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The Reachability and Related Decision Problems for Semi-Constructor TRSs

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Abstract
This paper shows that reachability is undecidable for confluent monadic and semi-constructor TRSs, and joinability and confluence are undecidable for monadic and semi-constructor TRSs. Here, a TRS is monadic if the height of the right-hand side of each rewrite rule is at most 1, and semi-constructor if all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms.

1 Introduction
In this paper, we consider the reachability problem for confluent monadic and semi-constructor TRSs posed by our previous paper [4]. Here, a TRS is monadic if the height of the right-hand side of each rewrite rule is at most 1, and semi-constructor if all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms. We give a negative answer to this problem. This undecidability result is compared with the decidability results of joinability and unification for the same class [4, 3].

Moreover, we show that joinability and confluence are undecidable for monadic and semi-constructor TRSs.

2 Preliminaries
We assume that the reader is familiar with standard definitions of rewrite systems [1] and we just recall here the main notations used in this paper.
Let $F$ be a finite set of operation symbols graded by an arity function $\ar: F \to \mathbb{N} = \{0, 1, 2, \ldots\}$, $F_n = \{ f \in F \mid \ar(f) = n \}$. We use $x, y$ as variables, $f$ as an operation symbol, $r, s, t$ as terms. Let $V(s)$ be the set of variables occurring in $s$. The height of a term is defined as follows: $\text{height}(a) = 0$ if $a$ is a variable or a constant and $\text{height}(f(t_1, \ldots, t_n)) = 1 + \max\{\text{height}(t_1), \ldots, \text{height}(t_n)\}$ if $n > 0$. The root symbol of a term is defined as $\text{root}(a) = a$ if $a$ is a variable and $\text{root}(f(t_1, \ldots, t_n)) = f$.

A position in a term is expressed by a sequence of positive integers, and positions are partially ordered by the prefix ordering $\preceq$. Let $O(s)$ be the set of positions of $s$. For a set of positions $W$, let $\operatorname{Min}(W)$ be the set of its minimal positions (w.r.t. $\leq$).

Let $s[p]$ be the subterm of $s$ at position $p$. For a sequence $(p_1, \ldots, p_n)$ of pairwise parallel positions and terms $t_1, \ldots, t_n$, we use $s[t_1, \ldots, t_n][p_1, \ldots, p_n]$ to denote the term obtained from $s$ by replacing each subterm $s[p_i]$ by $t_i$ ($1 \leq i \leq n$). For a set of function symbols $F$, let $O_F(s) = \{ p \in O(s) \mid \text{root}(s[p]) \in F \}$. For a string of unary function symbols $u = a_1a_2\cdots a_k$ and a term $t$, let $u(t)$ be an abbreviation for $a_1(a_2(\cdots a_k(t)))$.

A rewrite rule $\alpha \rightarrow \beta$ is a directed equation over terms. A TRS $R$ is a set of rewrite rules. Let $\rightarrow$ be the inverse of $\rightarrow$, $\leftrightarrow = \rightarrow \cup \leftarrow$, and $\downarrow = \rightarrow \cdot \leftarrow$. $t$ is reachable from $s$ if $s \rightarrow^* t$. $R$ is confluent on TRS $R$ if for every $s \rightarrow_R^* t, s \rightarrow_R^* t$ and $t \downarrow t$. A TRS $R$ is confluent if every $r$ is confluent on $R$. Let $\gamma: s_1 \leftarrow^{p\text{-inv}} s_n$. Let $|\gamma|$ be the number of steps of $\gamma$. $\gamma$ is called $p$-invariant if $q > p$ for any redex position $q$ of $\gamma$, and we write $\gamma: s_1 \leftarrow^{p\text{-inv}} s_n$.

The set $D_R$ of defined symbols for a TRS $R$ is defined as $D_R = \{ \text{root}(\alpha) \mid \alpha \rightarrow \beta \in R \}$. A term $s$ is semi-constructor if for every subterm $t$ of $s$, $t$ has no variable or root(t) is not a defined symbol.

