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When is a Stokes Line not a Stokes Line?
II. Examples involving differential equations

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1 Introduction

In the first paper of this trilogy (Howls 2005a), we have introduced the concept of a higher order Stokes phenomenon for functions that depend on an asymptotic parameter $\epsilon \rightarrow 0$ and an additional set of non-asymptotic variables $\mathbf{a}$. We explained the phenomenon in terms of a hyperasymptotic approach that can be extended to cover the class of functions that possess a Borel transform.

When solving differential equations it is often the case that an explicit, closed form, integral representation of the Borel transform either does not exist, or is not required. The purpose of this paper is to illustrate how the effect of the higher order Stokes phenomenon might be quantified under these circumstances.

In order to proceed with any exponential asymptotic analysis, at the very least it is essential that one must first find the exponents of the exponential prefactors or, synonymously the locations of the corresponding singularities $f_j(\mathbf{a})$ in the Borel plane. Assuming that this can be done, it is then necessary to deduce the Riemann sheet structure of the Borel plane. This is required so that when the condition for a Stokes phenomenon

$$S_{i>j} = \left\{\mathbf{a} \,:\, \epsilon^{-1}\left(f_j\left(\mathbf{a}\right) - f_i\left(\mathbf{a}\right)\right) > 0\right\}. \tag{1.1}$$

is satisfied, we may deduce whether the Stokes line is active or inactive. If the corresponding Borel singularities are on different Riemann sheets, then the Stokes line is inactive, otherwise it is active and a Stokes phenomenon will take place. The activity of a Stokes curve may change across a higher order Stokes curve in $\mathbf{a}$-space defined by

$$\frac{f_j(\mathbf{a}) - f_i(\mathbf{a})}{f_k(\mathbf{a}) - f_j(\mathbf{a})} \in R. \tag{1.2}$$

A single-dimensional integral representation provides a natural graphic solution to the determination of the activity of a Stokes curve. One simply has to plot the steepest descent contours passing through $f_i(\mathbf{a})$ and $f_j(\mathbf{a})$. If at the putative Stokes phenomeon these steepest paths connect and are topologically consistent with the contour of integration, then the Borel singularities are on the same Riemann sheet and the Stokes line is active. If there is no connection, then the Stokes curve is inactive.

To proceed in the absence of a suitable integral representation we make the following observation. The activity (or otherwise) of a Stokes curve can be linked to the Stokes multiplier that prefactors the exponentially small terms that are switched on (or off) across it. If a Stokes line is inactive then the Stokes multiplier can be interpreted as being zero.

From this viewpoint, the higher order Stokes phenomenon can be interpreted as causing a change in the value of a Stokes multiplier. As a higher order Stokes curve is crossed in **a**-space the Stokes multiplier associated with an asymptotic contribution will change abruptly, to or from zero, or in more general cases, between other complex values.

It is often possible to determine the activity of a Stokes curve from a monodromy argument, starting from one sector of the complex **a**-plane between Stokes lines with a specific expansion. Stepping over successive Stokes lines, we could travel to distant Stokes sectors and back by a variety of closed orbits, making assumptions about the non-zero or zero values of Stokes constants on each Stokes curve. If progress along these different routes led to contradictory asymptotic expressions in the starting and finishing sector then we might deduce that at least one of the Stokes constants was zero in the vicinity of the point of traversal and hence the Stokes curve was inactive there. In simple situations it is also possible to deduce the value of Stokes constants. However for arbitrary differential equations, the Stokes constants may take any complex value and more systematic methods may be necessary.

Here we choose to obtain the Stokes multipliers and hence the activity of the Stokes curves in regions of **a** delineated by the higher order Stokes lines from the explicit terms in the expansions. The methods of exponential asymptotics naturally lend themselves to this.

Assuming that we can find the relevant exponents (or Borel singularities) $f_j(\mathbf{a})$, we will proceed by relating this to the analytic properties as functions of **a** of the coefficients in the local expansions about the Borel singularities.

Other methods approaches are also possible. For one such method involving matching, see Chapman & Mortimer (2004). Other methods are discussed in Aoki *et al* (1994, 2002, 2002).

We shall discuss two examples, the first an ODE, the second a PDE.

2 Example: A Linear Ordinary Differential Equation

The following example can be found in Olde Daalhuis (2004), where further details and discussion may be found.

Consider the inhomogeneous second order differential equation

$$\epsilon^2 \frac{d^2 w}{dz^2} + z^2 w = \exp i \frac{(z^2/2 - 4z)}{\epsilon}, \qquad \epsilon \to 0^+. \tag{2.1}$$

Solutions of this equation will have three asymptotic contributions, two from the complementary functions and one from the particular integral.

The general solution of the corresponding homogeneous equation is of the form

$$w_c(z,\epsilon) = A(u)\sqrt{z} H^{(1)}_{1/4}\left(\frac{z^2}{2\epsilon}\right) + B(\epsilon)\sqrt{z} H^{(2)}_{1/4}\left(\frac{z^2}{2\epsilon}\right), \tag{2.2}$$

where $A(\epsilon)$ and $B(\epsilon)$ are arbitrary functions of ϵ and the H are Hankel functions.

The asymptotic expansion of the complementary function is thus

$$w_c(z,\epsilon) \sim \frac{\tilde{A}(\epsilon)}{\sqrt{z}} e^{iz^2/2\epsilon} \sum_{s=0}^{\infty} \frac{i^s d_s \epsilon^s}{z^{2s}} + \frac{\tilde{B}(u)}{\sqrt{z}} e^{-iz^2/2\epsilon} \sum_{s=0}^{\infty} \frac{(-i)^s d_s \epsilon^s}{z^{2s}}, \tag{2.3}$$

with

$$d_s = \frac{\Gamma(3/4 + s)}{s!\Gamma(3/4 - s)}, \tag{2.4}$$

and $\tilde{A}(\epsilon)$ and $\tilde{B}(\epsilon)$ are two arbitrary functions of ϵ different from $A(\epsilon)$ and $B(\epsilon)$ in (2.2). The non-asymptotic parameter of interest here is $\mathbf{a} = z$.

