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Exact WKB solutions at a regular singular point for 2×2 systems

Setsuro Fujiié
Mathematical Institute of Tohoku University
藤家雪朗 (東北大学大学院理学研究科数学専攻)

0 Introduction

This report is based on a joint work with L. Nedelec.

Recall first the radial Schrödinger equation

$$-h^2\frac{d^2u}{dx^2}+Q(x,h)u=0 \tag{1}$$

where the effective potential

$$Q(x,h)=V(x)+\frac{l(l+1)}{x^2}-E, \quad l\in\mathbb{N}=\{0,1,2,\dots\}$$

consists of the physical potential $V(x)$, the centrifugal potential $l(l+1)/x^2$ and the kinetic energy E. The numbers $\{l(l+1)\}_{l\in\mathbb{N}}$ are the eigenvalues of the Laplacian on the sphere S^2.

For this equation, the origin $x=0$ is a regular singular point and the Fuchs indices are $l+1$ and $-l$.

On the other hand, the WKB approximations (or Liouville Green functions) are given by

$$Q^{-1/4}\exp\left(\pm\int^x Q^{1/2}dx/h\right). \tag{2}$$

These functions behave like $x^{1/2\pm\sqrt{l(l+1)}}$ as x tends to 0 and the exponents differ from the Fuchs indices. This means that the WKB approximation (2), which is the leading term of the asymptotic expansion as $h \to 0$ (in a pole free and turning point free region), is not uniform with respect to x near the origin. This has been a problem since pointed out by Langer [5] (see [2] and [4] for treatments by different exact WKB methods).

Let us consider here the 2×2 system

$$\frac{h}{i}\frac{du}{dx} = \begin{pmatrix} x^2 - E & \gamma h/x \\ -\gamma h/x & -x^2 + E \end{pmatrix} u, \quad \gamma \in \frac{1}{2} + \mathbb{Z}. \tag{3}$$

This equation comes from a model of the Born-Oppenheimer approximation ([1]).

The origin $x = 0$ is a regular singular point also for this equation, and the Fuchs indices are $\pm\gamma$.

The WKB approximations, on the other hand, are of the form

$$\exp\left(\pm \int^x \sqrt{\alpha\beta}\,dx/h\right) \begin{pmatrix} (\alpha/\beta)^{1/4} \\ \mp i(\beta/\alpha)^{1/4} \end{pmatrix}$$

(see the WKB construction for systems in the next section). In this case, the exponents of these functions are $\pm\gamma$, which coincides with the Fuchs indices.

The aim of this report is to show that the exact WKB method established in [1] for latter type systems can be applied to construct a subdominant solution at a regular singular point as WKB solution. This enables us to connect, via Wronskian formula, the subdominant solution with other WKB solutions defined far away from the regular singular point.

1 Exact WKB method for 2×2 systems

In this section we review the exact WKB method used in [1] for 2×2 systems in a regular domain, i.e. in a domain with neither singularity nor turning point. This is a generalization of the exact WKB method of Gérard and Grigis [3] for the Schrödinger equations.

Let us consider the first order 2×2 system

$$\frac{h}{i}\frac{d\tilde{u}}{dx} = \tilde{A}(x,h)\tilde{u} \tag{4}$$

in a complex neighborhood Ω of a point $x = x_1 \in \mathbb{C}$. We assume that A is holomorphic in Ω depending regularly on h (i.e. $A(x,h) = A_0(x) + O(h)$), and

$$\mathrm{tr}\tilde{A} = 0, \quad \det\tilde{A} \neq 0.$$

After the change of the unknown vector $u = T(\phi, \omega)\tilde{u}$ by a matrix

$$T(\phi,\omega) = \begin{pmatrix} \cos\phi(x) & -\omega\sin\phi(x) \\ \omega^{-1}\sin\phi(x) & \cos\phi(x) \end{pmatrix},$$

with a suitable constant ω and a function $\phi(x)$, u satisfies (4) with $\tilde{A}$ replaced by an anti-diagonal matrix A:

$$\frac{h}{i}\frac{du}{dx} = A(x,h)u, \quad A = \begin{pmatrix} 0 & \alpha(x,h) \\ -\beta(x,h) & 0 \end{pmatrix}. \tag{5}$$

