretation [21]). The Wronskian identity precisely subsumes the resurgence relations found in the Borel-transform approach, whereas the newer exact quantization conditions (which achieve the solving task) have no clear counterpart in the Borel plane as far as we know. This direct approach also makes it easier to treat the general case at once, and it is now fully proven for homogeneous potentials [4].

## 2 Basic ingredients

### 2.1 The conjugate equations [38]

Just as a polynomial equation is better handled by treating all its conjugate roots together, the original differential equation (1) is to be supplemented by its *conjugates*, defined using

$$V^{[\ell]}(q) \stackrel{\text{def}}{=} e^{-i\ell\varphi} V(e^{-i\ell\varphi/2} q), \quad \lambda^{[\ell]} \stackrel{\text{def}}{=} e^{-i\ell\varphi} \lambda, \quad \text{with } \varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2}; \quad (2)$$

there are  $L$  distinct conjugate equations labeled by  $\ell \in \mathbb{Z}/L\mathbb{Z}$ , where

$$L = \begin{cases} N+2 & \text{generically} \\ \frac{N}{2} + 1 & \text{for an even polynomial } V(q). \end{cases} \quad (3)$$

### 2.2 Semiclassical asymptotics

• WKB formulae for solutions of the Schrödinger equation [38]: a solution of eq.(1) which is recessive (= decaying) for  $q \rightarrow +\infty$  is specified by the WKB behavior

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp -\int_Q^q \Pi_\lambda(q') dq', \quad \Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q) + \lambda)^{1/2}, \quad (4)$$

where  $\Pi_\lambda(q)$  is the classical forbidden-region momentum ( $\Pi_\lambda(q) dq$  is the Agmon metric). Reexpansions in descending powers of  $q \rightarrow +\infty$  yield

$$[V(q) + \lambda]^{-s+1/2} \sim \sum_\sigma \beta_\sigma(s; \vec{v}, \lambda) q^{\sigma-Ns} \quad (\sigma = \frac{N}{2}, \frac{N}{2}-1, \frac{N}{2}-2, \dots), \quad (5)$$

$$\psi_\lambda(q) \sim e^c q^{-N/4-\beta_{-1}(\vec{v})} \exp \left\{ -\sum_{\{\sigma>0\}} \beta_{\sigma-1}(\vec{v}) \frac{q^\sigma}{\sigma} \right\} \quad (6)$$

where  $\beta_{\sigma-1}(\vec{v}) \stackrel{\text{def}}{=} \beta_{\sigma-1}(0; \vec{v}, \lambda)$  (independent of  $\lambda$  for  $\sigma \geq 0$  when  $N > 2$ ).

Another solution, recessive for  $q \rightarrow +e^{-i\varphi/2}\infty$ , arises through the *first conjugate equation*:

$$\Psi_\lambda(q) \stackrel{\text{def}}{=} \psi_{\lambda^{[1]}}^{[1]}(e^{i\varphi/2} q). \quad (7)$$

As the asymptotic form (6) and its analogs for  $\Psi_\lambda(q)$  and their  $q$ -derivatives have overlapping sectors of validity, this **explicit exact Wronskian** follows:

$$\Psi_\lambda(q)\psi'_\lambda(q) - \Psi'_\lambda(q)\psi_\lambda(q) \equiv e^{c+c^{[1]}} 2i e^{i\varphi/4} e^{i\varphi\beta_{-1}(\vec{v})/2} (\neq 0). \quad (8)$$