The asymptotic expansion of the particular integral is obtained from substitution of the expansion

$$w_1(z,\epsilon) = \exp\left(i\frac{z^2/2 - 4z}{\epsilon}\right) \sum_{s=0}^{\infty} a_s(z)\epsilon^{s+2}, \tag{2.5}$$

into equation (2.1) to obtain

$$a_0(z) = \frac{1}{8(z-2)}, \qquad a_1(z) = \frac{-i}{16(z-2)^3} + \frac{i}{64(z-2)^2}, \tag{2.6}$$

with higher order terms given by

$$-8(z-2)a_{s+2} = a_s'' + 2i(z-4)a_{s+1}' + ia_{s+1}. \tag{2.7}$$

Clearly $a_s(z)$ has a pole of order $2s+1$ at $z=2$. We will use this information in the determination of the Stokes constants below. In contrast to what follows, here we have freedom in choosing the coefficients $a_s(z)$ in (2.8), since a_{s+1} is given in terms of derivatives. In what follows, the coefficients will possess the complicating factor of constants of integration.

We shall study the asymptotic behaviour of particular integral of (2.1) in the complex z-plane. We focus on the Stokes curves where the expansion (2.5) maximally dominates the other possible asymptotic contributions. As these Stokes curves are crossed traditionally these extra contributions appear or disappear in the complete asymptotic representation.

We thus form a template

$$w_1(z,\epsilon) \sim \exp\left(-\frac{f_1(z)}{\epsilon}\right)\epsilon^2\left(\sum_{s=0}^{\infty} a_s(z)\epsilon^s + \sum_j K_{1j}\exp\left(-\frac{F_{1j}(z)}{\epsilon}\right)\epsilon^{-\alpha_j}\sum_{s=0}^{\infty} b_{s,j}(z)\epsilon^s\right), \tag{2.8}$$

as $\epsilon \to 0^+$. where

$$f_1(z) = -i(z^2/2 - 4z) \tag{2.9}$$

and

$$F_{ij}(z) = f_j(z) - f_i(z) \tag{2.10}$$

The goal now is to determine the f_j, K_{1j} and $b_{s,j}$, $j \neq 1$. (Note that in more general cases the Stokes constants K_{1j} could depend on ϵ.)

If we used an integral representation of the particular integral we could proceed as in the Pearcey-type case as above. In the absence of a convenient integral representation (as will be the case in many situations) we could proceed as follows. The asymptotic expansion (2.8) is divergent. This is because of the presence of the singularities of at $f_j(z)$ in the Borel plane. The presence of this singularities is incorporated in the growth of the coefficients $a_s(z)$ as $s \to \infty$.

We will use the "factorial-over-power" ansatz for the higher orders of the a-coefficients

$$a_s(z) \sim \frac{K(z)\Gamma(s+\alpha)}{F_{1j}(z)^{s+\alpha}}, \tag{2.11}$$

as $s \to \infty$. Here the F_{1j} gives the spacing between f_1 and the singularities f_j in the Borel plane that control the divergence of the expansion of the particular integral. Note that (2.11) is a leading order approximation only. There are correction terms (as we will see below) and in more complication situations, alternative forms of this ansatz are required. When this ansatz is substituted into the recurrence relation (2.7), we find at leading order in s that

$$F'(z)^2 - 2i(z-4)F'(z) + 8(z-2) = 0. \tag{2.12}$$

Solving this equation we find two solutions

$$F_{12}(z) = i(z^2 - 4z + k_2) \quad \text{and} \quad F_{13}(z) = -4iz + k_3, \tag{2.13}$$

where k_2 and k_3 are constants. Hence the most general form of $a_s(z)$ is

$$a_s(z) \sim \frac{K_{12}(z)\Gamma(s+\alpha_2)}{F_{12}(z)^{s+\alpha_2}} + \frac{K_{12}(z)\Gamma(s+\alpha_3)}{F_{13}(z)^{s+\alpha_3}}, \tag{2.14}$$

We determine k_2 and the corresponding α_2 by recalling the fact that the exact $a_s(z)$ have a pole of order $2s+1$ at $z=2$. Hence, k_2 must be 4 so that

$$F_{12}(z) = i(z-2)^2 \quad \text{and} \quad \alpha_2 = 1/2. \tag{2.15}$$

The constant k_3 is determined by examining the homogeneous form of the original differential equation (2.1). This has a turning point at $z=0$. One signature of a turning point in the WKB-type expansions we are dealing with here is that the exponential prefactors, and hence the Fs relating to the complementary functions in the ansatz (2.15) be equal. Thus we set $F_{12}(0) = F_{13}(0)$ and deduce that $k_3 = 4i$.