Indeed, if

$$\tilde{A} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{6}$$

then A is also trace free and the $(1,1)$-entry is

$$a\cos 2\phi + \frac{1}{2}(\omega^{-1}b + \omega c)\sin 2\phi. \tag{7}$$

Hence A is anti-diagonal if we define $\phi(x)$ so that

$$\tan 2\phi = -\frac{2a}{\omega^{-1}b + \omega c}. \tag{8}$$

The function $\phi(x)$ defined by (8) is holomorphic in Ω, if the constant ω is suitably chosen, i.e. if the right hand side of (8) differs from $\pm i$. Then α and β are given by

$$\alpha = b\cos^2\phi - \omega^2 c\sin^2\phi - 2\omega a\cos\phi\sin\phi - ih\phi',$$

$$-\beta = c\cos^2\phi - \omega^{-2}b\sin^2\phi - 2\omega^{-1}a\cos\phi\sin\phi + ih\phi'.$$

In the following, we assume for simplicity that α and β are independent of h.

Put

$$z(x) = \int_{x_0}^{x} (\alpha\beta)^{1/2}dx, \quad H(z(x)) = \left(\frac{\beta(x)}{\alpha(x)}\right)^{1/4},$$

and

$$u = e^{\pm z(x)/h}\begin{pmatrix} H(z)^{-1} & H(z)^{-1} \\ \mp iH(z) & \pm iH(z) \end{pmatrix} w_{\pm}.$$

Then $w_\pm$ satisfy

$$\frac{dw_\pm}{dz} = \begin{pmatrix} 0 & H'_z/H \\ H'_z/H & \mp 2/h \end{pmatrix} w_\pm, \tag{9}$$

where H'_z stands for the derivative of H with respect to z. The point of this reduction is that the singular part of the perturbation as h tends to 0 appears only at the $(2,2)$ element.

We define formal series

$$w_{\mathrm{even},\pm} = \sum_{n=0}^{\infty} w_{2n,\pm}, \quad w_{\mathrm{odd},\pm} = \sum_{n=0}^{\infty} w_{2n+1,\pm} \tag{10}$$

by $w_{0,\pm} \equiv 1$ and for $n \geq 1$,

$$\begin{cases} (d/dz)w_{2n,\pm} & = (H'_z/H)w_{2n-1,\pm} \\ (d/dz \pm 2/h)w_{2n-1,\pm} & = (H'_z/H)w_{2n-2,\pm}, \end{cases} \tag{11}$$

with initial conditions $w_{n,\pm}(z_1) = 0$, $z_1 = z(x_1)$. Then

$$w_\pm = \begin{pmatrix} w_{\mathrm{even},\pm} \\ w_{\mathrm{odd},\pm} \end{pmatrix}$$

are formal solutions to (9), and consequently

$$u_\pm(x;x_1) = e^{\pm z(x)/h} \begin{pmatrix} H(x)^{-1} & H(x)^{-1} \\ \mp iH(x) & \pm iH(x) \end{pmatrix} \begin{pmatrix} w_{\mathrm{even},\pm} \\ w_{\mathrm{odd},\pm} \end{pmatrix}.$$

are formal solutions to (5). We have the following theorem. See [1] for the proof.

Theorem 1 1. *The formal series (10) are absolutely convergent in a neighborhood of x_1.*

2. *Let $\Omega_\pm$ be the set of $x \in \Omega$ such that there exsists a path from x_0 to x in Ω along which* $\mathrm{Re}\, z(x)$ *increases strictly. Then in $\Omega_\pm$ we have for each $N \in \mathbb{N}$*

$$w_{\mathrm{even},\pm} - \sum_{n-0}^{N-1} w_{2n,\pm} = O(h^N), \quad w_{\mathrm{odd},\pm} - \sum_{n-0}^{N-1} w_{2n+1,\pm} = O(h^{N+1}),$$

3. *The Wronskian (with respect to x) of two exact WKB solutions are given by*

$$\mathcal{W}(u_+(x,x_1), u_-(x;x_2)) = 2iw_{\mathrm{even},+}(x_2;x_1),$$

where $\mathcal{W}(f,g)$ is by definition the determinant of the matrix (f,g).