• Spectral asymptotics (with  $E \equiv -\lambda$ ): the Schrödinger operator over the whole real line with the confining potential  $V(|q|)$  has a purely discrete spectrum  $\{E_k\}_{k \in \mathbb{N}}$  ( $E_k \uparrow +\infty$ ). Due to *parity-splitting*,  $\mathcal{E}^+ \stackrel{\text{def}}{=} \{E_{2k}\}$  is the Neumann, and  $\mathcal{E}^- \stackrel{\text{def}}{=} \{E_{2k+1}\}$  the Dirichlet, spectrum (the boundary conditions being at  $q = 0$ ). As a corollary of the WKB formula (4) on eigenfunctions, the eigenvalues obey the *semiclassical Bohr–Sommerfeld* condition

$$S(E_k) \sim k + \frac{1}{2} \quad \text{for integer } k \rightarrow +\infty, \quad (9)$$

in terms of the *classical action function*  $S(E) \stackrel{\text{def}}{=} \oint_{\{p^2+V(q)=E\}} \frac{p dq}{2\pi}$ , which behaves as  $b_\mu E^\mu$  for  $E \rightarrow +\infty$  with  $\mu = \frac{1}{2} + \frac{1}{N}$  (the order).

Further expansion of eq.(9) to all orders in  $k^{-1}$  followed by reexpansion in descending powers of  $E$  results in a complete asymptotic ( $E \rightarrow +\infty$ ) eigenvalue formula, of the form [47]

$$\sum_{\alpha} b_{\alpha} E_k^{\alpha} \sim k + \frac{1}{2} \quad \text{for integer } k \rightarrow +\infty \quad (\alpha = \mu, \mu - \frac{1}{N}, \mu - \frac{2}{N}, \dots) \quad (10)$$

(the  $b_{\alpha}$  are polynomial in the  $\{v_j\}_{j \leq (\mu - \alpha)N}$ , and also parity-dependent but not above  $\alpha = -3/2$  nor if  $V$  is an even polynomial).

### 2.3 Spectral functions [43]

• Spectral zeta functions (parity-split for later use):

$$Z^{\pm}(s, \lambda) \stackrel{\text{def}}{=} \sum_{\substack{k \\ \text{even} \\ \text{odd}}} (E_k + \lambda)^{-s} \quad (\text{Re } s > \mu), \quad (\text{and } Z(s) = Z^{+}(s) + Z^{-}(s)) \quad (11)$$

extend to meromorphic functions of  $s \in \mathbb{C}$ , *regular* at  $s = 0$ , all due to eq.(10).

• Spectral determinants: *formally* meant to be

$$D^{\pm}(\lambda) = \text{“} \prod_{\substack{k \\ \text{even} \\ \text{odd}}} (\lambda + E_k) \text{”}, \quad (12)$$

they get rigorously specified through

$$\log D^{\pm}(\lambda) \stackrel{\text{def}}{=} -\partial_s Z^{\pm}(s, \lambda)_{s=0} \quad (\text{zeta-regularization}), \quad (13)$$

and further evaluated by *limit formulae* which we call *structure equations* (they are like Hadamard product formulae, but with **no undetermined constants whatsoever**, a crucial property here):

$$\log D^\pm(\lambda) = \lim_{K \rightarrow +\infty} \left\{ \sum_{k < K} \log(E_k + \lambda) + \frac{1}{2} \log(E_K + \lambda) \right. \\ \left. (k, K \begin{matrix} \text{even} \\ \text{odd} \end{matrix}) \quad \text{(counterterms:)} \quad - \frac{1}{2} \sum_{\{\alpha > 0\}} b_\alpha (E_K)^\alpha \left[ \log E_K - \frac{1}{\alpha} \right] \right\}. \quad (14)$$

As corollaries for  $D$  (and  $D^\pm$ ):

- $D$  is an **entire** function in  $(\lambda, \vec{v})$ , of order  $\mu$  in  $\lambda$ ;
- $\log D$  has a large- $\lambda$  expansion of a severely constrained (“standard” or “canonical”) form, or *generalized Stirling expansion* [29]:

$$\log D(\lambda) \sim \left[ a_1 \lambda (\log \lambda - 1) \right] + a_0 \log \lambda \left[ + \sum_{\mu_k \notin \mathbb{N}} a_{\mu_k} \lambda^{\mu_k} \right] \quad (15) \\ \text{(degree 1)} \quad \text{(degree 0)} \quad \text{(degrees } \mu_k \downarrow -\infty)$$

where obviously no term can occur with degree  $> \mu$  (growth restriction; e.g.,  $a_1 \equiv 0$  as soon as  $N > 2$ ); but above all, any extraneous **power terms**  $b_n \lambda^n$  ( $n \in \mathbb{N}$ ), including **additive constants** ( $b_0 \lambda^0$ ), are **banned** outright (canonical constraint).

