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Stability of 1-dimensional stationary solution to the compressible Navier-Stokes equations on the half space

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1. Introduction

This article is concerned with the compressible Navier-Stokes equation on the half space $\mathbb{R}_+^n (n \geq 2)$:

$$
\partial_t \rho + \text{div} (\rho u) = 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla p(\rho) = \mu \Delta u + (\mu + \mu') \nabla \text{div} u,
$$

(1.1)

$$
p(\rho) = K \rho^\gamma.
$$

Here $\mathbb{R}_+^n = \{ x = (x_1, x') ; x' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}, x_1 > 0 \}$; $\rho = \rho(x, t)$ and $u = (u^1(x, t), \cdots, u^n(x, t))$ denote the unknown density and velocity, respectively; $\mu, \mu', K$ and $\gamma$ are constants satisfying $\mu > 0, \frac{2}{n} \mu + \mu' \geq 0, K > 0$ and $\gamma > 1$. We consider (1.1) under the initial and boundary conditions

$$
u |_{x_1=0} = (u_0^1, 0, \cdots, 0),
$$

(1.2)

$$
\rho \to \rho_+, \ u \to (u_+^1, 0, \cdots, 0) \quad (x_1 \to \infty),
$$

$$
(\rho, u)|_{t=0} = (\rho_0, u_0),
$$

where $\rho_+$, $u_+^1$ and $u_0^1$ are given constants satisfying $\rho_+ > 0$ and $u_0^1 < 0$.

Kawashima, Nishibata and Zhu [4] investigated the conditions for $\rho_+$, $u_+^1$ and $u_0^1$ under which planar stationary motions occur. Namely, they showed that under suitable conditions for $\rho_+$, $u_+^1$ and $u_0^1$ there exists a stationary solution $(\tilde{\rho}, \tilde{u})$ of problem (1.1)–(1.2) in the form $\tilde{\rho} = \tilde{\rho}(x_1)$, $\tilde{u} = (\tilde{u}^1(x_1), 0, \cdots, 0)$. Furthermore, it was shown in [4] that $(\tilde{\rho}, \tilde{u})$ is asymptotically stable with respect to small one-dimensional perturbations, i.e.,
perturbations in the form \( \rho - \bar{\rho} = \rho(x_1, t) - \bar{\rho}(x_1), u - \bar{u} = (u^1(x_1, t) - \bar{u}^1(x_1), 0, \cdots, 0) \), provided that \( |u_+^1 - u_b^1| \) is sufficiently small.

In this article we will give a summary of the results in [3], where \((\bar{\rho}, \bar{u})\) is shown to be asymptotically stable with respect to multi-dimensional perturbations small in \( H^s(\mathbb{R}_+^n) \), provided that \( |u_+^1 - u_b^1| \) is sufficiently small. Here \( s \) is an integer satisfying \( s \geq \lceil n/2 \rceil + 1 \).

2. Stability Result

We first consider the one-dimensional stationary problem whose solutions represent planar stationary motions in \( \mathbb{R}_+^n \). We look for a smooth stationary solution \((\tilde{\rho}, \tilde{u})\) of \((1.1)-(1.2)\) of the form \( \tilde{\rho} = \overline{\rho}(x_1) > 0 \) and \( \tilde{u} = (\overline{u}^1(x_1), 0, \cdots, 0) \). Then the problem for \((\overline{\rho}, \overline{u}^1)\) is written as

\[
\begin{align*}
(\overline{\rho}, \bar{\rho})_{x_1} &= 0 \quad (x_1 > 0), \\
(\overline{\rho}(\bar{u}^1)^2)_{x_1} + p(\bar{\rho})_{x_1} &= (2\mu + \mu')\overline{u}_{x_1x_1}^1 \quad (x_1 > 0), \\
\overline{u}\big|_{x_1=0} &= \overline{u}_b^1, \\
\overline{\rho} &\to \rho_+, \quad \overline{u}^1 &\to u_+^1 \quad (x_1 \to \infty),
\end{align*}
\]  

(2.1)

where subscript \( x_1 \) stands for differentiation in \( x_1 \).