2 Asymptotics at a regular singular point

In this section, we study the asymptotic behavior of the exact WKB solutions to the system (5) near a regular singular point. Let us assume that α and β have a simple pole at $x = 0$ and put

$$\alpha(x,h) = \frac{h}{x}\tilde{\alpha}(\frac{x}{h},h), \quad \beta(x,h) = \frac{h}{x}\tilde{\beta}(\frac{x}{h},h), \tag{12}$$

where $\tilde{\alpha}(y,h)$ and $\tilde{\beta}(y,h)$ are analytic symbols at $y = 0$. In order that the Fuchs indices at the origin are independent of h, $c_1 = \tilde{\alpha}(0,h)$ and $c_2 = \tilde{\beta}(0,h)$ should be independent of h. The argument of this section works in this general setting, but in this report, we restrict ourselves to the quite simple case where $\tilde{\alpha}$ and $\tilde{\beta}$ are linear functions:

$$\tilde{\alpha}(y,h) = c_1 + b_1 y, \quad \tilde{\beta}(y,h) = c_2 + b_2 y, \tag{13}$$

and moreover we assume that b_1 and b_2 constants. This case permits us to know the necessary informations about the geometry of the Stokes curves and to give in a concrete way the angular domains around $x = 0$ where the asymptotic properties of the exact WKB solutions are valid. Moreover it is possible to compare our local semiclassical problem for (5) when x and h are small with the equivalent global two points connection problem for the non-semiclassical equation

$$\frac{y}{i}\frac{du}{dy} = (C + yB)\,u, \quad C = \begin{pmatrix} 0 & c_1 \\ -c_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 \\ -b_2 & 0 \end{pmatrix}. \tag{14}$$

The equation (14) has two singular points: $y = 0$ and $y = \infty$. 0 is a regular singular point and ∞ is a irregular singular point. The Fuchs indices at the origin $y = 0$ are the eigenvalues of C, i.e. $\pm\sqrt{c_1 c_2}$. Put $\gamma = \sqrt{c_1 c_2}$ and assume $\gamma > 0$. On the contrary, the asymptotic behavior of solutions at ∞ is dominated by the eigenvalues of B, i.e. $\pm\sqrt{b_1 b_2}$. We assume also $b_1 b_2 \neq 0$.

The two points connection problem is to study the asymptotic behavior as y tends to infinity of the subdominant solution, corresponding to the index $+\gamma$, which is characterized by its asymptotic behavior as y tends to 0 up to constant multiplication. The aim of this section is to do this by studying the semiclassical version (5) with the exact WKB method of the previous section.

Let us go back to the system (5). Then there are two turning points x_1 and x_2, i.e. zeros of $\det A$ near 0, which tend to 0 as h tends to 0:

$$x_j = y_j h, \quad y_j = -c_j/b_j \ (j = 1, 2).$$

Let us construct an exact WKB solution of + type which was introduced in the previous section, but with the base point x_1 of the symbol placed at the origin, where the equation is singular. It is necessary, therefore, to check that the construction is still possible and, in particular, to study the asymptotic properties of the solutions as x and h tend to 0.

Put as in section 1,

$$z(x,h) = \int^x \frac{(\tilde{\alpha}\tilde{\beta})^{1/2}}{t} dt = \gamma h \int^x \sqrt{\left(1 - \frac{t}{y_1 h}\right)\left(1 - \frac{t}{y_2 h}\right)} \, \frac{dt}{t},$$

$$H(x) = \left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)^{1/4} = \left(\frac{c_2}{c_1} \cdot \frac{1 - x/(y_2 h)}{1 - x/(y_1 h)}\right)^{1/4},$$

with branch

$$(\tilde{\alpha}\tilde{\beta})^{1/2}|_{x=0} = \gamma h, \quad \left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)^{1/4}|_{x=0} = \left(\frac{c_2}{c_1}\right)^{1/4} > 0.$$

We rewrite the recurrence equations (11) in the variable x in order to give the initial conditions at the origin instead of $z(0) = \infty$: $w_{0,+} \equiv 1$ and for $n \geq 1$,

$$\begin{cases} (d/dx) w_{2n,+} &= (H'_x/H) w_{2n-1,+} \\ \{d/dx \pm (2/h)(\tilde{\alpha}\tilde{\beta})^{1/2}/x\} w_{2n-1,+} &= (H'_x/H) w_{2n-2,+}, \end{cases} \tag{15}$$

with initial conditions $w_{n,+}(0) = 0$. Here H'_x stands for the derivative of H with respect to the x-variable.

Let

$$\theta_j = \arg y_j \ (j = 1, 2), \quad 0 \leq \theta_2 - \theta_1 \leq \pi$$

and Δ_ϵ be the union of two angular domains Δ_ϵ^1 and Δ_ϵ^2:

$$\Delta_\epsilon^1 = \{x \in \mathbb{C} \backslash \{0\}; \arg x \in (\theta_1 + \epsilon, \theta_2 - \epsilon)\}$$

$$\Delta_\epsilon^2 = \{x \in \mathbb{C} \backslash \{0\}; \arg x \in (\frac{\theta_1 + \theta_2 - 3\pi}{2} + \epsilon, \frac{\theta_1 + \theta_2 - \pi}{2} - \epsilon)\}.$$

Theorem 2 *1. Each function $w_{n,+}$ is holomorphic in a neighborhood D of the origin and the series*

$$w_{\mathrm{even},+} = \sum_{n=0}^{\infty} w_{2n,+}, \quad w_{\mathrm{odd},+} = \sum_{n=0}^{\infty} w_{2n+1,+} \tag{16}$$

converge absolutely in D.