Definition 1 A rule $\alpha \rightarrow \beta$ is monadic if $\text{height}(\beta) \leq 1$, semi-constructor if $\beta$ is semi-constructor. A TRS $R$ is monadic if every rule in $R$ is monadic, semi-constructor if every rule in $R$ is semi-constructor.
3 Undecidability of joinability for monadic and semi-constructor TRSs

We have shown that joinability is undecidable for linear semi-constructor TRSs [4]. In this section, we show that joinability for monadic and semi-constructor TRSs is undecidable by a reduction from the Post's Correspondence Problem (PCP). Let $P = \{(u_i, v_i) \in \Sigma^* \times \Sigma^* | 1 \leq i \leq n\}$ be an instance of the PCP. The corresponding TRS $R_P$ is constructed as follows. Let $F = F_0 \cup F_1 \cup F_2$ where $F_0 = \{0, c, d, \}$, $F_1 = \{a_i | 1 \leq i \leq n\} (= E) \cup \Sigma$, $F_2 = \{f, g\}$.

$$R_P = \{0 \rightarrow a_i(0) \mid 1 \leq i \leq n\} \cup \{0 \rightarrow f(c, d)\} \cup \{b \rightarrow a(b), b \rightarrow a(\$) \mid b \in \{c, d\}, a \in \Sigma\} \cup \{f(x, x) \rightarrow g(x, x)\} \cup \{e_i(g(u_i(x), v_i(y))) \rightarrow g(x, y) \mid 1 \leq i \leq n\}$$

$R_P$ is monadic. Here, $D_{R_P} = \{0, c, d, f\} \cup E$, so $R_P$ is semi-constructor.

**Lemma 2** $0 \rightarrow^{*}_{R_P} g(\$, )$ iff PCP $P$ has a solution.

**Proof.** $0 \rightarrow^{*}_{R_P} g(\$, )$ iff there exists $i_1: \cdots: i_m \in \{1, \ldots, n\}^*$ such that $0 \rightarrow^{m+1} e_{i_1} \cdots e_{i_m} (f(c, d)) \rightarrow^+ e_{i_m} \cdots e_{i_1} (f(u_{i_1} \cdots u_{i_m}(\$), u_{i_1} \cdots u_{i_m}(\$))) \rightarrow e_{i_m} \cdots e_{i_1} (g(u_{i_1} \cdots u_{i_m}(\$), u_{i_1} \cdots u_{i_m}(\$))) \rightarrow^{m} g(\$, ).$ Since $g(\$, )$ is a normal form, the following theorem holds.

**Theorem 3** Both joinability and reachability for monadic and semi-constructor TRSs are undecidable.

4 Undecidability of reachability for confluent monadic and semi-constructor TRSs

We give a stronger result for reachability, that is, reachability for confluent monadic and semi-constructor TRSs is undecidable. Note that joinability is decidable for the same class [4, 3]. Let $P = F \cup \{1\}$.

$$R_P = R_P \cup \{\$ \rightarrow 1\} \cup \{a(1) \rightarrow 1 \mid a \in \Sigma\} \cup \{e_i(g(1, x)) \rightarrow g(1, y), e_i(g(u_i(x), l)) \rightarrow g(x, 1), e_i(g(1, l)) \rightarrow g(1, 1) \mid 1 \leq i \leq n\}$$

$R_P$ is monadic. Here, $D_{R_P} = D_{R_P} \cup \{\$\} \cup \Sigma$, so $R_P$ is semi-constructor. First, we show the confluence of $R_P$.

4.1 Confluence of $R_P$

To show the confluence of $R_P$, we need some definitions and lemmata.

**Definition 4** The set of $\Sigma$-strings is defined as follows.

- $1, c, d$ and $\$ are $\Sigma$-strings.
- $a(t)$ is a $\Sigma$-string if $t$ is a $\Sigma$-string and $a \in \Sigma$.

**Lemma 5** For any $\Sigma$-string $s$, the following properties hold.

1. For any $\gamma: s \leftrightarrow t$, $t$ is a $\Sigma$-string.
2. $s \rightarrow^*$.

**Proof.**

1. By induction on $|\gamma|$.
2. By induction on the structure of $s$. □

**Corollary 6** Every $\Sigma$-string is confluent.

**Lemma 7** Let $\gamma: s \rightarrow^* t$ where $u \in \Sigma^+$. Then, if $\operatorname{root}(s) \notin \{1, c, d, \$\} \cup \Sigma$ and $u(s)_{t_1} = s$ then $\gamma$ is $p$-invariant.