Kawashima, Nishibata and Zhu [4] investigated problem (2.1) and gave a necessary and sufficient condition for the existence of solutions. Following [4], we introduce the Mach number at infinity defined by

\[ M_+ \equiv \frac{|u_+|}{\sqrt{p'(\rho_+)}}. \]

We also set

\[ \delta \equiv |u_+^1 - u_b^1|, \]

which measures the strength of the stationary solution.

**Proposition 2.1.** ([4]) Let \( u_+^1 < 0 \). Then problem (2.1) has a smooth solution \((\bar{\rho}, \bar{u}^1)\) if and only if \( M_+ \geq 1 \) and \( w_c u_+ > u_b \), where \( w_c \) is a certain positive number. The solution \((\bar{\rho}, \bar{u}^1)\) is monotonic, in particular, \( \bar{u}^1(x_1) \) is monotonically increasing when \( M_+ = 1 \). Furthermore, \((\bar{\rho}, \bar{u}^1)\) has the following decay properties as \( x_1 \to \infty \).

(i) If \( M_+ > 1 \), then for any nonnegative integer \( k \) there exists a constant \( C > 0 \) such that

\[ |\partial_{x_1}^k (\bar{\rho} - \rho_+, \bar{u}^1 - u_+^1)| \leq C \delta e^{-\sigma x_1} \]

for some positive constant \( \sigma \).
(ii) If $M_+ = 1$, then for any nonnegative integer $k$ there exists a constant $C > 0$ such that

$$|\partial_{x_1}^k (\tilde{\rho} - \rho_+, \tilde{u}^1 - u_+)| \leq C \frac{\delta^{k+1}}{(1 + \delta x_1)^{k+1}}.$$ 

Our interest is the stability properties of $(\tilde{\rho}, \tilde{u})$, $\tilde{u} = (\tilde{u}^1, 0, \cdots, 0)$, with respect to multi-dimensional perturbations. To state our stability result we introduce function spaces. For $0 < T \leq \infty$ and $\sigma \in \mathbb{Z}$, $\sigma \geq 0$, we define the Banach space

$$Z^\sigma(T) = X^\sigma(T) \times Y^\sigma(T)$$

where

$$X^\sigma(T) = \bigcap_{j=0}^{\lfloor \sigma \rfloor} C^j([0, T]; H^{\sigma-2j})$$

and

$$Y^\sigma(T) = X^\sigma(T) \cap H^j(0, T; \tilde{H}^{\sigma+1-2j}).$$

Here $\tilde{H}^m = H^m \cap H^1_0$ when $m \geq 1$ and $\tilde{H}^m = L^2$ when $m = 0$. The norm of $Z^\sigma(T)$ is defined by $||U||_{Z^\sigma(T)} = ||\phi||_{X^\sigma(T)} + ||\psi||_{Y^\sigma(T)}$ for $U = (\phi, \psi)$, where

$$||\phi||_{X^\sigma(T)} = \sup_{0 \leq t \leq T} ||\phi(t)||_\sigma, \quad ||\psi||_{Y^\sigma(T)} = \left( ||\phi||_{X^\sigma(T)}^2 + \int_0^T ||\psi(t)||_{\sigma+1}^2 dt \right)^{1/2}$$

with

$$||\phi(t)||_{\sigma, k} = \left( \sum_{j=0}^k ||\partial_t^j \phi(t)||_{H^{\sigma-2j}}^2 \right)^{1/2}, \quad ||\phi(t)||_\sigma = ||\phi(t)||_{\sigma, \lfloor \sigma \rfloor}.$$ 

We simply denote by $Z^\sigma$, $X^\sigma$ and $Y^\sigma$ when $T = \infty$.

