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Kyoto University
An Improved Recursive Decomposition Ordering for Term Rewriting Systems Revisited

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Abstract

Simplification orderings, like the recursive path ordering and the improved recursive decomposition ordering, are widely used for proving the termination property of term rewriting systems. The improved recursive decomposition ordering is known as the most powerful simplification ordering.

In this paper, we investigate the improved recursive decomposition ordering for proving termination of term rewriting systems. We completely show that the improved recursive decomposition ordering is closed under substitutions.

Keywords: Term rewriting system, Termination, Improved recursive decomposition ordering, Simplification ordering

1 Introduction

Term rewriting systems (TRSs, for short) are regarded as a computation model that reduces terms by applying directed equations, called rewrite rules. TRSs are widely used as a model of functional and logic programming languages and as a basis of automated theorem proving, symbolic computation, algebraic specification and verification [1, 15, 23].

The terminating property is fundamental notion of TRSs as computation models [4]. Since the terminating property of TRS is undecidable in general [5], several sufficient conditions for proving this property have been successfully developed in particular cases. These techniques can be classified into two approaches: semantic methods and syntactic methods.

Simplification orderings are representatives of syntactic methods [18, 21]. Many simplification orderings (for instance, the recursive path ordering (with status) (RPO(S), for short) [2, 10], the recursive decomposition ordering (with status) (RDO(S), for short) [8, 13, 14], the improved recursive decomposition ordering (with status) (IRD(S), for short) [17, 19] and so on) have been defined on TRSs. IRDS is among the most powerful simplification orderings [19, 20].

First, Jouannaud, Lescanne and Reining defined the recursive decomposition ordering with multiset status [8]. They said that the closure under substitutions of it is straightforward using definition of decomposition. However they did not give the formal proof of it.

The recursive decomposition ordering with arbitrary status (RDOS) was first described by Lescanne [13]. Complete proofs concerning the lexicographical status are given by Lescanne [14]. An implementation of recursive decomposition ordering with multiset status has been made in the first rewriting environment with tools for proving termination called REVE as it was a convenient tools for proposing extension of the precedence [12].

Rusinowitch [17] gave the definition of the improved recursive decomposition ordering (IRD) and investigated the relationship between several simplification orderings: the path of subterm ordering (PSO) [16], the recursive path ordering (RPO) and the recursive decomposition ordering (RDO). But they did not discuss that IRD is closed under substitutions.

Steinbach [19] gave the definition of the improved recursive decomposition ordering with status (IRDS) based on IRD defined by Rusinowitch [17] and compared of the power as well as the time behavior of all orderings suggested [18, 20, 22]. They showed that IRDS is a simplification ordering and IRDS is closed under substitutions [18, 19], however their proof was not complete. They used the proposition that for any substitution $\theta$, $dec_p(s) \gg_{EL} dec_q(t)$ implies $dec_p(s\theta) \gg_{EL} dec_q(t\theta)$ as key idea in their
proof without proving. But this proposition was not trivial. So, we need give the formal proof of it by
induction on \(|s| + |t|\) in this paper.

We proposed IRDS for higher-order rewrite systems, called the higher-order improved recursive
decomposition ordering (HIRDS, for short) [6, 7]. Our method was inspired by Jouannaud and Rubio's
idea for RPOS [9] and particular properties of IRDS. We showed that our ordering is a more powerful
ordering than their ordering. Furthermore we showed that HIRDS is closed under substitutions. However
our proof was very complicated and generalized, so we try to show that IRDS is closed under substitutions
in this paper. Furthermore we review that IRDS is a simplification ordering.

In section 2 we give the basic notations. Section 3 presents the definition of the improved recursive
decomposition ordering with status (IRDS) and we completely show that IRDS is closed under substitutions.
Also, we review that the IRDS is a simplification ordering.

2 Preliminaries

We mainly follow the basic notations of [11, 19]. An abstract reduction system (ARS for short) is a pair
\((A, \rightarrow)\) consisting of a set \(A\) and a binary relation \(\rightarrow \subseteq A \times A\). We say that ARS \((A, \rightarrow)\) is terminating
if there is no infinite sequence \(a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \ldots\) of elements in \(A\). A binary relation on a set \(A\) is
called a (strict) partial ordering over \(A\) if it is an irreflexive and transitive on \(A\). The partial ordering is
usually denoted by \(>\). A partial ordering \(>\) on a set \(A\) is well-founded if \(>\) has no infinite descending
sequences, i.e., there is no sequence of the form \(a_0 > a_1 > a_2 > \ldots\) of elements in \(A\).

A signature is a set of function symbols. Associated with \(f \in \mathcal{F}\) is a natural number denoting its
arity. Function symbols of arity 0 are called constants. Let \(\mathcal{T}(\mathcal{F}, \mathcal{V})\) be the set of all terms built from
\(\mathcal{F}\) and a countably infinite set \(\mathcal{V}\) of variables, disjoint from \(\mathcal{F}\). The set of variables occurring in a term \(t\)
is denoted by \(\mathcal{V}(t)\). The root symbol of a term \(t\) is defined as follows: \(\text{root}(t) = t\) if \(t\) is a variable and
\(\text{root}(t) = f\) if \(t = f(t_1, \ldots, t_n)\).

