The Reachability and Related Decision Problems for Semi-Constructor TRSs
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Abstract
This paper shows that reachability is undecidable for confluent monadic and semi-constructor TRSs, and joinability and confluence are undecidable for monadic and semi-constructor TRSs. Here, a TRS is monadic if the height of the right-hand side of each rewrite rule is at most 1, and semi-constructor if all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms.

1 Introduction
In this paper, we consider the reachability problem for confluent monadic and semi-constructor TRSs posed by our previous paper [4]. Here, a TRS is monadic if the height of the right-hand side of each rewrite rule is at most 1, and semi-constructor if all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms. We give a negative answer to this problem. This undecidability result is compared with the decidability results of joinability and unification for the same class [4, 3].

Moreover, we show that joinability and confluence are undecidable for monadic and semi-constructor TRSs.

2 Preliminaries
We assume that the reader is familiar with standard definitions of rewrite systems [1] and we just recall here the main notations used in this paper.
Let \( F \) be a finite set of operation symbols graded by an arity function \( \alpha: F \to N(= \{0, 1, 2, \ldots\}) \), \( F_n = \{ f \in F \mid \alpha(f) = n \} \). We use \( x, y \) as variables, \( f \) as an operation symbol, \( r, s, t \) as terms. Let \( V(s) \) be the set of variables occurring in \( s \). The height of a term is defined as follows: \( \text{height}(a) = 0 \) if \( a \) is a variable or a constant and \( \text{height}(f(t_1, \ldots, t_n)) = 1 + \max\{ \text{height}(t_1), \ldots, \text{height}(t_n) \} \) if \( n > 0 \). The root symbol of a term is defined as \( \text{root}(a) = a \) if \( a \) is a variable and \( \text{root}(f(t_1, \ldots, t_n)) = f \).

A position in a term is expressed by a sequence of positive integers, and positions are partially ordered by the prefix ordering \( \leq \). Let \( O(s) \) be the set of positions of \( s \). For a set of positions \( W \), let \( \text{Min}(W) \) be the set of its minimal positions (w.r.t. \( \leq \)).

Let \( s_p \) be the subterm of \( s \) at position \( p \). For a sequence \( (p_1, \ldots, p_n) \) of pairwise parallel positions and terms \( t_1, \ldots, t_n \), we use \( s[t_1, \ldots, t_n]_{(p_1, \ldots, p_n)} \) to denote the term obtained from \( s \) by replacing each subterm \( s_{p_i} \) by \( t_i (1 \leq i \leq n) \). For a set of function symbols \( F \), let \( O_F(s) = \{ p \in O(s) \mid \text{root}(s_p) \in F \} \). For a string of unary function symbols \( u = a_1a_2\cdots a_k \) and a term \( t \), let \( u(t) \) be an abbreviation for \( a_1(a_2(\cdots a_k(t))) \).

A rewrite rule \( \alpha \rightarrow \beta \) is a directed equation over terms. A TRS \( R \) is a set of rewrite rules. Let \( \leftarrow \) be the inverse of \( \rightarrow \), \( \leftrightarrow = \rightarrow \cup \leftarrow \), and \( \downarrow = \rightarrow^+ \leftarrow^* \). A term \( t \) is reachable from \( s \) if \( s \rightarrow^* t \). \( R \) is confluent on TRS \( R \) if for every \( s \rightarrow^* t \rightarrow^* h, t \downarrow h \). A TRS \( R \) is confluent if every \( R \) is confluent on \( R \). For a rewrite sequence \( \gamma: s_1 \leftarrow^{k_1} s_2 \leftarrow^{k_2} \cdots \leftarrow^{k_{p-1}} s_n \), the number of steps of \( \gamma \) is called p-invariant if \( q > p \) for any redex position \( q \) of \( \gamma \), and we write \( \gamma: s_1 \leftarrow^{p-\text{inv}} s_n \).