**Theorem 2.2.** Let $s$ be an integer satisfying $s \geq \lfloor n/2 \rfloor + 1$ and let $(\tilde{\rho}, \tilde{u})$ be the solution of (2.1). Then there exists a positive number $\delta_0$ such that if $|u_b^1 - u_+^1| < \delta_0$, then $(\tilde{\rho}, \tilde{u})$ is stable with respect to perturbations small in $H^s(\mathbb{R}^n_+)$ in the following sense: there exist $\epsilon_0 > 0$ and $C > 0$ such that if the initial perturbation $(\rho(0) - \tilde{\rho}, u(0) - \tilde{u}) \in H^s$ and satisfies a suitable compatibility condition, then perturbation $(\rho(t) - \tilde{\rho}, u(t) - \tilde{u})$ exists in $Z^s$, and it satisfies

$$\|(\rho(t) - \tilde{\rho}, u(t) - \tilde{u})\|_{H^s} \leq C \|(\rho(0) - \tilde{\rho}, u(0) - \tilde{u})\|_{H^s}.$$
for all $t \geq 0$ and
\[
\lim_{t \to \infty} \|\partial_x (\rho(t) - \overline{\rho}, u(t) - \overline{u})\|_{H^{\epsilon-1}} = 0,
\]
provided that $\|(\rho(0) - \overline{\rho}, u(0) - \overline{u})\|_{H^s} \leq \varepsilon_0$. In particular,
\[
\lim_{t \to \infty} \|(\rho(t) - \overline{\rho}, u(t) - \overline{u})\|_{\infty} = 0.
\]

**Remarks.** (i) The stability of $(\overline{\rho}, \overline{u})$ was firstly investigated in [4] and they proved Theorem 2.1 for $n = 1$, i.e., $(\overline{\rho}, \overline{u})$ is stable with respect to small perturbations in the form $\rho - \overline{\rho} = \rho(x_1, t) - \overline{\rho}(x_1), \ u - \overline{u} = (u^1(x_1, t) - \overline{u}^1(x_1), 0, \ldots, 0)$.

(ii) We here consider large time behavior of solutions of (1.1)–(1.2) only under the conditions for $\rho_+, u^1_+$ and $u^2_+$ given in Proposition 2.1. As is easily imagined, if one of these conditions would be disturbed, then complicated phenomena might occur. In fact, Matsumura [5] proposed a classification of all possible time asymptotic states in terms of boundary data for one-dimensional problem. Some parts of this classification were already proved rigorously. See [5].

3. Outline of the Proof

Let us rewrite the problem into the one for perturbations. We set $(\phi, \psi) = (\rho - \overline{\rho}, u - \overline{u})$. Then problem (1.1)–(1.2) is transformed into
\[
\begin{align*}
\partial_t \phi + u \cdot \nabla \phi + \rho \mathrm{div} \psi &= F, \\
\rho (\partial_t \psi + u \cdot \nabla \psi) + L \psi + p'(+\rho) \nabla \phi &= G, \\
\psi|_{x_1 = 0} &= 0; \quad (\phi, \psi) \to (0, 0) \quad (x_1 \to \infty), \\
(\phi, \psi)|_{t=0} &= (\phi_0, \psi_0)
\end{align*}
\]
(3.1)

where
\[
\begin{align*}
L \psi &= -\mu \Delta \psi - (\mu + \mu') \nabla \mathrm{div} \psi, \\
F &= -\psi \cdot \nabla \overline{\rho} - \phi \mathrm{div} \overline{u}, \\
G &= -(\rho \psi + \phi \overline{u}) \cdot \nabla \overline{u} - (p'(\rho) - p'(\overline{\rho})) \nabla \overline{\rho}.
\end{align*}
\]

The proof of Theorem 2.1 is thus reduced to showing the global existence of solution $(\phi, \psi)$ of (3.1) in the class $Z^s$, where $s$ is an integer satisfying $s \geq [n/2] + 1$. 
Let us firstly consider the local existence of solutions. The local existence can be proved by applying the result in [2]. In fact, problem (3.1) is a hyperbolic-parabolic system satisfying the assumptions in [2] that guarantees the local solvability in $H^s$ for $s \geq [n/2] + 1$. Therefore, we obtain the following

**Proposition 3.1.** Let $s$ be an integer satisfying $s \geq s_0 = \lfloor \frac{n}{2} \rfloor + 1$. Assume that the initial value $(\phi_0, \psi_0)$ satisfies the following conditions.