A substitution is a map \(\theta\) from \(\mathcal{V}\) to \(\mathcal{T}(\mathcal{F}, \mathcal{V})\) with the property that the set \(\{x \in \mathcal{V} \mid \theta(x) \neq x\}\)
is finite. If \(\theta\) is a substitution and \(t\) a term then \(\theta(t)\) denotes the result of applying \(\theta\) to \(t\). We call \(\theta(t)\)
an instance of \(t\).

We introduce a fresh constant symbol \(\square\), named hole. A context \(C[]\) is a term in \(\mathcal{T}(\mathcal{F} \cup \{\square\}, \mathcal{V})\)
containing precisely one hole. If \(C[]\) is a context and \(t\) a term then \(C[t]\) denotes the result of replacing
the hole in \(C[]\) by \(t\).

A binary relation \(R\) on terms is closed under substitutions if \(s \mathcal{R} t\) implies \(\theta(s) \mathcal{R} \theta(t)\), for any substitution
\(\theta\). And a binary relation \(R\) on terms is closed under contexts if \(s \mathcal{R} t\) implies \(C[s] \mathcal{R} C[t]\), for any context
\(C[]\). \(|t|\) denotes the size of term \(t\), i.e., the total number of function symbols and variables occurring in \(t\).
Terms are identified with finite labeled trees. A position in a term can be viewed as a finite sequence of
natural numbers, pointing out a path from the root of this tree. \(P(t)\) denotes the set of all positions of a term \(t\).
\(P_T(t)\) denotes the set of all terminal positions (positions of all leaves) of the term \(t\). The letter \(v\) denotes
root positions. We write \(w \leq z\) if \(w\) is a prefix of \(z\). The subterms of \(t\) at position \(p\) is denoted by \(t_p\), and we write \(t \geq t_p\).
If \(t \geq t_p\) and \(t \neq t_p\) then \(t_p\) is called the proper subterm of \(t\), denoted by \(t \triangleright t_p\).

A rewrite rule on \(\mathcal{T}(\mathcal{F}, \mathcal{V})\) is a pair of terms \(l \rightarrow r\) such that \(l \notin \mathcal{V}\) and \(\mathcal{V}(r) \subseteq \mathcal{V}(l)\). A term rewriting
system (TRS, for short) \((\mathcal{F}, \mathcal{R})\) where \(\mathcal{F}\) is a set of function symbols and \(\mathcal{R}\) is a set of rewrite
rules on \(\mathcal{T}(\mathcal{F}, \mathcal{V})\). \((\mathcal{F}, \mathcal{R})\) is often abbreviated as \(\mathcal{R}\) and in that case \(\mathcal{F}\) is defined to be the set of function
symbols that appear in \(\mathcal{R}\). We often present a TRS as a set of rewrite rules, without making explicit
its signature, assuming that the signature consists of the function symbols occurring in the rewrite rules.
The smallest rewrite relation on \(\mathcal{T}(\mathcal{F}, \mathcal{V})\) that contains \(\mathcal{R}\) is denoted by \(\rightarrow\). So \(s \rightarrow r\) if there exists
a rewrite rule \(l \rightarrow r\) in \(\mathcal{R}\), a substitution \(\theta\), and a context \(C\) such that \(s = C[\theta]\) and \(t = C[\theta(t)]\). The
subterm \(\theta(t)\) of \(s\) is called a redex and we say that \(s\) rewrites to \(t\) by contracting redex \(\theta\). We call \(s \rightarrow r\)
a rewrite or reduction step.

Given a binary relation \(\triangleright\), the multiset extension \(\triangleright\) is defined as the transitive closure of the following
relation \(\triangleright\) on multiset. \(M \cup \{s\} \triangleright M \cup \{t_1, \ldots, t_n\}\) where \(n \geq 0\) and \(s \triangleright t_i\) for any \(i \in \{1, \ldots, n\}\).
Assume \(\triangleright\) is a well-founded ordering on a set \(A\). Then \(\triangleright\) is a well-founded ordering on the multisets
of elements in \(A\). We say that a binary relation \(R\) on terms has the subterm property if \(C[t]Rt\) for any
context \(C[]\) \(\neq \square\) and term \(t\).
3 Improved Recursive Decomposition Ordering Revisited

Throughout this section we are dealing with finite signatures only.

Definition 3.1 ([2, 4, 11]) A simplification ordering on \( T(F, V) \) is a partial ordering that is closed under substitutions, contexts and has the subterm property.

Since we are dealing with finite signatures only, we obtain the following result.

Theorem 3.2 ([2, 4, 11]) Simplification orderings are well-founded.

We obtain the following theorem from the result of Dershowitz [2].