For  $N = 2$ , eqs.(14), resp. (15), essentially restore classic results: the analytical continuation by the *Euler–Maclaurin formula*, resp. the *Stirling expansion*, for  $-\partial_s \zeta(s, \lambda)_{s=0} \equiv \log \sqrt{2\pi} / \Gamma(\lambda)$  (Lerch formula), with  $\zeta(s, \lambda) =$  the Hurwitz zeta function. Those two founding formulae of all asymptotics involve zeta-regularization, as the Lerch formula shows; by inference, WKB theory (a branch of asymptotics) also ought to be intimately tied to zeta-regularization, but *this* aspect traditionally goes *unnoticed*.

### 3 Fundamental exact identities

#### 3.1 Canonical normalization of recessive solution

The solution  $\psi_\lambda(q)$  described by eqs.(4), (6) to be recessive for  $q \rightarrow +\infty$ , still awaits normalization. It proves useful to fix it also at  $q = +\infty$ , as

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(q') dq' \quad (q \rightarrow +\infty). \quad (16)$$

However,  $\int_q^{+\infty} \Pi_\lambda(q') dq'$  ("the Agmon distance from  $q$  to  $+\infty$ ") diverges: we then wish to define this "improper action integral" as

$$\left[ \int_q^{+\infty} (V(q') + \lambda)^{1/2-s} dq' \right]_{s \rightarrow 0} \tag{17}$$

(analytical continuation from  $\{\text{Re } s > \mu\}$  to  $s = 0$ ), but  $s = 0$  is a *singular point* in general: a simple pole of residue  $N^{-1}\beta_{-1}(\vec{v})$  (cf. eq.(6)). Now:

- $\beta_{-1}(s; \vec{v}) = 0$  quite frequently (for any polynomial  $V(q)$  of odd degree  $N$ ; also for any even polynomial of degree  $N = 4M$ ; for any  $V(q) = q^N$ ,  $N \neq 2$ ); then  $s = 0$  becomes a regular point and the prescription (17) is fine;

- when  $\beta_{-1}(s; \vec{v}) \neq 0$ , a stronger regularization is suggested by the quantum-classical correspondence. "Classical spectral functions" are naturally definable, by mimicking the quantum formulae (11), (13), as

$$Z_{\text{cl}}(s, \lambda) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{dq dp}{2\pi} (p^2 + V(|q|) + \lambda)^{-s} \tag{18}$$

$$\log D_{\text{cl}}(\lambda) \stackrel{\text{def}}{=} -\partial_s Z_{\text{cl}}(s, \lambda)_{s=0}. \tag{19}$$

These formulae together imply a classical counterpart to the formal eq.(12),

$$\text{"} \int_{-\infty}^{+\infty} (V(|q|) + \lambda)^{1/2} dq \text{"} = \log D_{\text{cl}}(\lambda); \tag{20}$$

using parity symmetry, this further splits to give the pair [50, Sec.1.2.2]

$$D_{\text{cl}}^\pm(\lambda) = \Pi_\lambda(0)^{\pm 1/2} \exp \text{"} \int_0^{+\infty} \Pi_\lambda(q) dq \text{"}; \tag{21}$$