(a) $(\phi_0, \psi_0) \in H^s$ and $(\phi_0, \psi_0)$ satisfies the $\hat{s}$-th order compatibility condition, where $\hat{s} = \lfloor s - 1/2 \rfloor$.

(b) $\inf_x \rho_0(x) \geq -\frac{1}{4} \inf_x \tilde{\rho}(x_1)$.

Then there exists a positive number $T_0$ depending on $\| (\phi_0, \psi_0) \|_{H^s}$ and $\inf_x \tilde{\rho}(x_1)$ such that problem (3.1) has a unique solution $(\phi, \psi) \in Z^s(T_0)$ satisfying $\phi(x, t) \geq -\frac{1}{2} \inf_x \tilde{\rho}(x_1)$ for all $(x, t) \in \mathbb{R}^n_+ \times [0, T_0]$. Furthermore, there exist constants $C > 0$ and $\gamma > 0$ depending on $s$, $\| (\phi_0, \psi_0) \|_{H^s}$ and $\inf_x \tilde{\rho}(x_1)$ such that

$$\| (\phi, \psi) \|_{Z^s(T_0)}^2 \leq C \{ 1 + \| (\phi_0, \psi_0) \|_{H^s}^2 \}^\gamma \| (\phi_0, \psi_0) \|_{H^s}^2$$

We next derive a priori estimates to show the global existence of solution. We define $E_\sigma(t)$ and $D_\sigma(t)$ by

$$E_\sigma(t) = \left( \sup_{0 \leq \tau \leq t} \left\{ |\psi(\tau)|^2_{\sigma} + \| \phi(\tau) \|^2_{H^\sigma} + |\partial_{\tau} \phi(\tau)|^2_{\sigma-1} \right\} \right)^{1/2}$$

and

$$D_\sigma(t) = \left\{ \left( \int_0^t \| \partial_x \psi \|^2_{H^\sigma} + \| \phi |_{x_1=0} \|^2_{L^2(\mathbb{R}^{n-1})} d\tau \right)^{1/2} \right\}$$

for $\sigma = 0$,

$$D_\sigma(t) = \left\{ \left( \int_0^t \| \partial_x \psi \|^2_{H^\sigma} + \| \phi |_{x_1=0} \|^2_{L^2(\mathbb{R}^{n-1})} + \| \partial_{\tau} \phi \|^2_{\sigma-1} + \| \partial_{\tau} \psi \|^2_{\sigma-1} d\tau \right)^{1/2} \right\}$$

for $\sigma \geq 1$.

In what follows we will denote the solution $(\phi, \psi)$ and the initial value $(\phi_0, \psi_0)$ by $U = (\phi, \psi)$, $U_0 = (\phi_0, \psi_0)$.

Theorem 2.2 follows from Proposition 3.1 and the following a priori estimate.
Proposition 3.2. Let \( U = (\phi, \psi) \) be a solution of (3.1) on \([0, T]\). Assume that \( E_s(t) < 1 \) for all \( t \in [0, T] \). Then there exist constants \( \varepsilon_0 > 0 \) and \( C > 0 \), which are independent of \( T > 0 \), such that

\[
E_s(t)^2 + D_s(t)^2 \leq C \|U_0\|_{H^s}^2
\]

for all \( t \in [0, T] \), provided that \( \|U_0\|_{H^s} < \varepsilon_0 \).

Outline of the proof of Proposition 3.2

As in the one-dimensional problem studied in [4], the point in the proof of Proposition 3.2 is to derive a suitable bound for the \( L^2 \) norm of \((\phi, \psi)\). Due to the fact that the stationary solution has no shear components, one can obtain the \( L^2 \) bound in the same way as in the one-dimensional case in [4].