Another way of obtaining this result is to note that, as we will see below, the recurrence relation for the coefficients $b_{s,j}(z)$ show that the $b_{s,j}(z)$ coefficients themselves have poles at $z=0$. However, the terms in the asymptotic expansion of $w_1(z,u)$ are analytic in z near $z=0$, see (2.6), (2.7). Hence, the exponentially 'small' terms in (2.8) must cancel each other as $z \to 0$. Hence we have

$$F_{13}(0) = -4i(z-1), \quad \alpha_3 = \alpha_2 = \frac{1}{2}, \quad K_2(0) = -K_3(0) \tag{2.16}$$

We can now deduce the exponential behaviours f_j in the expansion of particular integral. From (2.10), (2.13), (2.15) and (2.16) we have

$$\begin{aligned} f_1(z) &= -i(z^2/2 - 4z), \\ f_2(z) &= F_{12}(z) + f_1(z) = i(z^2/2 + 4), \\ f_3(z) &= F_{13}(z) + f_1(z) = -i(z^2/2 - 4) \end{aligned} \tag{2.17}$$

Note that f_2 and f_3 give rise to exponential prefactors that are of the same form as that arising in the expansion of the complementary function. They differ only by a constant shift in a $1/\epsilon$ term. In fact we could have written the expansion of the complementary function (comp func expansion eqn) in terms of the f_j here by a redefinition of the constants $\tilde{A}(\epsilon)$ and $\tilde{B}(\epsilon)$.

We are now in a position to draw the Stokes curves for the particular integral expansion for $\epsilon \to 0^+$. Recall from the first paper in the trilogy (Howls 2005a) that $S_{i>j}$ is the Stokes curve on which the expansion prefactored by an exponential involving f_i maximally dominates the expansion prefactored by an exponential involving f_j. We substitute (2.17) into the necessary condition for a Stokes curve (1.1) and find that the Stokes curves are given by

$$|\Im z| = |\Re z|, \qquad |\Im z| = |\Re z - 2|, \qquad \Re z = 1, \tag{2.18}$$

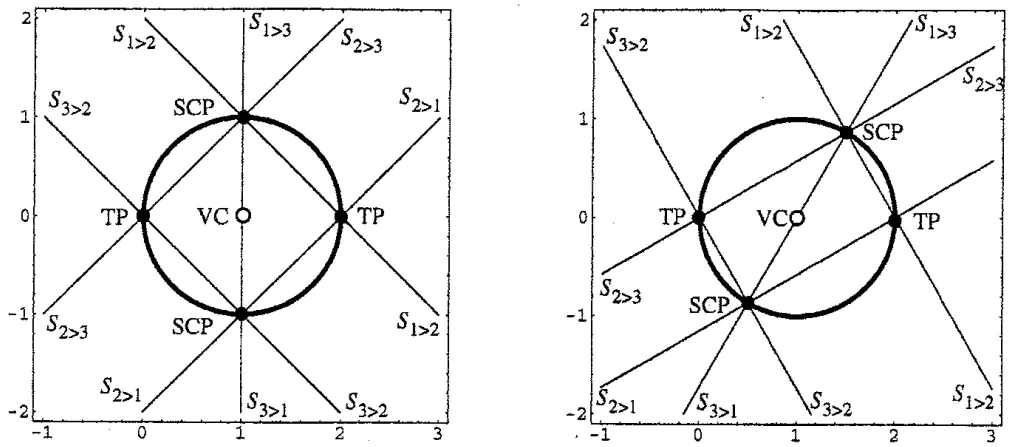

Figure 1: The candidate Stokes curves in the complex $\mathbf{a} = z$ plane defined by (1.1) based on exponents (2.17) with $\epsilon > 0$ (left) and $\arg \epsilon = -\pi/6(right)$. The bold circle in both diagrams is the corresponding higher order Stokes line which from (1.2), is invariant under changes of $\arg \epsilon$. Stokes line sprout from turning points (TP) at $z = 0$ and $z = 2$, and also from the virtual turning point or virtual caustic (VC) at $z = 1$. The candidate Stokes lines cross at Stokes crossing points (SCP) at $z = 1 \pm i$ when $\epsilon > 0$. The candidate Stokes curves and SCP obviously rotate with $\arg \epsilon$. The higher order stokes curve is invariant.

Plotting these curves for $\epsilon \to 0^+$, we arrive at figure 1.

Note that not only do the Stokes curves meet at the double turning points at $z = 0$ and $z = 2$, but also cross at at the regular Stokes crossing points (SCP) $z = 1 \pm i$. The higher order Stokes curve can be plotted by substituting $f_1(z)$, $f_2(z)$ and $f_3(z)$ from (2.17) into the definition (1.2). It transpires that the curve is simply the circle

$$|z - 1| = 1, \tag{2.19}$$

and passes through the Stokes crossing points (as it must).

We now begin an examination of the activity of the Stokes curves. It transpires that we only need to consider one point in each of the regions delineated by the higher order stokes curve to determine the activity of the lines passing through that region.

When solution $w_1(z, u)$ crosses the Stokes curve $S_{1>2}$ it can switch on a subdominant contribution of a Stokes constant times $w_2(z, u)$, which is the expansion of one solution of the homogeneous version of (2.1). The expansion of $w_2(z, u)$ has the small ϵ asymptotic behaviour

$$w_2(z, \epsilon) \sim \frac{1}{\sqrt{z}} e^{-f_2(z)/\epsilon} \epsilon^{3/2} \sum_{s=0}^{\infty} b_s(z) \epsilon^s, \tag{2.20}$$

where the b_s satisfy the recurrence relation

$$b'_{s+1} = \frac{-i}{2}\left(\frac{b''_s}{z} - \frac{b'_s}{z^2} + \frac{3b_s}{4z^3}\right) \tag{2.21}$$

which is obtained from the the substitution of (3.4) into (2.2). Note that the form of the recurrence relation will introduce (at this stage unknown) constants of integration for each coefficient $b_s(z)$. In (2.20) the factor $\epsilon^{3/2}$ is just the product of ϵ^2 and $\epsilon^{-\alpha_2}$ in (2.8).