2. When $(x,h) \to (0,0)$ in $\Delta_\epsilon \times (0,h_0]$, we have

$$w_{\mathrm{even},+} - \sum_{n=0}^{N-1} w_{2n,+} = \begin{cases} O((|x|/h)^{2N}) & \text{as } |x|/h \to 0, \\ O((h/|x|)^{2N}) & \text{as } h/|x| \to 0, \end{cases}$$

$$w_{\mathrm{odd},+} - \sum_{n=0}^{N-1} w_{2n+1,+} = \begin{cases} O((|x|/h)^{2N+1}) & \text{as } |x|/h \to 0, \\ O((h/|x|)^{2N+2}) & \text{as } h/|x| \to 0, \end{cases}$$

Corollary 3 *Let*

$$u(x,h) = e^{z(x)/h} \begin{pmatrix} H(z(x))^{-1} & H(z(x))^{-1} \\ -iH(z(x)) & iH(z(x)) \end{pmatrix} \begin{pmatrix} w_{\mathrm{even},+} \\ w_{\mathrm{odd},+} \end{pmatrix},$$

then u is a solution to (5) with A given by (12) and (13). Moreover, when $(x,h) \to (0,0)$ in $\Delta_\epsilon \times (0,h_0]$, we have

$$u(x,h) \sim e^{z(x)/h} \begin{pmatrix} H(x)^{-1} \\ -iH(x) \end{pmatrix}$$

both as $|x|/h \to 0$ and $h/|x| \to 0$. In particular,

$$u(x,h) \sim c x^{\gamma} \begin{pmatrix} (c_1/c_2)^{1/4} \\ (c_2/c_1)^{1/4} \end{pmatrix} \quad \text{as } |x|/h \to 0, \tag{17}$$

for some constant c

The last formula (17) means that u is a subdominant solution at the origin.

We prove here the second part of Theorem 2. For this, we need the following lemma:

Lemma 4 *Let z_1, z_2 be complex numbers whose arguments ϕ_1 and ϕ_2 satisfy*

$$\epsilon < \phi_1, \phi_2 < 2\pi - \epsilon, \quad \pi + 2\epsilon < \phi_1 + \phi_2 < 3\pi - 2\epsilon \tag{18}$$

for a positive ϵ. Then there exists a positive constant δ such that

$$\frac{\mathrm{Re}\sqrt{(1-\sigma/z_1)(1-\sigma/z_2)}}{1+\sigma} \geq \delta \quad (0 < \sigma < +\infty),$$

where the square root is defined to be 1 when $\sigma = 0$.

Remark: Off course the condition (18) should be regarded as modulo 2π. For example, it can be replaced by

$$-2\pi + \epsilon < \phi_1 < -\epsilon, \ \epsilon < \phi_2 < 2\pi - \epsilon, \ -\pi + 2\epsilon < \phi_1 + \phi_2 < \pi - 2\epsilon. \tag{19}$$

Proof: As σ increases, the argument $\arg(1 - \sigma/z_j)$ increases if $0 < \phi_j \leq \pi$ and decreases if $\pi < \phi_j < 2\pi$, and

$$\lim_{\sigma \to 0} \arg(1 - \sigma/z_j) = 0, \quad \lim_{\sigma \to +\infty} \arg(1 - \sigma/z_j) = \pi - \phi_j.$$

Let $\psi(\sigma)$ be the argument of $\sqrt{(1-\sigma/z_1)(1-\sigma/z_2)}$, which is the mean of $\arg(1-\sigma/z_1)$ and $\arg(1-\sigma/z_2)$. Then

$$\begin{array}{ll} 0 \leq \psi(\sigma) \leq \pi - (\phi_1+\phi_2)/2 & \text{if } 0 < \phi_j \leq \pi \ (j = 1, 2), \\ \pi - (\phi_1+\phi_2)/2 \leq \psi(\sigma) \leq 0 & \text{if } \pi \leq \phi_j < 2\pi \ (j = 1, 2), \\ (\pi - \phi_2)/2 \leq \psi(\sigma) \leq (\pi - \phi_1)/2 & \text{if } 0 < \phi_1 \leq \pi \leq \phi_2 < 2\pi. \end{array}$$