Proposition 3.3. There exists a constant \( M > 0 \) such that if

\[
E_s(t) \leq M
\]

for all \( t \in [0, T] \), then

\[
E_0(t)^2 + D_0(t)^2 \leq C \{\|U_0\|_2^2 + R_0(t)^2\},
\]

uniformly in \( t \in [0, T] \), where \( C > 0 \) is independent of \( T \) and

\[
R_0(t)^2 = - \int_0^t \{ (\rho \psi \cdot \nabla \overline{u}, \psi) + ((p(\rho) - p(\overline{\rho})) - p'(\rho)) \phi, \mathrm{div} \overline{u} \} + \left( \frac{\rho}{\rho' L} \phi \frac{L}{\overline{u}}, \psi \right) \} d\tau.
\]

Proof. As in [4], we introduce an energy functional based on the energy function defined by

\[
\rho \mathcal{E} = \rho \left\{ \frac{1}{2} |u|^2 + \Phi(\rho) \right\}, \quad \Phi(\rho) = \int^\rho \frac{p(\zeta)}{\zeta^2} d\zeta.
\]

Note that \( \Phi(\rho) \) is a strictly convex function of \( \frac{1}{\rho} \). We then define

\[
\rho \tilde{\mathcal{E}} = \rho \left\{ \frac{1}{2} |\psi|^2 + \Psi(\rho, \tilde{\rho}) \right\},
\]

where

\[
\Psi(\rho, \tilde{\rho}) = \Phi(\rho) - \Phi(\tilde{\rho}) - \partial_{\frac{1}{\rho}} \Phi(\tilde{\rho}) \left( \frac{1}{\rho} - \frac{1}{\tilde{\rho}} \right)
\]

\[
= \int_{\rho}^{\tilde{\rho}} \frac{p(\zeta) - p(\tilde{\rho})}{\zeta^2} d\zeta.
\]
As shown in [4], $\rho \Psi(\rho, \tilde{\rho})$ is equivalent to $|\rho - \tilde{\rho}|^2$ for suitably small $|\rho - \tilde{\rho}|$, and hence, there are positive constants $c_0$ and $c_1$ such that

\begin{equation}
(3.3) \quad c_0^{-1}|U| \leq \rho \tilde{E} \leq c_0 |U|,
\end{equation}

where $U = (\phi, \psi)$, $\phi = \rho - \tilde{\rho}$ with $|\phi| \leq c_1$.

Since $H^{s} \hookrightarrow L^\infty$ we can find a number $M > 0$ such that if $E_s(t) \leq M$, then $\|\phi(t)\|_{\infty} \leq c_1$ and $\inf \phi(x, t) \geq -\frac{1}{4} \inf \rho(x_1)$ for all $t \in [0, T]$.

A direct calculation shows
\[
\partial_t (\rho E) + \text{div} (\rho u E + (p(\rho) - p(\tilde{\rho}))(\psi) = \mu \text{div} (\frac{1}{2} |\nabla \psi|^2) + (\mu + \mu') \text{div} (\psi \text{div} \psi) - \mu |\nabla \psi|^2 - (\mu + \mu') (\text{div} \psi)^2 + \mathcal{R}_0,
\]

where $\mathcal{R}_0 = \mathcal{R}_0(x, t)$ is the function defined by
\[
\mathcal{R}_0 = -\rho (\psi \cdot \nabla \overline{u}) \cdot \psi - (p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho}) \phi) \text{div} \overline{u} - \frac{1}{\tilde{\rho}} \phi \psi \cdot L \overline{u}.
\]

Proposition 3.3 now follows from this identity and (3.3). This completes the proof.