**Proof.** By induction on $|\gamma|$. □

**Definition 8** The set of $E$-strings is defined as follows.

- $0, f(t_1, t_2)$ and $g(t_1, t_2)$ are $E$-strings if $t_1, t_2$ are $\Sigma$-strings.
- $e_i(t)$ is an $E$-string if $t$ is an $E$-string and $i \in \{1, \ldots, n\}$.

**Lemma 9** For any $E$-string $s$, the following properties hold.

1. For any $\gamma: s \leftrightarrow t$, $t$ is an $E$-string.
2. $s \rightarrow^* g(1, 1)$.

**Proof.**

1. By induction on $|\gamma|$.
2. By induction on the structure of $s$. Basis: For any $E$-strings $s_1, s_2$, $f(s_1, s_2) \rightarrow^* f(1, 1) \rightarrow g(1, 1)$ and $g(s_1, s_2) \rightarrow^* g(1, 1)$ by Lemma 5(2), and $0 \rightarrow f(c, d) \rightarrow^* g(1, 1)$. Thus, $s \rightarrow^* g(1, 1)$ if $s = f(s_1, s_2)$, $g(s_1, s_2) \in 0$. Induction step: Let $s = e_i(s')$ for some $i \in \{1, \ldots, n\}$. By the induction hypothesis, $s' \rightarrow^* g(1, 1)$. Thus, $e_i(s') \rightarrow^* g(1, 1)$. □

**Corollary 10** Every $E$-string is confluent.
The following lemma is used as a component of the proof of Lemma 12.

Lemma 11 For any \( i \in \{1, \ldots, n\} \) and terms \( r_1, r_2 \), the following properties hold.

1. If \( s \cong e_i(g(r_1, r_2)) \rightarrow^{*} t \) then there exist terms \( t_1, t_2 \) such that \( t \rightarrow^{*} g(t_1, t_2) \).
2. If \( g(s_1, s_2) \cong e_i(g(r_1, r_2)) \rightarrow^{*} g(t_1, t_2) \) and \( g(r_1, r_2) \) is confluent then \( g(s_1, s_2) \downarrow g(t_1, t_2) \).

Proof.

(1) Let \( t = e_i(g(t_1', t_2')) \). If \( r_1 \) is a \( \Sigma \)-string then \( t_1' \rightarrow^{*} 1 \) by Lemma 5. Otherwise, \( r_1 \neq 1 \). Thus, \( r_1 = u_i(r_1') \) for some term \( r_1' \) by \( e_i(g(r_1, r_2)) \rightarrow s \). By Lemma 7, \( t_1' = u_i(t_1') \), where \( r_1' \rightarrow^{*} t_1' \). Similarly, \( t_2' \rightarrow^{*} 1 \) or \( t_2' = u_i(t_2') \) for some term \( t_2' \). Thus, \( t \rightarrow^{*} g(t_1, t_2) \), where \( t_1 \in \{1, t_1'\} \) and \( t_2 \in \{1, t_2'\} \).

(2) By the definition of \( R_P \), \( e_i(g(r_1, r_2)) \rightarrow^{*} e_i(g(s_1', s_2')) \rightarrow^{*} g(s_1', s_2') \) and \( e_i(g(r_1, r_2)) \rightarrow^{*} e_i(g(t_1', t_2')) \rightarrow^{*} g(t_1', t_2') \). Thus, \( s_1' \rightarrow^{*} r_1 \rightarrow^{*} t_1', s_2' \rightarrow^{*} t_2', s_1 = 1 \).

Case of \( s_1' = t_1' = 1 \) : Obviously, \( s_1'' = s_1 = s_2 = 1. \)

Case of \( s_1' = 1 \) and \( t_1' = u_i(t_1') \) : Obviously, \( s_1'' = s_1 = 1. \)

Case of \( s_1' = u_i(s_1') \) and \( t_1' = 1 \) : Similar to the previous one.

Case of \( s_1' = u_i(s_1') \) and \( t_1' = u_i(t_1') \) : By confluence of \( g(r_1, r_2), r_1 \) is confluent. Thus, \( u_i(s_1) \downarrow u_i(t_1) \). If \( s_1 \) is a \( \Sigma \)-string then \( s_1 \downarrow t_1 \) by Corollary 6. Otherwise, \( s_1 \downarrow t_1 \) by Lemma 7.