Likewise, on crossing the Stokes curve $S_{1>3}$, $w_1(z,u)$ can switch on a constant times $w_3(z,u)$, which is the expansion of the second solution of the homogeneous version of equation (at beginning). With the small ϵ asymptotic behaviour

$$w_3(z,\epsilon) \sim \frac{1}{\sqrt{z}} e^{-f_3(z)/\epsilon} \epsilon^{3/2} \sum_{s=0}^{\infty} c_s(z)\epsilon^s, \tag{2.22}$$

with recurrence relation

$$c'_{s+1} = \frac{i}{2}\left(\frac{c''_s}{z} - \frac{c'_s}{z^2} + \frac{3c_s}{4z^3}\right). \tag{2.23}$$

Again each $c_s(z)$ will involve constants of integration.

To determine the constants of integration in the $b_s(z)$ and $c_s(z)$ we again resort to the analytic properties of the $a_s(z)$. We use the complete expansion for the late terms of the $a_n(z)$ of which (2.14) included but the first terms.

$$a_n(z) \sim \frac{K_{12}}{2\pi i} \sum_{s=0}^{\infty} \frac{b_s(z)\Gamma(n-s+1/2)}{\sqrt{z}F_{12}(z)^{n-s+1/2}} + \frac{K_{13}}{2\pi i} \sum_{s=0}^{\infty} \frac{c_s(z)\Gamma(n-s+1/2)}{\sqrt{z}F_{13}(z)^{n-s+1/2}}, \tag{2.24}$$

as $n \to \infty$. We know this to be the form of the expansion of the late terms since the Borel plane (and from this the Borel transform) involves only three singularities, and (2.24). This illustrates the power of the Borel-plane approach.

Recalling the reason for this calculation, we remark that the value of the Stokes constants (multipliers) K_{21} and K_{31} will depend on the location of z with respect to the higher order Stokes curve.

The logic for determining the activity of the Stokes lines then proceeds as follows

- We observe from (2.16) that $F_{13}(1) = 0$. However, from (2.6) and (2.7) the exact coefficients $a_n(z)$ have no apparent singularity at $z = 1$. Hence we must have $K_{13} = 0$ for z in the vicinity of $z = 1$. With the Stokes constants formulated as above, K_{13} can only change value across a higher order Stokes curve. Hence $K_{13} = 0$ inside the circle S_{higher}, that is, for $|z-1| < 1$.

- An asymptotic analysis of the first few exact coefficients $a_n(z)$ from (2.6)-(2.7) shows that

$$a_n(z) \sim \frac{\Gamma(n+1/2)i^n}{\Gamma(1/2)2^{2n+3}z^{n+1}}, \tag{2.25}$$

as $|z| \to \infty$. We now compare the powers of z in (2.25) with (2.24) together with the respective forms of $F_{12}(z)$ and $F_{13}(z)$. We deduce that the term in (2.24) involving $F_{13}(z)$ must dominate the right hand side of (2.24) as $|z| \to \infty$. A comparison of the prefactors of (2.24) with (2.25) suggests that $K_{13}c_0 = \sqrt{\pi i}/2$. Without loss of generality, we can take $K_{31} = \sqrt{\pi i}/2$ and $c_0 = 1$. These results are valid everywhere outside the higher order Stokes curve $|z-1| > 1$.

- Observation of the form of the first few exact coefficients $a_n(z)$ from (2.6)-(2.7) shows that

$$a_n(z) = \frac{\text{a polynomial in } z \text{ of degree } n}{(z-2)^{2n+1}} \tag{2.26}$$

This allows us to identify that the coefficients of integration for the c_s in (2.23) are all zero. If any of them were non-zero, this would lead to overall positive powers of z appearing in the expansion (2.23) of $a_n(z)$ as $|z| \to \infty$. This would not agree with the exact form of (2.6).

- It follows from (2.6), (2.7), (2.24), (2.26) that the blow up of $a_n(z)$ near $z = 2$ is dominated by

$$a_n(z) \sim \frac{\Gamma(n+1/2)\sqrt{i}}{8\Gamma(1/2)F_{12}(z)^{n-s+1/2}}. \tag{2.27}$$

Comparison of leading orders of (2.24), (2.27) suggests $K_{12}b_0 = i\sqrt{\pi i/8}$. Again, without loss of generality, we take $K_{12} = i\sqrt{\pi i/8}$ and $b_0 = 1$. It turns our that these results are valid for all $z \in C$.

- Consideration of the constants of integration arising in the $b_s(z)$ shows that any non-zero value would also lead to non-analytic behaviour on the right hand side of (2.24) as $|z| \to \infty$. This would violate the analyticity of $a_n(z)$ (2.26) as $|z| \to \infty$ and so they must all be zero.

- We can conclude that when $|z-1| > 1$

$$a_n(z) \sim \frac{(-i)^n}{4\sqrt{2\pi z}} \sum_{s=0}^{\infty} \frac{(1/4,s)\Gamma(n-s+1/2)}{z^{2s}(z-2)^{2n-2s+1}} + \frac{i^n}{8\sqrt{\pi(z^2-z)}} \sum_{s=0}^{\infty} \frac{(1/4,s)\Gamma(n-s+1/2)}{z^{2s}(4z-4)^{n-s}}, \tag{2.28}$$

as $n \to \infty$.

- If $|z-1| < 1$ we have

$$a_n(z) \sim \frac{(-i)^n}{4\sqrt{2\pi z}} \sum_{s=0}^{\infty} \frac{(1/4,s)\Gamma(n-s+1/2)}{z^{2s}(z-2)^{2n-2s+1}}, \tag{2.29}$$

as $n \to \infty$.