To estimate higher order derivatives, we rewrite (3.1) as
\begin{align*}
\partial_t \phi + u \cdot \nabla \phi + \rho_+ \text{div} \psi &= f, \\
\partial_t \psi + \frac{1}{\rho_+} L \psi + \frac{p'(\rho_+)}{\rho_+} \nabla \phi &= g, \\
\psi|_{x_1 = 0} &= 0, \\
(\phi, \psi) \rightarrow (0, 0) \quad (x_1 \rightarrow \infty), \\
(\phi, \psi)|_{t = 0} &= (\phi_0, \psi_0),
\end{align*}

where $L \psi = -\mu \Delta \psi - (\mu + \mu') \nabla \text{div} \psi$, $f = \tilde{f} + \hat{f}$ and $g = -\tilde{u} \cdot \nabla \psi + \tilde{g} + \hat{g}$. Here \(\tilde{\hat{f}} = -\phi \text{div} \psi\), \(\tilde{\hat{f}} = -\overline{\rho} - \rho_+\) \text{div} \psi - \psi \cdot \nabla \overline{\rho} - \phi \text{div} \overline{u}\), and \(\hat{g} = \overline{\hat{g}}(1) + \overline{\hat{g}}(2) + \overline{\hat{g}}(3), \hat{g} = \hat{g}(1) + \hat{g}(2) + \hat{g}(3)\) with
\[
\begin{align*}
\hat{g}(1) &= \hat{P}(\rho, \rho_+) \phi \nabla \phi, \\
\hat{g}(2) &= \frac{1}{\rho \rho_+} \phi L \psi, \\
\hat{g}(3) &= -\psi \cdot \nabla \psi, \\
\overline{\hat{g}}(1) &= \overline{\hat{P}}(\rho, \rho_+) (\overline{\rho} - \rho_+) \nabla \phi + \overline{P}(\rho, \tilde{\rho}) \phi \nabla \overline{\rho}, \\
\overline{\hat{g}}(2) &= \frac{1}{\rho \rho_+} (L \overline{u}) \phi + \frac{1}{\rho \rho_+} (\overline{\rho} - \rho_+) L \psi, \\
\overline{\hat{g}}(3) &= -\psi \cdot \nabla \overline{u}, \\
P(\rho_1, \rho_2) &= \int_0^1 p''(\rho_2 + \theta (\rho_1 - \rho_2)) d\theta, \\
\overline{\hat{P}}(\rho_1, \rho_2) &= \frac{p''(\rho_1)}{\rho_1 \rho_2} - \frac{p''(\rho_1, \rho_2)}{\rho_2}.
\end{align*}
\]
Before proceeding further, we introduce some notations. We define \( N_{\sigma} \geq 0 \) by
\[
N_{\sigma}(t)^{2} = \int_{0}^{t} \left( ||\hat{f}||_{\sigma}^{2} + ||\tilde{g}||_{\sigma-1}^{2} + ||\psi \cdot \nabla \phi||_{\sigma-1}^{2} \right) d\tau
+ \sum_{1 \leq 2j + |\alpha'| \leq \sigma} \int_{0}^{t} |(\partial_{\tau}^{j} \partial_{x}^{|\alpha'|} \psi)| d\tau
+ \sum_{1 \leq 2j + |\alpha| \leq \sigma} \int_{0}^{t} |(\partial_{\tau}^{j} \partial_{x}^{|\alpha|} \psi)| d\tau
+ \sum_{2j + |\alpha| + \ell \leq \sigma-1} \int_{0}^{t} ||\partial_{\tau}^{j} \partial_{x}^{|\alpha|} \partial_{x_{1}}^{\ell+1} \psi||_{2}^{2} d\tau,
\]

where \([C, D] = [C, D] = CD - DC\). We also define \( R_{\sigma} \geq 0 \) for \( \sigma \geq 1 \) by
\[
R_{\sigma}(t)^{2} = R_{\sigma-1}(t)^{2} + \int_{0}^{t} \left( ||\hat{f}||_{\sigma}^{2} + ||\tilde{g}||_{\sigma-1}^{2} + ||\tilde{u} \cdot \nabla \phi||_{\sigma-1}^{2} \right) d\tau
+ \sum_{1 \leq 2j + |\alpha'| \leq \sigma} \int_{0}^{t} |(\partial_{\tau}^{j} \partial_{x}^{|\alpha'|} \tilde{g})| d\tau
+ \sum_{1 \leq 2j + |\alpha| \leq \sigma} \int_{0}^{t} |(\partial_{\tau}^{j} \partial_{x}^{|\alpha|} \phi)| d\tau
+ \sum_{2j + |\alpha| + \ell \leq \sigma-1} \int_{0}^{t} ||\partial_{\tau}^{j} \partial_{x}^{|\alpha|} \partial_{x_{1}}^{\ell+1} \tilde{u} \cdot \nabla \phi||_{2}^{2} d\tau.
\]