$\log D_{\text{cl}}^\pm(\lambda)$  are also the divergent parts of the large- $\lambda$  expansions (15) for  $\log D^\pm(\lambda)$ , so they are canonical as well. The idea is then to *define*  $\int_0^{+\infty} \Pi_\lambda(q) dq$  to be  $\frac{1}{2} \log D_{\text{cl}}(\lambda)$  (and  $\int_q^{+\infty} = \int_0^{+\infty} - \int_0^q$ ). This in turn fixes the recessive solution, with a normalization  $e^{\mathcal{C}} \neq 1$  in eq.(6) (an *anomaly* factor, which would turn up elsewhere if not here):

$$\mathcal{C} \equiv \frac{1}{N} [-(2 \log 2)\beta_{-1}(s; \vec{v}) + \partial_s \left( \frac{\beta_{-1}(s; \vec{v})}{1-2s} \right)]_{s=0}. \tag{22}$$

As examples of the resulting improper action integrals: [52]

$$\begin{aligned} \int_0^{+\infty} (q^N + \lambda)^{1/2} dq &= -\frac{1}{2} \pi^{-1/2} \Gamma(\frac{1}{2} + \mu) \Gamma(-\mu) \lambda^\mu \quad (N \neq 2; \mu = \frac{1}{2} + \frac{1}{N}) \\ \int_0^{+\infty} (q^2 + \lambda)^{1/2} dq &= -\frac{1}{4} \lambda (\log \lambda - 1) \quad (\text{anomalous when } \lambda \neq 0) \\ \int_0^{+\infty} (q^4 + vq^2)^{1/2} dq &= -\frac{1}{3} v^{3/2}. \end{aligned} \tag{23}$$

Remarks:

- a “classical determinant”  $D_{\text{cl}}$  (or  $D_{\text{cl}}^{\pm}$ ) is *not* an entire function of  $\lambda$  like  $D$ : the discrete spectrum of zeros of  $D$  classically becomes a continuous branch cut for  $D_{\text{cl}}$ .
- our treatment of the normalization for  $\beta_{-1} \neq 0$  was initially flawed [47, 48]: we erroneously prejudged the natural answer to remain  $\mathcal{C} = 0$ , so we had temporary inconsistencies of normalization in the anomalous case; these affect eqs.(15–19), (26) in [47] and Secs.1.1, 2.1, 2.2 in [50], but cancel themselves spontaneously thereafter; the error is repaired from [49] onwards.

### 3.2 Basic exact identities

Under the canonical normalization for the recessive solution, and with  $' = \frac{d}{dq}$ ,

$$D^+(\lambda) \equiv -\psi'_\lambda(0), \quad \text{resp.} \quad D^-(\lambda) \equiv \psi_\lambda(0). \quad (24)$$

The idea, assuming  $N > 2$ , is first to prove the logarithmic  $\lambda$ -derivatives of eqs.(24) by analytically computing traces of the resolvent kernel, and to  $\lambda$ -integrate back: then, no integration constants can reappear precisely because all four logarithms have the canonical large- $\lambda$  behavior (15), in particular

$$\psi'_\lambda(0) \sim -D_{\text{cl}}^+(\lambda), \quad \text{resp.} \quad \psi_\lambda(0) \sim D_{\text{cl}}^-(\lambda) \quad (25)$$

(from eqs.(21) and (16) at  $q = 0$ , also valid for  $\lambda \rightarrow +\infty$ ). In depth, eqs.(24) arose by controlling *all integration constants* without exception, from eq.(14) onwards.

The basic exact identities (24) can now be substituted into the explicitly known Wronskian (8) computed at  $q = 0$ , resulting in a *functional relation between spectral determinants* or “Wronskian identity”, also fundamental:

$$e^{+i\varphi/4} D^{[1]+}(e^{-i\varphi} \lambda) D^-(\lambda) - e^{-i\varphi/4} D^+(\lambda) D^{[1]-}(e^{-i\varphi} \lambda) \equiv 2i e^{+i\varphi\beta_{-1}(\bar{v})/2}. \quad (26)$$

## 4 Exact quantization conditions

At first glance, the exact functional relations (26) look grossly underdetermined; like the original Wronskian formula (8), they amount to one equation for two unknown functions. The exact information here seems to present a “missing link” throughout. This is however a delusion, because those unknown functions have a truly special form here, embodied in the structure



equations (14) (themselves consequences of zeta-regularization), and this precisely fills that information gap.