The set \( D_R \) of defined symbols for a TRS \( R \) is defined as \( D_R = \{ \text{root}(a) \mid \alpha \rightarrow \beta \in R \} \). A term \( s \) is semi-constructor if for every subterm \( t \) of \( s \), \( t \) has no variable or \( \text{root}(t) \) is not a defined symbol.

Definition 1 A rule \( \alpha \rightarrow \beta \) is monadic if \( \text{height}(\beta) \leq 1 \), semi-constructor if \( \beta \) is semi-constructor. A TRS \( R \) is monadic if every rule in \( R \) is monadic, semi-constructor if every rule in \( R \) is semi-constructor.
3 Undecidability of joinability for monadic and semi-constructor TRSs

We have shown that joinability is undecidable for linear semi-constructor TRSs [4]. In this section, we show that joinability for monadic and semi-constructor TRSs is undecidable by a reduction from the Post's Correspondence Problem (PCP). Let $P = \{(u_i, v_i) \in \Sigma^* \times \Sigma^* | 1 \leq i \leq n\}$ be an instance of the PCP. The corresponding TRS $R_P$ is constructed as follows. Let $F = F_0 \cup F_1 \cup F_2$ where $F_0 = \{0, c, d, \} , F_1 = \{e_i | 1 \leq i \leq n\}$ and $F_2 = \{f, g\}$.

$$R_P = \{0 \rightarrow e_i(0) | 1 \leq i \leq n\} \cup \{0 \rightarrow f(c, d)\}
\cup \{b \rightarrow a(b), b \rightarrow a(\}) | b \in \{c, d\}, a \in \Sigma\}
\cup \{f(x, x) \rightarrow g(x, x)\}
\cup \{e_i(g(u_i(x), v_i(y)) \rightarrow g(x, y) | 1 \leq i \leq n\}$$

$R_P$ is monadic. Here, $D_{R_P} = \{0, c, d, f\} \cup E$, so $R_P$ is semi-constructor.

**Lemma 2** $0 \not\rightarrow R_P g(\$_{1} \cdots \$_{m})$ iff PCP $P$ has a solution.

**Proof.** $0 \rightarrow R_P g(\$ \$_{1} \cdots \$_{m})$ iff there exists $i_1, \ldots , i_m \in \{1, \ldots , n\}$ such that $0 \rightarrow R_P e_{i_m} \cdots e_{i_1} f(c, d) \rightarrow e_{i_m} \cdots e_{i_1} (f(u_{i_1} \cdots u_{i_m}(\$), u_{i_1} \cdots u_{i_m}(\$))) \rightarrow e_{i_m} \cdots e_{i_1} (g(u_{i_1} \cdots u_{i_m}(\$), u_{i_1} \cdots u_{i_m}(\$))) \rightarrow R_P g(\$ \$_{1} \cdots \$_{m}) \rightarrow R_P g(\$ \$_{1} \cdots \$_{m})$.

Since $g(\$ \$_{1} \cdots \$_{m})$ is a normal form, the following theorem holds.

**Theorem 3** Both joinability and reachability for monadic and semi-constructor TRSs are undecidable.

4 Undecidability of reachability for confluent monadic and semi-constructor TRSs

We give a stronger result for reachability, that is, reachability for confluent monadic and semi-constructor TRSs is undecidable. Note that joinability is decidable for the same class [4, 3]. Let $F = F \cup \{1\}$.

$$\hat{R}_P = R_P \cup \{\$ \rightarrow 1\} \cup \{a(1) \rightarrow 1 | a \in \Sigma\}
\cup \{e_i(g(1, u_i(y))) \rightarrow g(1, y),
\therefore e_i(g(1, u_i(y))) \rightarrow g(1, y)\}
\cup \{e_i(g(1, 1)) \rightarrow g(x, 1),
\therefore e_i(g(1, 1)) \rightarrow g(x, 1)\}
\cup \{\$ \rightarrow 1\} \cup \{1 \rightarrow \$\}$$

$\hat{R}_P$ is monadic. Here, $D_{\hat{R}_P} = D_{R_P} \cup \{\$\} \cup \Sigma$, so $\hat{R}_P$ is semi-constructor. First, we show the confluence of $\hat{R}_P$.