In any case, under the condition (18), we have $|\psi(\sigma)| < (\pi - \epsilon)/2$ for all $\sigma > 0$. It follows that the real part of $\sqrt{(1-\sigma/z_1)(1-\sigma/z_2)}/\sigma$ is bounded from below by a positive constant. □

Proof of Theorem 2: The recurrence equations (15) can be written in the integral form:

$$w_{2n,+} = J[w_{2n-1,+}], \quad w_{2n-1,+} = I[w_{2n-2,+}],$$

where

$$J[f] = \int_0^x \frac{H'_x(\xi)}{H(\xi)} f(\xi) d\xi,$$

$$I[f] = \int_0^x \exp\left\{-\frac{2}{h}\int_\xi^x \frac{\sqrt{\tilde{\alpha}(t)\tilde{\beta}(t)}}{t}dt\right\} \frac{H'_x(\xi)}{H(\xi)} f(\xi)d\xi.$$

In our special case, the integral operators J and I are of the form

$$J[f] = \frac{y_1 - y_2}{4\gamma^2}\int_0^{x/h} \frac{f(h\eta)d\eta}{(1-\eta/y_1)(1-\eta/y_2)},$$

$$I[f] = \frac{y_1 - y_2}{4\gamma^2}\times$$

$$\int_0^{x/h} \exp\left\{-2\gamma \int_\eta^{x/h} \frac{\sqrt{(1-s/y_1)(1-s/y_2)}}{s}ds\right\} \frac{f(h\eta)d\eta}{(1-\eta/y_1)(1-\eta/y_2)}.$$

Let $x = re^{i\theta}$ and put $\eta = \rho e^{i\theta}$, $s = \sigma e^{i\theta}$. Then we have

$$|J[f]| \leq \frac{|y_1 - y_2|}{4\gamma^2}||f||_\infty \int_0^{r/h} \frac{d\rho}{|(1-\rho e^{i\theta}/y_1)(1-\rho e^{i\theta}/y_2)|},$$

$$|I[f]| \leq \frac{|y_1 - y_2|}{4\gamma^2}||f||_\infty \times$$

$$\int_0^{r/h} \exp\left\{-2\gamma \int_\rho^{r/h} \frac{\mathrm{Re}\sqrt{(1-\sigma e^{i\theta}/y_1)(1-\sigma e^{i\theta}/y_2)}}{\sigma}d\sigma\right\}$$

$$\frac{d\rho}{|(1-\rho e^{i\theta}/y_1)(1-\rho e^{i\theta}/y_2)|}.$$

We apply Lemma 4 with $z_j = y_j/e^{i\theta}$, $\phi_j = \arg z_j = \theta_j - \theta$ $(j = 1, 2)$. We can easily check that $0 \leq \phi_2 - \phi_1 \leq \pi$, and

$$x \in \Delta_\epsilon^1 \quad \Rightarrow \quad \epsilon - \pi < \phi_1 < -\epsilon, \quad \epsilon < \phi_2 < \pi - \epsilon,$$

$$x \in \Delta_\epsilon^2 \quad \Rightarrow \quad \epsilon < \phi_1, \phi_2 < 2\pi - \epsilon, \quad \pi + 2\epsilon < \phi_1 + \phi_2 < 3\pi - 2\epsilon.$$

Hence

$$|J[f]| \leq \frac{|y_1 - y_2|}{4\gamma^2\delta^2}||f||_\infty \int_0^{r/h} \frac{d\rho}{(1+\rho)^2},$$

$$|I[f]| \leq \frac{|y_1 - y_2|}{4\gamma^2\delta^2}||f||_\infty \int_0^{r/h} e^{2\gamma\delta(\rho - r/h)} \frac{d\rho}{(1+\rho)^2}.$$

Then Theorem 2 follows from

$$\int_0^{r/h} \frac{d\rho}{(1+\rho)^2} = \begin{cases} O(r/h) & (r/h \to 0), \\ O(1) & (h/r \to 0), \end{cases}$$

$$\int_0^{r/h} e^{2\gamma\delta(\rho - r/h)} \frac{d\rho}{(1+\rho)^2} = \begin{cases} O(r/h) & (r/h \to 0), \\ O(h^2/r^2) & (h/r \to 0). \end{cases}$$

□

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