### 4.1 Degenerate cases

- $N = 2$ :  $[-\frac{d^2}{dq^2} + (q^2 + \lambda)] \psi(q) = 0$  (harmonic oscillator).

This is the most singular case:  $\beta_{-1} = \lambda/2 \neq 0$ , and it has  $\lambda$ -dependence. Now,  $\varphi = \pi$  makes eq.(26) degenerate: for  $\lambda$  real, it separates into real and imaginary parts, which recombine as  $D^+(\lambda)D^-(-\lambda) = 2 \cos \frac{\pi}{4}(\lambda-1)$ . The zeros are then obvious, and they clearly split between  $D^+(\lambda)$  or  $D^-(-\lambda)$  according to their sign; thereupon, the structure equations (14) yield the explicit solutions

$$D^+(\lambda) = \frac{2^{-\lambda/2} 2\sqrt{\pi}}{\Gamma(\frac{1+\lambda}{4})}, \quad D^-(\lambda) = \frac{2^{-\lambda/2} \sqrt{\pi}}{\Gamma(\frac{3+\lambda}{4})}. \tag{27}$$

- $N$  even: the generalized eigenvalue problems  $[-\frac{d^2}{dq^2} + (q^N + \Lambda q^{\frac{N}{2}-1})] \psi(q) = 0$  are also exactly solvable [50, Sec.4]: setting  $\nu = \frac{1}{N+2}$ , eqs.(26) and (14) likewise lead to the relevant Neumann and Dirichlet determinants for  $q \in [0, +\infty)$ , as

$$D_N^+(\Lambda) = -\frac{2^{-\Lambda/N} (4\nu)^{\nu(\Lambda+1)+1/2} \Gamma(-2\nu)}{\Gamma(\nu(\Lambda-1) + 1/2)}, \quad D_N^-(\Lambda) = \frac{2^{-\Lambda/N} (4\nu)^{\nu(\Lambda-1)+1/2} \Gamma(2\nu)}{\Gamma(\nu(\Lambda+1) + 1/2)}; \tag{28}$$

whereas over the whole real  $q$ -axis,

$$\det [-\frac{d^2}{dq^2} + (q^N + \Lambda q^{\frac{N}{2}-1})] = \begin{cases} D_N^+(\Lambda)D_N^-(\Lambda) & \text{if } N \equiv 2 \pmod{4} \\ (\sin \pi\nu)^{-1} \cos \pi\nu\Lambda & \text{if } N \equiv 0 \pmod{4}. \end{cases} \tag{29}$$

Remark: actually, the potentials  $(q^N + \Lambda q^{\frac{N}{2}-1})$  are *supersymmetric* at the generalized eigenvalues  $\Lambda$  (see [50] and the footnote under that reference).

### 4.2 The generic situation

The preceding treatment does not generalize straightforwardly. Instead, the mere division of the functional relation (26) at  $\lambda = -E_k$  by its first conjugate partner leads to the pair of formulae

$$2 \arg D^{[1]\pm}(-e^{-i\varphi} E_k) - \varphi \beta_{-1}(\vec{v}) = \pi[k + \frac{1}{2} \pm \frac{N-2}{2(N+2)}] \quad \text{for } k \begin{matrix} \text{even} \\ \text{odd} \end{matrix}, \tag{30}$$

which have the outer form of Bohr–Sommerfeld quantization conditions but are *exact, albeit implicit* (the left-hand sides invoke the determinants of the first conjugate spectra, equally unknown).