Similarly, \( s_2 \downarrow t_2 \). Thus, \( g(s_1, s_2) \downarrow g(t_1, t_2) \). \( \square \)

Now, we show the confluence of \( R_P \).

Lemma 12 \( R_P \) is confluent.

Proof. We show that for any \( \gamma : s \rightarrow^{*} r \rightarrow^{*} t \), \( s \downarrow t \) by induction on \( \text{height}(r) \).

Basis: If \( r \in \{c, d\} \) then \( s \downarrow t \) by Corollary 6, else if \( r = 0 \) then \( s \downarrow t \) by Corollary 10. Otherwise, \( s = r = t \) since \( r \) is a normal form.

Induction step: If \( \gamma \) is \( \varepsilon \)-invariant then \( s \downarrow t \) by the induction hypothesis. So, we consider that \( \gamma \) has an \( \varepsilon \)-reduction. Let \( \gamma_1 : r \rightarrow^{*} s \) and \( \gamma_2 : r \rightarrow^{*} t \). Without lost of generality, we assume that \( \gamma_1 \) has an \( \varepsilon \)-reduction and \( \text{root}(r) \in \Sigma \cup \{f\} \cup E \).

Case of root\((r) \in \Sigma \): \( \gamma_2 : r = a(r_1) \rightarrow^{*} a(1) \rightarrow 1 = s \) holds for some \( a \in \Sigma \) and \( r_1 \). By Lemma 5, \( t \rightarrow^{*} 1 \).

Case of root\((r) \in \Sigma \): \( \gamma_2 : r = f(r_1, r_2) \rightarrow^{*} g(s_1, s_2) \rightarrow^{*} s \) holds for some terms \( r_1, r_2, s_1, s_2 \). If \( \gamma \) is \( \varepsilon \)-invariant then \( t = f(t_1, t_2) \) where \( r_1 \rightarrow^{*} t_1 \) and \( r_2 \rightarrow^{*} t_2 \). In this case, \( s \rightarrow^{*} g(r_0, r_0) \rightarrow^{*} t \) for some \( r_0 \) by Figure 1(i). If \( \gamma \) has an \( \varepsilon \)-reduction then \( \gamma_2 : r = f(r_1, r_2) \rightarrow^{*} f(r''_1, r''_2) \rightarrow^{*} g(r''_1, r''_2) \rightarrow^{*} t_1, t_2 = t \) holds for some terms \( r''_1, r''_2 \). In this case, \( s \rightarrow^{*} g(r_0, r_0) \rightarrow^{*} t \) for some \( r_0 \) by Figure 1(ii).

Case of root\((r) \in E \): \( \gamma_2 : r = e_i(r_1) \rightarrow^{*} e_i(g(s_1', s_2')) \rightarrow^{*} g(s_1', s_2') \rightarrow^{*} s \) holds for some terms \( r_1, s_1', s_2', s_1, s_2 \) and \( i \in \{1, \ldots, n\} \). If \( \gamma \) is \( \varepsilon \)-invariant then \( t = e_i(t_1) \) where \( r_1 \rightarrow^{*} t_1 \). By the induction hypothesis, there exists a term \( t' \) such that \( e_i(g(s_1', s_2')) \rightarrow^{*} t' \rightarrow^{*} t \). By Lemma 11(i), \( t' \rightarrow^{*} g(t_1', t_2') \) for some \( t_1', t_2' \). Here, \( g(s_1', s_2') \) is confluent by the induction hypothesis and \( r_1 \rightarrow^{*} g(s_1', s_2') \). Thus, \( s \downarrow g(t_1', t_2') \) by Lemma 11(2). (See Figure 1(iii).) If \( \gamma_2 \) has an \( \varepsilon \)-reduction then \( \gamma_2 : r = e_i(r_1) \rightarrow^{*} e_i(g(t_1', t_2')) \rightarrow^{*} g(t_1', t_2') \rightarrow^{*} g(t_1, t_2) \rightarrow^{*} t \) holds for some terms \( t_1', t_2', t_1, t_2, t_1', t_2 \). There exists a term \( s' \) such that \( s \rightarrow^{*} s' \rightarrow^{*} e_i(g(t_1', t_2')) \) as shown in Figure 1(iii).