4.1 Confluence of $\hat{R}_P$

To show the confluence of $\hat{R}_P$, we need some definitions and lemmata.

**Definition 4** The set of $\Sigma$-strings is defined as follows.

- $1, c,d$ and $\$ are $\Sigma$-strings.
- $a(1)$ is a $\Sigma$-string if $t$ is a $\Sigma$-string and $a \in \Sigma$.

**Lemma 5** For any $\Sigma$-string $s$, the following properties hold.

1. For any $\gamma : s \leftrightarrow t$, $t$ is a $\Sigma$-string.
2. $s \rightarrow 1$.

**Proof.** By induction on $|\gamma|$.

- By induction on the structure of $s$.

**Corollary 6** Every $\Sigma$-string is confluent.

**Lemma 7** Let $\gamma : u(s) \rightarrow t$ where $u \in \Sigma^*$. Then, if root($s$) $\notin \{1, c, d, \}$ and $u(s), u(s)_{p} = s$ then $\gamma$ is $p$-invariant.

**Proof.** By induction on $|\gamma|$.

**Definition 8** The set of $E$-strings is defined as follows.

- $0, f(t_1, t_2)$ and $g(t_1, t_2)$ are $E$-strings if $t_1, t_2$ are $\Sigma$-strings.
- $e_i(t)$ is an $E$-string if $t$ is an $E$-string and $i \in \{1, 2, \ldots , n\}$.

**Lemma 9** For any $E$-string $s$, the following properties hold.

1. For any $\gamma : s \leftrightarrow t$, $t$ is an $E$-string.
2. $s \rightarrow 1$.$(1, 1)$.\)

**Proof.**

(1) By induction on $|\gamma|$.

(2) By induction on the structure of $s$. Basis : For any $E$-strings $s_1, s_2$, $f(s_1, s_2) \rightarrow f(1, 1) \rightarrow g(1, 1)$ and $g(s_1, s_2) \rightarrow g(1, 1)$ by Lemma 5(2), and $0 \rightarrow f(c, d) \rightarrow g(1, 1)$. Thus, $s \rightarrow 1$.

Induction step : Let $s = e_i(s')$ for some $i \in \{1, 2, \ldots , n\}$. By the induction hypothesis, $s' \rightarrow 1$. Thus, $e_i(s') \rightarrow 1$.\)

**Corollary 10** Every $E$-string is confluent.
The following lemma is used as a component of the proof of Lemma 12.

**Lemma 11** For any $i \in \{1, \cdots, n\}$ and terms $r_1, r_2$, the following properties hold.

(1) If $s \not\sim e_i(g(r_1, r_2)) \rightarrow^* t$ then there exist terms $t_1, t_2$ such that $t \rightarrow^* g(t_1, t_2)$.

(2) If $g(s_1, s_2) \not\sim e_i(g(r_1, r_2)) \rightarrow^* g(t_1, t_2)$ and $g(r_1, r_2)$ is confluent then $g(s_1, s_2) \downarrow g(t_1, t_2)$.

**Proof.**

(1) Let $t = e_i(g(t_1', t_2'))$. If $r_1$ is a $\Sigma$-string then $t_1 \rightarrow^{*} 1$ by Lemma 5. Otherwise, $r_1 \not\in 1$. Thus, $r_1 = u_i(r_1')$ for some term $r_1'$ by $e_i(g(r_1, r_2)) \rightarrow s$. By Lemma 7, $t_1' = u_i(t_1')$, where $r_1' \rightarrow^{*} t_1'$. Similarly, $t_2' \rightarrow^{*} 1$ or $t_2' = u_i(t_2')$ for some term $t_2'$. Thus, $t \rightarrow^{*} g(t_1, t_2)$, where $t_1 \in \{t_1', t_2\}$ and $t_2 \in \{t_1', t_2\}$.