However, we can write the *totality* of such conditions for the  $L$  conjugate problems (Sec.2.1), together with the structure equations that express all the determinants from their spectra: namely, for  $\ell \in \mathbb{Z}/L\mathbb{Z}$  and  $k, K \overset{\text{even}}{\text{odd}}$ ,

$$i^{-1} \left[ \log D^{[\ell+1]\pm}(-e^{-i\varphi} E_k^{[\ell]}) - \log D^{[\ell-1]\pm}(-e^{+i\varphi} E_k^{[\ell]}) \right] - (-1)^\ell \varphi \beta_{-1}(\vec{v}) \quad (31)$$

$$= \pi \left[ k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \right],$$

$$\log D^{[\ell]\pm}(\lambda) \equiv \lim_{K \rightarrow +\infty} \left\{ \sum_{k < K} \log(E_k^{[\ell]} + \lambda) + \frac{1}{2} \log(E_K^{[\ell]} + \lambda) \right. \quad (32)$$

$$\left. - \frac{1}{2} \sum_{\{\alpha > 0\}} b_\alpha^{[\ell]}(E_K^{[\ell]})^\alpha \left[ \log E_K^{[\ell]} - \frac{1}{\alpha} \right] \right\}.$$

As a result, the exact spectra now appear as solutions of nonlinear, parity-split, selfconsistent or *fixed-point* equations of the form

$$\mathcal{M}^+ \{ \mathcal{E}^{\bullet+} \} = \mathcal{E}^{\bullet+}, \quad \text{resp.} \quad \mathcal{M}^- \{ \mathcal{E}^{\bullet-} \} = \mathcal{E}^{\bullet-}, \quad (33)$$

where  $\mathcal{E}^{\bullet\pm}$  denote the disjoint unions of the conjugate spectra  $\mathcal{E}^{[\ell]\pm}$ .

Numerical tests, done for  $N \leq 6$  and not too big coefficients  $\vec{v}$ , suggest that the fixed-point equations (31)–(32) can be cast into *contractive* form. Assuming this conjecture, the problems (33) get solved simply by forward iterations, which induce convergence toward the exact spectra starting from trial spectra that need only be *semiclassically correct* (to a certain order).

### 4.3 Example: homogeneous potentials $q^N$ [44]–[46]

This is the simplest case (all conjugate spectra coincide, and  $\lambda$  is the only variable), and specially so if  $N > 2$  (see Sec.4.1 for  $N = 2$ , [46] for  $N = 1$ ).

For  $N > 2$ , the exact quantization conditions (31) reduce to a single pair (e.g., see fig.1 for  $N = 4$ ),

$$2 \arg D^\pm(-e^{-i\varphi} E_k) = \pi \left[ k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \right] \quad \text{for } k = \begin{matrix} 0,2,4,\dots \\ 1,3,5,\dots \end{matrix} \quad (34)$$

while the structure equations (32) simplify as

$$\arg D^\pm(-e^{-i\varphi} E) \equiv \sum_{\substack{k \\ \text{even} \\ \text{odd}}} \arg(E_k - e^{-i\varphi} E) \quad (E > 0), \quad (35)$$