Here, root\((s') = g \) by root\((s) = g \). By the induction hypothesis and \( r_1 \rightarrow^{*} g(t_1', t_2'), g(t_1', t_2') \) is confluent. Thus, \( s' \downarrow t \) by Lemma 11(ii). (See Figure 1(iv).) \( \square \)

4.2 Reachability for confluent monadic and semi-constructor TRSs

Lemma 13 For any \( \gamma : s \rightarrow^{*} R_P t \), if \( s \) has 1 as its subterm then so does \( t \).

Proof. Since for any \( \alpha \rightarrow \beta \in R_P \), \( V(\alpha) = V(\beta) \) and if \( \alpha \) has 1 as its subterm then so does \( \beta \). \( \square \)

Lemma 14 \( 0 \rightarrow^{*} R_P g(s, s) \) iff \( 0 \rightarrow^{*} R_P g(s, s) \).

Proof. Only if part: Let \( \gamma : 0 \rightarrow^{*} R_P g(s, s) \). We assume to the contrary that \( \gamma \) must have \( R_P \) \( R_P \) reduction, i.e., \( \gamma : 0 \rightarrow^{*} R_P g(s, s) \) for some \( s, t \). By the definition of \( R_P \), \( t \) has 1 as its subterm. By Lemma 13, \( g(s, s) \) has 1 as its subterm, a contradiction. If part: By \( R_P \subseteq R_P \).

By Lemmata 2, 12, and 14, the following theorem holds.

Theorem 15 Reachability for confluent monadic and semi-constructor TRSs is undecidable.
5 Undecidability of confluence of monadic and semi-constructor TRSs

We show that confluence of monadic and semi-constructor TRSs is undecidable.

Let $F' = F_0' \cup F_1'$ where $F_0' = \{2\}$, $F_1' = \{h\}$.

$$R = \{h(a) \rightarrow h(0), h(g(s, s)) \rightarrow 2\}$$

$\hat{R}_P \cup R$ is monadic. Here, $D_R = \{h\}$, so $\hat{R}_P \cup R$ is semi-constructor.

Lemma 16 For any $s$ with root($s$) $\in F'$, the following properties hold.

1. If $s \rightarrow^{*}_{\hat{R}_P \cup R} t$ then root($t$) $\in F'$.
2. If $0 \rightarrow^{*}_{\hat{R}_P} g(s, s)$ then $s \rightarrow^{*}_{\hat{R}_P \cup R} 2$.

The proof is straightforward, so omitted.

Lemma 17 Let $s \rightarrow^{*}_{\hat{R}_P \cup R} t$, $\text{Min}(O_F(s)) = \{p_1, \ldots, p_m\}$, and $\text{Min}(O_F(t)) = \{q_1, \ldots, q_n\}$. Then, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow^{*}_{\hat{R}_P} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$ or $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} = t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$.

Proof. Let $s \rightarrow^{*}_{\hat{R}_P \cup R} t$. If there exists $i \in \{1, \ldots, m\}$ such that $p_i \leq p$ then $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} = t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$ by Lemma 16(1).

By contradiction, otherwise, obviously $s \rightarrow^{*}_{\hat{R}_P} t$. Since every function symbol in $F'$ does not occur in $\hat{R}_P$, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow^{*}_{\hat{R}_P} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$. $\square$

Lemma 18 $\hat{R}_P \cup R$ is confluent iff $0 \rightarrow^{*}_{\hat{R}_P} g(s, s)$.

Proof. Only if part: By $\text{h(0)} \rightarrow^{*}_{\hat{R}_P} \text{h(g(s, s))} \rightarrow^{*}_{\hat{R}_P} 2$, confluence ensures that $\text{h(0)} \rightarrow^{*}_{\hat{R}_P \cup R} 2$. Since $2$ is a normal form, $\text{h(0)} \rightarrow^{*}_{\hat{R}_P \cup R} 2$. Thus, there exists a shortest sequence $\gamma$ that satisfies $\gamma : \text{h(0)} \rightarrow^{*}_{\hat{R}_P \cup R} \text{h(g(s, s))} \rightarrow^{*}_{\hat{R}_P} 2$. Since $\gamma$ is shortest, $\text{h(0)} \rightarrow^{*}_{\hat{R}_P \cup R} \text{h(g(s, s))}$. Thus, there exists $\gamma' : 0 \rightarrow^{*}_{\hat{R}_P \cup R} \text{g(s, s)}$. Obviously, every function symbol occurring in $\gamma'$ belongs to $\hat{P}$. Thus, $0 \rightarrow^{*}_{\hat{R}_P} \text{g(s, s)}$.