(2) By the definition of $R_P$, $e_i(g(r_1, r_2)) \rightarrow^{\epsilon-inv}
\begin{align*}
e_i(g(s_1', s_2')) &\rightarrow g(s_1', s_2') \rightarrow^{\epsilon-inv} g(s_1, s_2) \\
e_i(g(s_1', s_2')) &\rightarrow g(t_1', t_2') \rightarrow^{\epsilon-inv} g(t_1, t_2)
\end{align*}

Thus, $s_1' \rightarrow^{*} r_1 \rightarrow t_1'$. If $s_1' \rightarrow^{*} t_1'$, then $s_1' = t_1'$.

Case of $s_1' = t_1' = 1$: Obviously, $s_1'' = s_1 = t_1'' = t_1 = 1$.

Case of $s_1' = 1$ and $t_1' = u_i(t_1')$: Obviously, $s_1'' = s_1 = 1$. By Lemma 5, $t_1 \rightarrow^{*} 1$.

Case of $s_1' = u_i(s_1')$ and $t_1' = u_i(t_1')$: Similar to the previous one.

Thus, $s_1' \rightarrow^{*} 1$.

Case of $s_1' = u_i(s_1')$ and $t_1' = u_i(t_1')$: By confluence of $g(r_1, r_2)$, $r_1$ is confluent. Thus, $u_i(s_1) \downarrow u_i(t_1)$. If $s_1$ is a $\Sigma$-string then $s_1 \rightarrow^{*} 1$ by Corollary 6. Otherwise, $s_1 \rightarrow^{*} 1$ by Lemma 7.

Similarly, $s_2 \rightarrow^{*} 1$. Thus, $g(s_1, s_2) \rightarrow g(t_1, t_2)$. □

**Lemma 12** $R_P$ is confluent.

**Proof.** We show that for any $\gamma: s \not\sim r \rightarrow^* t$, $s \downarrow t$ by induction on $\text{height}(r)$.

**Basis:** If $r \in \{c, d\}$ then $s \downarrow t$ by Corollary 6, else if $r = 0$ then $s \downarrow t$ by Corollary 10. Otherwise, $s = r$ since $r$ is a normal form.

**Induction step:** If $\gamma$ is $\epsilon$-invariant then $s \downarrow t$ by the induction hypothesis. So, we consider that $\gamma$ has an $\epsilon$-reduction. Let $\gamma_1: r \rightarrow^{*} s$ and $\gamma_2: r \rightarrow^{*} t$.

Without loss of generality, we assume that $\gamma_1$ has an $\epsilon$-reduction and $\text{root}(r) \in \Sigma \cup \{f\} \cup E$.

Case of root($r$) $\in \Sigma$: $\gamma: r = a(r_1) \rightarrow^{*} a(1) \rightarrow 1 = s$ holds for some $a \in \Sigma$ and $r_1$. By Lemma 5, $t \rightarrow^{*} 1$.

Case of root($r$) $\in \Sigma$: $\gamma: r = a(r_1) \rightarrow^{*} a(1) \rightarrow 1 = s$ holds for some $a \in \Sigma$ and $r_1$. By Lemma 5, $t \rightarrow^{*} 1$.