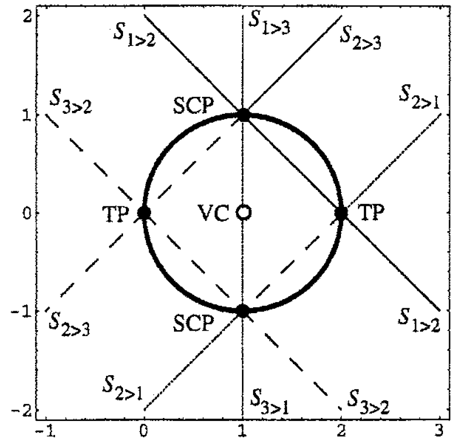

Figure 2: The Stokes curves in the complex $\mathbf{a} = z$ based on exponents (2.17) with $\epsilon > 0$ and considering function defined to be the particular integral of (2.1) at $z = 3/2$. The thin line denotes Stokes curves that are active and across which a Stokes phenmenon takes place. The dotted lines denote Stokes curves that are inactive. The dashed curves denote active Stokes curves which are nevertheless irrelevant to the asymptotics of the function in question.

- These results can be confirmed numerically using the systematic techniques of hyper-asymptotics and choosing points inside and outside the higher order Stokes curve (see Olde Daalhuis 2004 for details).

We can now discuss the activity of the Stokes curves in relation to the higher order Stokes phenomenon.

The Stokes curves $S_{2>1}$ and $S_{3>1}$, that is $z = 1 + ri$, $r < 0$, can never be active since the solutions of the homogeneous equation (2.2) are independent of the inhomogeneity (Howls & Olde Daalhuis 2003).

Above, we showed that the Stokes multiplier $K_{13} = 0$ for $|z - 1| < 1$. Thus the Stokes line $S_{1>3}$ from $z = 1$ to $z = 1 + i$ is also inactive. However that portion of $S_{1>3}$, where $z = 1 + ri$, $r > 1$, is active, since $K_{13} \neq 0$ for $|z - 1| < 1$.

We may check this by following the asymptotics of the particular integral $w_1(z)$ that has (2.7) as its complete asymptotic expansion in the triangle with vertices $z = 0$, $z = 1$, and $z = 1 + i$. We will walk around the Stokes crossing point $z = 1 + i$ in the clock wise direction and examine the monodromy of the expansion.

- When we cross the Stokes curve $S_{2>3}$ nothing happens, since solution $w_2(z)$ is not contained in the asymptotics of $w_1(z)$ at this crossing. Hence, although the part of the Stokes curve

$S_{2>3}$ between the turning point $z = 0$ and Stokes crossing point $z = 1 + i$ is active but is irrelevant.

- As we cross the Stokes curve $S_{1>2}$, $w_1(z)$ switches on $K_{21}w_2(z)$.
- On crossing the Stokes curve $S_{1>3}$ $w_1(z)$ switches on $K_{31}w_3(z)$.
- We cross the Stokes curve $S_{2>3}$ again. This time $w_2(z)$ is active, and it switches off $K_{31}w_3(z)$. Thus $S_{2>3}$ is still active and is now a relevant Stokes curve.
- We cross the Stokes curve $S_{1>2}$ again $w_1(z)$ switches off $K_{21}w_2(z)$.
- Monodromy of the particular integral then demands that the Stokes curve from $z = 1$ to $z = 1 + i$ cannot be active.

Conversely, if the the Stokes curve $z = 1 + ri$, $r > 1$ were to be inactive, then nothing would have happened at step (3), and this would have led to a contradiction. The role of the higher order Stokes curve in this example is now clear. The Stokes line $S_{1>3}$ is seen to change activity across the higher order Stokes line at the SCP $z = 1 + i$.

Note that in this example we can use information about the origins of the expansions as a short cut to assessing the activity of the Stokes curves. The process can be simplified dramatically if we recall that the w_1 is solution of the full inhomogeneous equation (2.1). Hence the homogeneous solutions w_2 and w_3 cannot ever contain a contribution from a specific w_1 in their re-expansions (see Howls and Olde Daalhuis 2003). Consequently all the Stokes curves $S_{2>1}$ and $S_{3>1}$ can never be active. Likewise, for the same reason since there are only three possible contributions $S_{2>3}$ and $S_{3>2}$ must always be active. In a more complicated problem that could not be written as an inhomogeneous problem this simplification would not occur.

Note that if we consider the values of the $f_j(z)$ in (3.1) we observe that $f_1(1) = f_3(1)$. Hence we should expect a turning point at $z = 1$, with associated divergences in $a_n(1)$ or $c_n(1)$. This does not happen, since we have established that $K_{13} = 0$ at $z = 1$, i.e., the Borel singularity $f_1(1)$ is on a different Riemann sheet to $f_3(1)$. Since $z = 1$ has the characteristics of a turning point, but no divergence, it is called a "virtual turning point", see Aoki et al (2001). The existence of virtual turning points will be of some significance in the third paper.

3 Example: A Linear Partial Differential Equation

In this example we illustrate the use of analytical properties of coefficient expansion to determine the Stokes multipliers and investigate the higher order Stokes phenomenon in a linear partial differential equation where $\mathbf{a} = (x, t)$. Again a monodromy argument may be used to examine the activity of Stokes curves. However, here we again study the phenomenon from the point of view of Stokes multipliers deduced from the analytical properties of the expansion coefficients. The presence of an additional non-asymptotic parameter naturally complicates the calculations, and several of the steps in the example outlined below may be undetermined in more general situations. Nevertheless it is instructive to see how the arguments of the ODE example above need to be generalised.

Our goal is to compute the active and inactive Stokes lines. The key problems remain the same. First one must determine the exponents/Borel singularities $f(\mathbf{a})$, then one must determine sufficiently many of the coefficients in the expansion of interest to gain insight into their analytic properties. Then one uses a resurgence ansatz for the asymptotic form of the coefficient, including Stokes multiplers, that in turn are evaluated by comparing analytic properties with the exact coefficients.