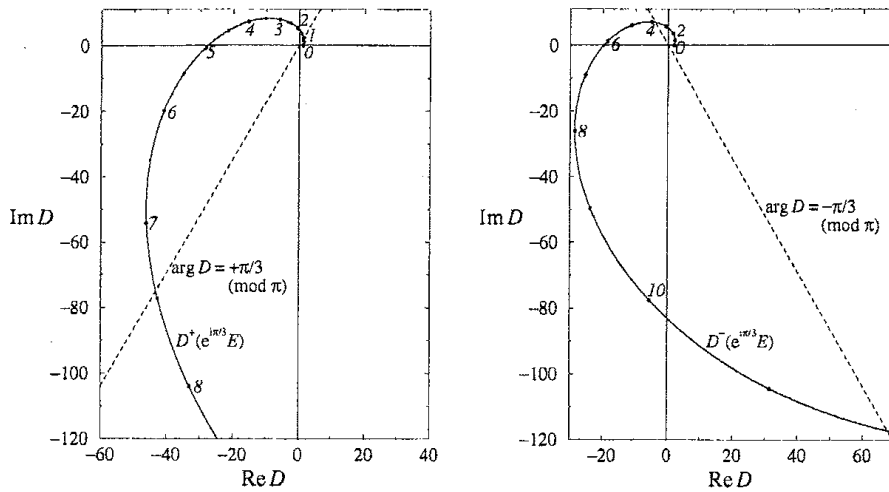


Figure 1: The exact quantization formulae (34) (+, left; -, right) for the homogeneous quartic potential  $q^4$ : the curves  $\{D^\pm(e^{i\pi/3} E)\}_{E>0}$  (solid lines labeled by  $E$ -values) cross the (dashed) lines of prescribed argument in the complex  $D$ -plane.

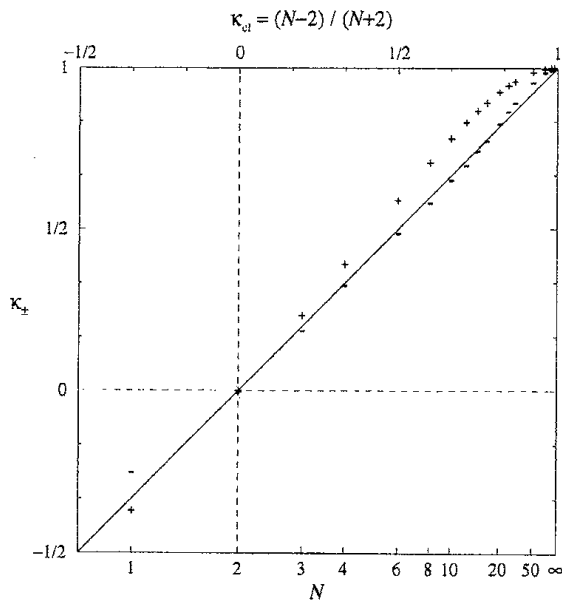


Figure 2: Numerically measured linear contraction factors  $\kappa_\pm$  at the fixed points (i.e., the exact  $\pm$  spectra), upon iterations of the exact quantization conditions (34)–(35) for homogeneous potentials  $|q|^N$ . As horizontal scale we use the value taken by  $\kappa_\pm$  classically (when  $D^\pm$  is replaced by  $D_{cl}^\pm$  as in eq.(21)):  $\kappa_{cl}(N) \equiv \frac{N-2}{N+2}$ .

and the asymptotic boundary condition is  $b_\mu E_k^\mu \sim k + \frac{1}{2}$  for  $k \rightarrow +\infty$ .

The latter fixed-point equations have now been proven *globally contractive* on suitable sequence spaces [4], by methods specific to the homogeneous case (see fig.2). (For the more general equations (31)–(32), contractivity remains a conjecture.)

Those very same equations (34)–(35) have also been found to yield the ground states of some exactly solvable models in 2D statistical mechanics [21, 40] and in conformal quantum field theories [7] (*Bethe Ansatz* equations). A deep explanation for this coincidence of equations (and, by uniqueness, of their solutions) in so different dynamical settings is not yet available. This “ODE/IM correspondence” (as a dictionary between Schrödinger Ordinary Differential Equations and Integrable Models) currently applies to homogeneous problems (even singular ones) plus the supersymmetric equations of Sec.4.1 on the Schrödinger side, and only to the ground (non-excited) states in integrable models. There is now no integrable-model realization of the general inhomogeneous quantization conditions (31)–(32); by extension, we still consider these as Bethe Ansatz equations, and the general 1D Schrödinger equation (1) as exactly solvable through them.