4.2 Reachability for confluent monadic and semi-constructor TRSs

**Lemma 13** For any $\gamma: s \rightarrow^{*} r \rightarrow^{*} t$, if $s$ has 1 as its subterm then so does $t$.

**Proof.** Since for any $\alpha \rightarrow \beta \in R_P$, $\forall(\alpha) = \forall(\beta)$ and if $\alpha$ has 1 as its subterm then so does $\beta$. □

**Lemma 14** $0 \rightarrow^{*} r_{P}, g(s, s) \not\rightarrow^{*} g(s, s)$.

**Proof.** Only if part: Let $\gamma: 0 \rightarrow^{*} g(s, s)$. We assume to the contrary that $\gamma$ must have $R_P \setminus R_P$ reduction, i.e., $\gamma: 0 \rightarrow^{*} r_{P}, s \rightarrow^{*} R_P \setminus R_P, t \rightarrow^{*} R_P g(s, s)$ for some $s, t$. By the definition of $R_P$, $t$ has 1 as its subterm. By Lemma 13, $g(s, s)$ has 1 as its subterm, a contradiction. If part: $R_P \subseteq R_P$. □

By Lemmata 2, 12, and 14, the following theorem holds.

**Theorem 15** Reachability for confluent monadic and semi-constructor TRSs is undecidable.
5 Undecidability of confluence of monadic and semi-constructor TRSs

We show that confluence of monadic and semi-constructor TRSs is undecidable.

Let $F' = F_0' \cup F_1'$ where $F_0' = \{2\}$, $F_1' = \{h\}$.

$$R = \{ h(x) \rightarrow h(0), h(g(s, s)) \rightarrow 2 \}$$

$R \cup R$ is monadic. Here, $D_R = \{h\}$, so $R \cup R$ is semi-constructor.

Lemma 16 For any $s$ with root($s$) $\in F'$, the following properties hold.

1. If $s \rightarrow^{*}_{R \cup R} t$ then root($t$) $\in F'$.
2. If $0 \rightarrow^{*}_{R} g(s, s)$ then $s \rightarrow^{*}_{R \cup R} 2$.

The proof is straightforward, so omitted.

Lemma 17 Let $s \rightarrow^{*}_{R \cup R} t$, Min($O_F$(s)) = \{p_1, \ldots, p_m\}, and Min($O_F$(t)) = \{q_1, \ldots, q_n\}. Then, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow_{R} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$ or $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} = t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$.

Proof. Let $s \rightarrow^{*}_{R \cup R} t$. If there exists $i \in \{1, \ldots, m\}$ such that $p_i \leq p$ then $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} = t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$. Otherwise, obviously $s \rightarrow_{R \cup R} t$. Since every function symbol in $F'$ does not occur in $R \cup R$, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow^{*}_{R \cup R} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$.

Lemma 18 $R \cup R$ is confluent iff $0 \rightarrow^{*}_{R \cup R} g(s, s)$.

Proof. Only if part: By $h(0) \rightarrow^{*}_{R \cup R} h(g(s, s)) \rightarrow^{*}_{R \cup R} 2$, confluence ensures that $h(0) \rightarrow^{*}_{R \cup R} 2$. Since $2$ is a normal form, $h(0) \rightarrow^{*}_{R \cup R} 2$. Thus, there exists a shortest sequence $\gamma$ that satisfies $h(0) \rightarrow^{*}_{R \cup R} h(g(s, s)) \rightarrow^{*}_{R \cup R} 2$. Since $\gamma$ is shortest, $h(0) \rightarrow^{*}_{R \cup R} h(g(s, s))$. Thus, there exists $\gamma' : 0 \rightarrow^{*}_{R \cup R} g(s, s)$.

Obviously, every function symbol occurring in $\gamma'$ belongs to $R$. Thus, $0 \rightarrow^{*}_{R \cup R} g(s, s)$. By Lemma 14, $0 \rightarrow^{*}_{R \cup R} g(s, s)$.