We shall study the equation

$$u_t - u_x = \epsilon^2 u_{xxx} - \frac{1}{1+ix}, \qquad u(x,0) = -i\ln(1+ix). \tag{3.1}$$

defined on $-\infty < x < +\infty$, $t > 0$ with $\epsilon \to 0^+$. It will be necessary to consider at least x to be a complex variable. This equation has also been studied in Chapman & Mortimer (2004) using a matching and resummation approach. It is convenient to extract the first term of the expansion and to change variables to (s,τ) thus

$$u(x,t) = -i\ln(1+ix) + v(s,\tau), \qquad \text{where } s = x+t,\ \tau = t. \tag{3.2}$$

Then (3.1) becomes

$$v_\tau = \epsilon^2 v_{sss} - \frac{2\epsilon^2}{(1+i(s-\tau))^3}, \qquad v(s,0) = 0. \tag{3.3}$$

where $v(s,\tau)$ has a leading order power series expansion

$$v(s,\tau) \sim \sum_{n=1}^{\infty} a_n(s,\tau)\epsilon^{2n}. \tag{3.4}$$

Equations for the coefficients a_n can be obtained by substituting (3.4) into (3.3) to obtain

$$\frac{\partial a_1}{\partial \tau} = \frac{-2}{(1+i(s-\tau))^3}, \qquad a_1(s,0) = 0, \tag{3.5}$$

and

$$\frac{\partial a_n}{\partial \tau} = \frac{\partial^3}{\partial s^3} a_{n-1}, \qquad a_n(s,0) = 0, \quad n = 2,3,4,\cdots. \tag{3.6}$$

Thus the a_n satisfy the recurrence relation

$$a_1(s,\tau) = \frac{i}{(1+i(s-\tau))^2} - \frac{i}{(1+is)^2}, \qquad a_n(s,\tau) = \int_0^\tau \frac{\partial^3 a_{n-1}(s,\tilde{\tau})}{\partial s^3}\, d\tilde{\tau}. \tag{3.7}$$

We thus observe that $a_1(s,0)$ has no singularity. However at time $\tau = 0^+$, two singularities in $a_1(s,\tau)$ are spontaneously created at $s = i$ and $s - \tau = i$. This is consistent with the work of Costin and Tanveer (2003)

As in the first example we form a template solution, now of the form

$$v(s,\tau;\epsilon) \sim \exp\left(-\frac{f_e(s,\tau)}{\epsilon}\right)\left(\sum_{n=0}^{\infty} a_n(s,\tau)\epsilon^{2n} + \sum_j K_{ej}\exp\left(-\frac{F_{ej}(z)}{\epsilon}\right)\epsilon^{-\alpha_j}\sum_{s=0}^{\infty} b_{s,j}(z)\epsilon^s\right), \tag{3.8}$$

as $\epsilon \to 0^+$. where

$$f_e(s,\tau) = 0 \tag{3.9}$$

and

$$F_{ij}(z) = f_j(z) - f_i(z) \tag{3.10}$$

The goal is again to determine the F_{ej} and Stokes multipliers K_{ej}.

Following the first examples, we consider the ansatz

$$a_n(s,\tau) \sim \frac{K(s,\tau)\Gamma(2n+\tilde{\alpha})}{F(s,\tau)^{2n+\alpha}}, \quad \text{as } n \to \infty, \tag{3.11}$$

where the prefactor that includes the Stokes multiplier, $K(s,\tau)$ is now a function of s and τ. Note that in contrast to the ODE example above, $\alpha \neq \tilde{\alpha}$ as we will see below. For more general cases Mortimer (2004) it may also be necessary to allow α and $\tilde{\alpha}$ in (3.11) to also be functions of s and τ. (The $2n$ in the gamma function arises with hindsight from a preliminary calculation.) Substituting this ansatz into (3.6) we obtain

$$F_\tau = (F_s)^3. \tag{3.12}$$

Note that we have no explicit boundary data for this equation. However we are interested in solutions of this (3.12) that blow up at the same places as $a_n(s,\tau)$, that is at $s = i$ and at $s - \tau = i$.

For the blow up at $s = i$, by comparison of the powers of derivatives in (3.12) we pose a solution of the form

$$F(s,\tau) = \frac{(s-i)^{3/2}}{g(s,\tau)}. \tag{3.13}$$

The function $g(s,\tau)$ is non-singular at $s = i$. Substitution of this ansatz into (3.11) gives

$$a_n(s,\tau) \sim \frac{K(s,\tau)\Gamma(2n+\tilde{\alpha})\,(g(s,\tau))^{2n+\alpha}}{(s-i)^{3n+3\alpha/2}}. \tag{3.14}$$

Comparison of this form with the blow up of $a_1(s,\tau)$ at $s = i$, shows that

$$\alpha = -2/3 \tag{3.15}$$

Thus

$$a_n(s,\tau) \sim \frac{K(s,\tau)\Gamma(2n+\tilde{\alpha})\,(g(s,\tau))^{2n-2/3}}{(s-i)^{3n-1}} \quad \text{as } n \to \infty. \tag{3.16}$$

At, this point, those readers familiar with exponential asymptotics may question the comparison of an ansatz for late terms $n \to \infty$ with the exact form of the first few terms. The resolution of this apparently illogical action is that we are exploiting the fact that there are addition variables, s and τ, which give rise to analytic behaviour that is seeded in the first few terms and propagated via (3.7) to higher order terms. Of equal importance to this argument is that the form of the recurrence relation for the higher order a_n (3.7), introduces no additional singularities in x or t.