## 5 Extensions: realized vs desirable

### 5.1 Solving for the unknown functions $\psi$ [47]–[51]

We keep focusing on the recessive solution  $\psi_\lambda$  of eq.(1) as an example. By letting now the reference half-line vary as  $[Q, +\infty)$ , we obtain parametric translates of the basic identities (24), essentially  $\psi_\lambda(Q) \equiv D_Q^-(\lambda)$ ,  $\psi'_\lambda(Q) \equiv -D_Q^+(\lambda)$ . The computation of  $\psi_\lambda(Q)$  thus amounts to a parametric spectral problem, and this is now exactly solvable through parametric fixed-point equations like (33), e.g.,  $\mathcal{M}_Q^-\{\mathcal{E}_Q^{\bullet-}\} = \mathcal{E}_Q^{\bullet-}$  for  $\psi_\lambda(Q)$ . (This has also undergone successful numerical tests.)

### 5.2 Toward quantum perturbation theory [8, 52]

By rescalings *à la* Symanzik, the perturbative regime for an anharmonic potential like  $V(q) = q^2 + gq^4$  amounts to the  $v \rightarrow +\infty$  regime for the potential  $q^4 + vq^2$ . This limit is not manifest, and is actually singular, in our exact quantization formalism. We recently obtained explicit large- $v$  behaviors for

such spectral determinants, basically as

$$\frac{\det^\pm(-\frac{d^2}{dq^2} + q^N + vq^M + \lambda)}{\det^\pm(-\frac{d^2}{dq^2} + vq^M + \lambda)} \sim \frac{\det_{\text{cl}}^\pm(-\frac{d^2}{dq^2} + q^N + vq^M + \lambda)}{\det_{\text{cl}}^\pm(-\frac{d^2}{dq^2} + vq^M + \lambda)} \quad (36)$$

(for  $v \rightarrow +\infty$ , with  $N > M$ ); i.e., the singular (divergent) large- $v$  behavior is entirely concentrated in the *classical parts* of the determinants; these now express in terms of improper action integrals (cf. Sec.3.1),

$$\frac{\det_{\text{cl}}^\pm(-\frac{d^2}{dq^2} + q^N + vq^M + \lambda)}{\det_{\text{cl}}^\pm(-\frac{d^2}{dq^2} + vq^M + \lambda)} \equiv \frac{\exp \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2} dq}{\exp \int_0^{+\infty} (vq^M + \lambda)^{1/2} dq}, \quad (37)$$

and the latter can be estimated in the end [52].

### 5.3 Generalizations?

It is most desirable to push exact solvability beyond polynomial potentials. According to some exact results, rational potentials [2, 7] and trigonometric polynomials [53, 16] appear as realistic targets, as well as some differential equations of order  $> 2$  [1, 41, 22, 3]. The general Heun class is also a possible direction [31], which includes other important equations (Mathieu, Lamé, ...).

Of course, an even greater challenge is to tackle higher-dimensional Schrödinger-type problems, in line with the original Balian–Bloch surmise itself [5]. Exact asymptotic methods are still in the exploratory stage there, see for instance [14, 35, 10, 37, 34].

Therefore the subject is far from being closed!

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<sup>1</sup>In [43], eq.(6.25) should have read  $\zeta'(-1) = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{12}(1 - \gamma - \log 2\pi)$  (with no consequence elsewhere).

<sup>2</sup>Misprints in [46]: in eqs.(18),  $D^\pm(e^{-i\varphi} \lambda)$  should read  $D^\pm(-e^{-i\varphi} \lambda)$  (twice) and just underneath,  $[0, e^{-i\varphi} \infty)$  should read  $[0, -e^{-i\varphi} \infty)$ .

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<sup>3</sup>The Corrigendum [49] to [47] is also required for [48], see end of Sec.3.1 above.

<sup>4</sup>Corrections for [50]: 1) the term “quasi-exactly solvable” was wrongly put for “supersymmetric” throughout (without practical consequence; the two notions happen to agree when  $N = 2$  and  $6$ , but not beyond); 2) regarding the Airy zeros in Table 1,  $Z_1^+(0) \approx 0.0861126$ ,  $e^{-Z_1^+(0)} \approx 0.9174909$ ,  $e^{-Z_1^-(0)} \approx 1.2585417$ .