If part: Let $s \rightarrow^{*}_{R \cup R} r \rightarrow^{*}_{R \cup R} t$. By Lemma 17, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow^{*}_{R \cup R} r[2, \ldots, 2]_{\{a_1, \ldots, a_l\}} \rightarrow^{*}_{R \cup R} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$, where Min($O_F$(r)) = \{a_1, \ldots, a_l\}, Min($O_F$(s)) = \{p_1, \ldots, p_m\}, and Min($O_F$(t)) = \{q_1, \ldots, q_n\}. Since $R \cup R$ is confluent by Lemma 12, $s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}} \rightarrow^{*}_{R \cup R} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$. By $0 \rightarrow^{*}_{R \cup R} g(s, s)$ and Lemma 16(2), $s \rightarrow^{*}_{R \cup R} s[2, \ldots, 2]_{\{p_1, \ldots, p_m\}}$ and $t \rightarrow^{*}_{R \cup R} t[2, \ldots, 2]_{\{q_1, \ldots, q_n\}}$. Thus, $s \rightarrow^{*}_{R \cup R} t$.

By Lemmata 2 and 18, the following theorem holds.

Theorem 19 Confluence of monadic and semi-constructor TRSs is undecidable.

6 Confluence of flat TRSs

In [2], the undecidability of confluence of flat TRSs has been claimed, but we found that the proof is incorrect. In this section, we explain its flaw.

Definition 20 [2] A rule $\alpha \rightarrow \beta$ is flat if height($\alpha$) $\leq 1$ and height($\beta$) $\leq 1$.

In [2], first the undecidability of reachability has been obtained by showing that $0 \rightarrow^{*}_{R} 1$ iff there
exists a solution for PCP for the following TRS $R_1$.

$$R_1 = R_0 \cup$$

\[ \{ 0 \rightarrow f(q_A^3, q_A^4, q_B^5, q_B^6, q_A^9, q_B^9, q_B^{10}), \]

\[ f(x_1, x_2, x_1, y_{11}, y_{11}, y_{12}, y_1, y_{12}) \rightarrow \]

\[ g(x_1, x_2, x_1, y_{11}, y_{11}, y_{12}, y_1, y_{12}), \]

\[ g(x_0, x_0, y_{17}, y_{17}, y_{18}, y_{18}, y_{10}, y_{10}) \rightarrow 1 \} \]

Here, $R_0$ has many rules, so omitted (see [2], p.267).

Next, the undecidability of confluence has been obtained by showing the claim that $R_1 \cup R_0$ is confluent iff $0 \rightarrow^{*} R_1 1$ for the following TRS $R_2$.

$$R_2 = \{ 2 \rightarrow 0, 2 \rightarrow 1 \} \cup \{ c \rightarrow 0 \mid c \in \Xi_0 \setminus \{ 0, 1 \} \}$$

\[ \cup \{ d(x) \rightarrow 0, d(1) \rightarrow 1 \mid d \in \Xi_0 \} \]

\[ \cup \{ f(z_1, \cdots, z_8) \rightarrow 1, g(z_1, \cdots, z_8) \rightarrow 1 \mid \]

one of the $z_i$ is $1$, \]

the others are distinct variables\}

Here, $\Xi = \Xi_0 \cup \Xi_1 \cup \{ f, g \}$, which is a set of function symbols occurring in $R_2$. $\Xi_0, \Xi_1$ have many symbols, so omitted (see [2], p.267). Note that $\Xi_0$ has

$q_A^3, q_A^4, q_B^5, q_B^6, q_A^9, q_B^9, q_B^{10}.$

However, the proof of the only-if part of the claim is incorrect. The proof claims that if $0 \rightarrow^{*} R_1 1$ does not hold then $R_1 \cup R_2$ is not confluent because of the peak $0 \leftarrow R_2 2 \rightarrow R_2 1$. But, the claim overlooks that $0 \rightarrow^{*} R_1 f(q_A^3, q_A^4, q_B^5, q_B^6, q_A^9, q_B^9, q_B^{10}) \rightarrow^{*} R_2 f(0,0,0,0,0,0,0) \rightarrow R_1 g(0,0,0,0,0,0,0) \rightarrow R_1 1$. Thus, the undecidability of confluence of flat TRS has not been shown. Now, Jacquemard claims that the proof can be corrected.

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References