**Proposition 3.4.** Let \( 1 \leq \sigma \leq s \). Assume that (3.2) holds. Then there exists a constant \( C > 0 \) such that
\[
E_{\sigma}(t)^{2} + D_{\sigma}(t)^{2} \leq C \{ ||U_{0}||_{H^{s}}^{2} + R_{\sigma}(t)^{2} + N_{\sigma}(t)^{2} \}.
\]

To prove Proposition 3.4 we introduce a notation
\[
|v|_{k} = \left( \sum_{|\alpha| = k} \| \partial_{x}^{|\alpha|} v \|_{2}^{2} \right)^{1/2}.
\]
We also define \( T_{j,|\alpha'|} \) by
\[
T_{j,|\alpha'|} v = \partial_{t}^{j} \partial_{x}^{|\alpha'|} v.
\]
Proposition 3.4 follows from the following inequalities.

**Proposition 3.5.** Let $\sigma$ be a nonnegative integer satisfying $\sigma \leq s$.

(i) Let $j$ and $\alpha'$ satisfy $2j + |\alpha'| = \sigma$. Then

$$
\|T_{j,\alpha'} U(t)\|_{H^{s}}^2 + \int_{0}^{t} \|L^{1/2}T_{j,\alpha'} \psi\|_{2}^2 \, d\tau \leq C\{\|U_{0}\|_{H^{s}}^2 + R_{\sigma}(t)^2 + N_{\sigma}(t^2)\},
$$

where $\|L^{1/2}\psi\|_{2}^2 = \mu\|\nabla\psi\|_{2}^2 + (\mu + \mu')\|\text{div} \psi\|_{2}^2$.

(ii) Let $j$ and $\alpha'$ satisfy $2j + |\alpha'| = \sigma - 1$. Then

$$
\|L^{1/2}T_{j,\alpha'} \psi(t)\|_{2}^2 + \oint_{0}^{t} \|T_{j+1,\alpha'} \psi\|_{2}^2 \, d\tau \leq \eta D_{\sigma}(t)^2 + C_{\eta}N_{\sigma}(t)^2
$$

for any $\eta > 0$. Here and in what follows $N_{\sigma}(t)^2$ denotes

$$
N_{\sigma}(t)^2 = \|U_{0}\|_{H^{s}}^2 + E_{\sigma-1}(t)^2 + D_{\sigma-1}(t)^2 + R_{\sigma}(t)^2 + N_{\sigma}(t^2).
$$

(iii) Let $j$ and $\alpha'$ satisfy $2j + |\alpha'| + \ell = \sigma - 1$. Then

$$
\|T_{j,\alpha'} \partial_{x_{1}}^{l+1} \phi(t)\|_{2}^2 + \int_{0}^{t} \|T_{j,\alpha'} \partial_{x_{1}}^{l+1} \phi\|_{2}^2 \, d\tau
\leq \eta D_{\sigma}(t)^2 + C_{\eta}\{N_{\sigma}(t)^2 + \int_{0}^{t} \|T_{j+1,\alpha'} \partial_{x_{1}}^{l} \psi\|_{2}^2 + \|\partial_{x}\partial_{x'}T_{j,\alpha'} \partial_{x_{1}}^{l} \psi\|_{2}^2 \, d\tau\}
$$

for any $\eta > 0$.