We also need to incorporate the boundary condition in (3.6), namely that $a_n(s,0) = 0$ in the vicinity of $s = 1$. Lower order correction terms to the ansatz (3.11) (by definition!) cannot balance the leading order and so we set

$$g(s,\tau) \sim A\tau^{\beta}, \qquad \text{as } \tau \to 0. \tag{3.17}$$

Note that many other functional forms for g are possible here, however this choice agrees with the exact values to leading order in τ. In more complicated examples we might not be so fortunate. A consequence of this choice is that

$$\beta = 1/2. \tag{3.18}$$

With these investigations in mind, we can revise our solution (3.13) to (3.12).

$$F(s,\tau) = \frac{2i(s-i)^{3/2}}{3\sqrt{3\tau}} h(s,\tau), \tag{3.19}$$

and obtain from (3.12) the nonlinear PDE:

$$(h + 2/3(s-i)h_s)^3 = h - 2\tau h_\tau. \tag{3.20}$$

By substitution of a Taylor series expansion for the solution h, it is easy to see that the only solution of this PDE that is analytic near $(s,\tau) = (i,0)$ is $h(s,\tau) = \pm 1$. Hence, the solution F of (3.12) that is singular near to $s = i$ and that agrees with the behaviour as $\tau \to 0$ is given by F_{es} where

$$F_{es} = \frac{2i(s-i)^{3/2}}{3\sqrt{3\tau}}. \tag{3.21}$$

We may therefore infer the existence of Stokes curves $S_{e>p}$ where $F_{es} > 0$, where a contribution from singularity s may be switched on.

Obviously other solutions of (3.12) exist. In particular we seek a solution that is singular near $s - \tau = i$, the other singularity of $a_n(s,\tau)$, that may be deduced from an examination of the first few terms of (3.7). To that end we try a solution of the form

$$F(s,\tau) = (1 + i(s-\tau))\, k(s,\tau). \tag{3.22}$$

where $k(s,\tau)$ is analytic at $s-\tau=i$. For convenience we make the temporary coordinate change $x=1+i(s-\tau)$, $t=i\tau$. Then $F=xk(x,t)$ and

$$(k+xk_x)^3 = k + x(k_x - k_t). \tag{3.23}$$

Again, the only analytic solution near $(x,t)=(0,0)$ is $k(x,t)=\pm 1$. A second solution F to (3.12) is thus F_{ep} where

$$F_{ep} = 1 + i(s-\tau). \tag{3.24}$$

Hence we deduce the existence of another Borel singularity p and Stokes curves $S_{e>p}$ across which a contribution from singularity p may be switched on.

The two solutions (3.21) and (3.24) correspond to the the two singularities present in the $a_n(s,\tau)$. As before, the Fs given by these expressions are the distances in the Borel plane between the origin and the corresponding singularities, s and p respectively, all these points mapping from the exponents of the prefactors of the expansions in question. From these results we may also deduce the location of a further turning point defined by

$$F_{sp} = F_{ep} - F_{es} = 1 + i(s-\tau) - \frac{2i(s-i)^{3/2}}{3\sqrt{3\tau}} = 0 \tag{3.25}$$

Here, F_{sp} is the complex length of third side of the triangle formed by the three Borel points e, s, and p involved in this problem. Clearly $F_{sp}=0$ when $s=i+3t$ (or $x=i+2t$)and again this implies the existence of Stokes curves $S_{s>p}$ where singularity s can switch on asymptotic contribution from singularity p.

We are now in a position sketch all the associated Stokes curves, and to assess their activity from the associated higher order Stokes curve (1.2) $F_{sp}/F_{ep}>0$ All these curves are displayed in figure 3.

If we analytically continue to complex x, but keeping t fixed and real, we observe a similar picture to figure 1. The higher order Stokes curve sprouts from two turning points (TP) at $s=i$ and $s=\tau+i$ corresponding to singularities of the coefficients. The Stokes curves cross the higher order Stokes at Stokes crossing points (SCP).

We now compute the Stokes multipliers in the different regions of (s,τ) delineated by the higher order Stokes curve. Following the usual exponential asymptotic techniques (Olde Daalhuis 1998) we compute $a_1(s,\tau),\cdots,a_6(s,\tau)$ (here we regard the index 6 as a large number!) These coefficients rapidly increase in length, but near $s=i$, the following asymptotic form can be recognized

$$a_n(s,\tau) \sim \frac{i(-\tau)^{n-1}\Gamma(3n-1)}{(s-i)^{3n-1}\Gamma(n)}, \qquad \text{as } s\to i. \tag{3.26}$$

Note that this dominant behaviour near $s=i$ also satisfies the recurrence relation in (3.7). Thus (3.16) holds for all n. This is the F_{s_1} behaviour. It also follows that $\tilde{\alpha}=-1/2$.

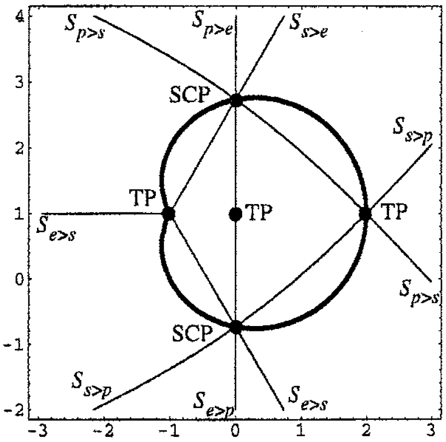

Figure 3: The candidate Stokes curves in the complex x-plane at constant t for the solutions of (3.1) for the indicated boundary conditions, based on exponents (3.21) and (3.24) with $\epsilon > 0$. The higher order Stokes curve is the kidney-shaped thin line. This picture scales with t, the higher order Stokes curves and turning points TP, coalescing at the point $x = i$ when $t = 0$.