By Lemma 14, $0 \rightarrow^{*}_{\hat{R}_P} \text{g(s, s)}$ and $\text{h(0)} \rightarrow^{*}_{\hat{R}_P \cup R} 2$. By Lemma 17, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow^{*}_{\hat{R}_P} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$, where $\text{Min}(O_F(r)) = \{o_1, \ldots, o_i\}$, $\text{Min}(O_F(s)) = \{p_1, \ldots, p_m\}$, and $\text{Min}(O_F(t)) = \{q_1, \ldots, q_n\}$. Since $\hat{R}_P \cup R$ is confluent by Lemma 12, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow^{*}_{\hat{R}_P} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$. By $0 \rightarrow^{*}_{\hat{R}_P} \text{g(s, s)}$ and Lemma 16(2), $s \rightarrow^{*}_{\hat{R}_P \cup R} s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}}$ and $t \rightarrow^{*}_{\hat{R}_P \cup R} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$. Thus, $s \rightarrow^{*}_{\hat{R}_P \cup R} t$. $\square$

By Lemmata 2 and 18, the following theorem holds.

Theorem 19 Confluence of monadic and semi-constructor TRSs is undecidable.

6 Confluence of flat TRSs

In [2], the undecidability of confluence of flat TRSs has been claimed, but we found that the proof is incorrect. In this section, we explain its flaw.

Definition 20 [2] A rule $\alpha \rightarrow \beta$ is flat if height($\alpha$) $\leq 1$ and height($\beta$) $\leq 1$.

In [2], first the undecidability of reachability has been obtained by showing that $0 \rightarrow^{*}_{R_r} 1$ iff there
exists a solution for PCP for the following TRS \( R_1 \).

\[ R_1 = R_0 \cup \{ 0 \to f(q_1^{(2)}, q_1^{(4)}, q_3^{(13)}, q_3^{(14)}, q_3^{(14)}, q_3^{(15)}) \} \]

Here, \( R_0 \) has many rules, so omitted (see [2], p.267).

Next, the undecidability of confluence has been obtained by showing the claim that \( R_1 \cup R_0 \) is confluent iff \( 0 \to \overline{1} \) for the following TRS \( R_2 \).

\[ R_2 = \{ 2 \to 0, 2 \to 1 \} \cup \{ c \to 0 \mid c \in \Xi_0 \setminus \{0,1\} \} \]

\[ \cup \{ d(x) \to 0, d(1) \to 1 \mid d \in \Xi_1 \} \]

\[ \cup \{ f(z_1, \cdots, z_8) \to 1, g(z_1, \cdots, z_8) \to 1 \mid \]

one of the \( z_i \) is 1,

the others are distinct variables\}

Here, \( \Xi = \Xi_0 \cup \Xi_1 \cup \{f, g\} \), which is a set of function symbols occurring in \( R_1 \). \( \Xi_0, \Xi_1 \) have many symbols, so omitted (see [2], p.267). Note that \( \Xi_0 \) has \( q_1^{(3)}, q_1^{(4)}, q_1^{(5)}, q_1^{(6)}, q_1^{(13)}, q_1^{(14)}, q_1^{(15)}, q_1^{(16)} \).

However, the proof of the only-if part of the claim is incorrect. The proof claims that if \( 0 \to \overline{1} \) does not hold then \( R_1 \cup R_2 \) is not confluent because of the peak \( 0 \leftarrow R_3 1 \) \( 2 \to R_3 1 \). But, the claim overlooks that \( 0 \to R_3 \)

\[ f(q_1^{(3)}, q_1^{(4)}, q_1^{(5)}, q_1^{(6)}, q_1^{(13)}, q_1^{(14)}, q_1^{(15)}, q_1^{(16)}) \to R_3 \]

\[ f(0,0,0,0,0,0,0,0) \to R_3 g(0,0,0,0,0,0,0,0) \to R_3 1 \]  

Thus, the undecidability of confluence of flat TRSs has not been shown. Now, Jacquemard claims that the proof can be corrected.

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**References**