(iv) Let $j$ and $\alpha'$ satisfy $2j + |\alpha'| + \ell = \sigma - 1$ and set $\frac{D\phi}{Dt} = \partial_{t}\phi + u \cdot \nabla \phi$. Then

$$
\int_{0}^{t} \|T_{j,\alpha'} \frac{D\phi}{Dt}\|_{\ell+1}^2 \, d\tau \leq \eta D_{\sigma}(t)^2 + C_{\eta}\{N_{\sigma}(t)^2 + \int_{0}^{t} \|T_{j+1,\alpha'} \partial_{x_{1}}^{l} \psi\|_{2}^2 + \|\partial_{x}\partial_{x'}T_{j,\alpha'} \partial_{x_{1}}^{l} \psi\|_{2}^2 \, d\tau\}
$$

for any $\eta > 0$.

(v) Let $j$ and $\alpha'$ satisfy $2j + |\alpha'| + \ell = \sigma - 1$. Then

$$
\int_{0}^{t} \|T_{j,\alpha'} \psi\|_{\ell+2}^2 + \|T_{j,\alpha'} \phi\|_{\ell+1}^2 \, d\tau \leq C\left\{\|T_{j+1,\alpha'} \psi\|_{2}^2 + \|\partial_{x}\partial_{x'}T_{j,\alpha'} \partial_{x_{1}}^{l} \psi\|_{2}^2 \right\} \left\{\|T_{j,\alpha'} (\overline{u} \cdot \nabla \psi)\|_{2}^2 + \|T_{j,\alpha'} \overline{g}\|_{2}^2 + \|T_{j,\alpha'} \tilde{g}\|_{2}^2\right\} \, d\tau.
$$

(vi) Let $j$ and $\alpha'$ satisfy $2j + 1 \leq \sigma$. Then

$$
\|\partial_{t}^{j+1} \phi(t)\|_{2}^2 + \int_{0}^{t} \|\partial_{\tau}^{j+1} \phi\|_{2}^2 \, d\tau \leq \eta D_{\sigma}(t)^2 + C_{\eta}N_{\sigma}(t)^2
$$
for any $\eta > 0$.

**Proof.** Proposition 3.5 can be proved by the energy method as in [1, 6]. The details can be found in [3].

It remains to estimate $R_{\sigma}$ and $N_{\sigma}$. To estimate $R_{0}$ we will use a special case of Hardy's inequality

$$\left\| \frac{1}{x_{1}} \int_{0}^{x_{1}} v(y) \, dy \right\|_{L^{2}(0,\infty)} \leq C \| v \|_{L^{2}(0,\infty)}.$$  \hspace{1cm} (3.5)

In a similar manner as in [1, 4], applying (3.5) and the decay estimates in Proposition 2.1 together with the Gagliardo-Nirenberg inequality, one can show that

$$R_{0}(t)^{2} \leq C\{\delta D_{0}(t)^{2} + E_{s}(t)D_{s}(t)^{2}\}.$$  

Here we note that we also use the monotonicity of $\tilde{u}^{1}(x_{1})$ when $M_{+} = 1$.

For $\sigma \geq 1$, one can show, as in [1], that

$$R_{\sigma}(t)^{2} + N_{\sigma}(t)^{2} \leq C\{D_{\sigma-1}(t)^{2} + \delta D_{\sigma}(t)^{2} + E_{s}(t)D_{s}(t)^{2}\},$$

provided that $E_{s}(t) < \min\{M, 1\}$. Therefore, it follows that if $\delta$ is sufficiently small and $E_{s}(t) < \min\{M, 1\}$ then

$$E_{s}(t)^{2} + D_{s}(t)^{2} \leq C\{\|U_{0}\|_{H^{s}}^{2} + E_{s}(t)D_{s}(t)^{2}\},$$

and hence, we conclude that

$$E_{s}(t)^{2} + D_{s}(t)^{2} \leq C\|U_{0}\|_{H^{s}}^{2},$$

provided that $\|U_{0}\|_{H^{s}}$ is sufficiently small. This completes the proof of Proposition 3.2.

**References**