Similarly we discover from the recurrence relation (3.7) that for all n

$$a_n(s,\tau) \sim \frac{i(-1)^n\Gamma(2n)}{(s-\tau-i)^{2n}}, \qquad \text{as } s-\tau \to i. \tag{3.27}$$

This is the F_{p_1} behaviour. Note that this asymptoic behaviour along cannot be valid near $\tau = 0$ since it does not satisfy the initial condition that $a_n(s,0) = 0$.

Now we assemble the asymptotic behaviours (3.26) and (3.27) in the vicinities of the distant Borel singularities and write down the leading orders of the Borel re-expansions of the coefficients in the form

$$a_n(s,\tau) \sim K_1(s,\tau)\frac{i(-\tau)^{n-1}\Gamma(3n-1)}{(s-i)^{3n-1}\Gamma(n)} + K_2(s,\tau)\frac{i(-)^n\Gamma(2n)}{(s-\tau-i)^{2n}}, \qquad \text{as } n \to \infty, \tag{3.28}$$

The only unknowns are now the Stokes multipliers $K_1(s,\tau)$ and $K_2(s,\tau)$, which determine the activity of the Stokes lines. Note that by comparison of (3.28) with (3.26), $K_1(s,\tau) \approx 1$ near $s = i$ and from a comparision with (3.27), $K_2(s,\tau) \approx 1$ near $s - \tau = i$.

We may check these values numerically. Taking the exact coefficients for $n = 6, 7$, with typical values $s = -4.2$, $\tau = 0.02$ ($x = -4.22$, $t = 0.02$) in (3.28) we obtain two equations for the two unknowns K_1 and K_2. These may be solved trivially to yield

$$K_1(s,\tau) = 0.01263 - 0.00297i, \qquad K_2(s,\tau) = (0.18849 + 2.93520i) \times 10^{-14}. \tag{3.29}$$

Thus, to numerical accuracy, $K_2(s,\tau) = 0$ for (s,τ) outside the higher order Stokes curve.

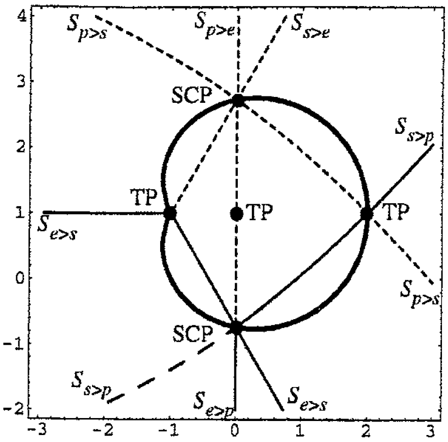

Figure 4: The activity of Stokes curves in the complex x-plane at constant t for the solutions of (3.1) for the indicated boundary conditions, based on exponents (3.21) and (3.24) with $\epsilon > 0$. The thin solid lines are active Stokes curves, the dashed lines are active but irrelevant Stokes curves, the dotted lines are inactive Stokes curves. Note that the Stokes curve $S_{e>p}$ changes activity at the Stokes crossing point near $x = -i$.

Similarly we take $n = 6, 7$, $s = 4.2$ and $\tau = 4.2$ ($x = 0$, $t = 4.2$)in (3.28) and solve this system to obtain

$$K_1(s,\tau) = 1.41100 - 0.12928i, \qquad K_2(s,\tau) = 0.9999998 - 0.0000002i. \tag{3.30}$$

Thus $K_2(s,\tau) \neq 0$ for (s,τ) inside the higher order Stokes curve.

We can thus deduce that the Stokes curve $S_{e>p}$ is inactive inside the higher order Stokes curve, but active outside the higher order Stokes curve. A monodromy argument can be used to determine that the curve $S_{p>e}$ must also be inactive inside the higher order Stokes curve, otherwise a contradiction will occur if one considers an analytic continuation around a circular path enclosing the middle TP of figure 3.

If one starts with only the contribution from e in the semi-infinite triangular region denoted A in figure 3, one may deduce the activity of all other Stokes curves from a monodromy argument. The final activity of the Stokes curves is displayed in figure 4.

4 Discussion

This paper has considered how to calculate the effect of a higher order Stokes line directly from the coefficients in an expansion. The method illustrated here requires knowledge of the positions of the Borel singularities as a function of the variables, together with analytic knowledge of the coefficients as a function of **a**. In the examples used we have been fortunate enough to obtain this information. Of course it is not too difficult to find equations where this information may not be available, or only available numerically. In these situations other techniques must be employed. However, if one is unable even to calculate the position of a subset of the Borel singularities, since these are effectively the exponents of the expansion prefactors, one will not even be able to write down an expansion involving exponentially small terms. Arguably, the question of activity of Stokes lines in these cases is somewhat redundant,

Other techniques for examining the activity of Stokes lines in the absence of initial integral representations also exist. Chapman and Mortimer (2004) have considered a direct approach using expansion coefficients similar to that of this paper. Rather than using the Borel structure, they seek a full expansion of the higher order late terms. An explicit resummation is sought to obtain an integral representation of the remainder of this expansion. The integral is then examined using the method of steepest descents.

Aoki *et al* (1994, 2001, 2002) have also considered the activity of higher order Stokes lines using and exact WKB approach. They have written down detailed guidelines for activity in terms of the proximity of Stokes lines to virtual caustics and Stokes crossing points.

So far we have only considered the effect of the higher order Stokes line for complex values of the parameters **a**. Readers may then be left with the impression that the effects of the higher order Stokes phenomenon at real values of spatial or temporal variables is insignificant. In the next paper we demonstrate that this is not the case.

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