Theorem 3.3 Let \( R \) be a TRS and let \( > \) be a simplification ordering on \( T(F, V) \). If \( l > r \) for any \( l \rightarrow r \in R \) then \( R \) is terminating.

Proof. Assume that \( s \rightarrow_{R} t \), where \( s \) and \( t \) are terms. There exists a rewrite rule \( l \rightarrow r \) in \( R \), a substitution \( \theta \) and a context \( C[\] \) such that \( s = C[\theta] \) and \( t = C[\theta] \). By the assumption \( l > r \) and definition 3.1, hence \( s = C[\theta] > C[\theta] = t \) holds. Since \( > \) is well-founded on \( T(F, V) \) by theorem 3.2, \( R \) is terminating. \( \square \)

The improved recursive decomposition ordering depends on a partial ordering \( >_{F} \) on the signature \( F \), the so-called precedence.

A status function \( \tau \) is assumed, mapping every \( f \in F \) to either mul or lex, for some permutation \( \pi \) on \( n \) elements, where \( n \) is arity of \( f \). For a partial ordering \( > \) on \( T(F, V) \) the partial ordering \( >_{\tau(f)} \) is defined on sequences of length \( n \) : \( \tau(f) = \text{mul} \) describes multiset extension and \( \tau(f) = \text{lex} \) describes lexicographic comparison according to the permutation \( \pi \). The results of an application of the function \( \text{args} \) to a term \( t = f(t_{1}, \ldots t_{n}) \) depend on the status of \( f \) : If \( \tau(f) = \text{mul} \), then \( \text{args}(t) \) is the multiset \( \{t_{1}, \ldots t_{n}\} \) and otherwise, \( \text{args}(t) \) is the tuple \( (t_{1}, \ldots t_{n}) \).

For \( u \in P_{T}(t) \), a path-decomposition \( \text{dec}_{u}(t) = \{t \mid v \in \text{P}_{T}(u) \} \) is a set of subterms of \( t \). Note that \( i.v \in P_{T}(f(t_{1}, \ldots t_{n})) \) implies \( v \in P_{T}(t) \). We also define a decomposition \( \text{dec} \{t_{1}, \ldots t_{n}\} = \{\text{dec}_{u}(t_{i}) \mid i \in \{1, \ldots n\}, u \in P_{T}(t_{i}) \} \). A decomposition is a multiset of all path-decomposition of the terms \( t_{1}, \ldots, t_{n} \). For the path-decomposition \( \text{dec}_{u}(t) \), a set of subterms of \( \text{dec}_{u}(t) \), \( \text{sub} \{\text{dec}_{u}(t), s\} = \{s' \in \text{dec}_{u}(t) \mid s >_{\text{EL}} s'\} \).

We give the improved recursive decomposition ordering with status (IRDS) defined by Steinbach [19, 20] as following.

Definition 3.4 (IRDS) Let \( s \) and \( t \) be terms. For a precedence \( >_{F} \) and a status \( \tau \) the improved recursive decomposition ordering with status (IRDS) on \( T(F, V) \) is defined as follows:

\[ s >_{\text{IRDS}} t \iff \text{dec} \{s\} >_{\text{EL}} \text{dec} \{t\} \] where \( >_{\text{EL}} \) is the multiset extension of \( >_{\text{EL}} \).

\[ \text{dec}_{u}(s) \ni u >_{\text{EL}} v' \in \text{dec}_{u}(v) \] is defined by the following (a), (b) and (c).

(a) root(u') \( >_{F} \) root(v'), or
(b) root(u') = root(v'), \( \tau(\text{root}(u')) = \text{mul} \) and either
\[ \cdot \text{sub} \{\text{dec}_{p}(u), u'\} >_{\text{EL}} \text{sub} \{\text{dec}_{q}(v), v'\}, \] or
\[ \cdot \text{sub} \{\text{dec}_{p}(u), u'\} = \text{sub} \{\text{dec}_{q}(v), v'\} \text{ and} \]
\[ \text{dec} \{\text{args}(u')\} >_{\text{EL}} \text{dec} \{\text{args}(v')\}, \] or
(c) root(u') = root(v'), \( \tau(\text{root}(u')) \neq \text{mul} \), \( \text{args}(u') >_{\text{IRDS}} \text{args}(v') \) and \( \{u'\} >_{\text{IRDS}} \text{args}(v') \).

Next, we give the example of comparison using IRDS.

Example 3.5 We consider the term \( s = \neg X \supset (Y \supset Z) \) and \( t = Y \supset (X \lor Z) \) where \( X, Y, Z \in V \), \( F = \{\neg, \supset, \lor\} \) and \( \tau(f) = \text{mul} \) for any \( f \in F \). We give the precedence as follows: \( \neg \supset \supset \lor V \).

We have \( P_{T}(s) = \{11, 21, 22\} \) and \( P_{T}(t) = \{1, 21, 22\} \). See figure 1.

Then \( \text{dec} \{s\} = \{\text{dec}_{11}(s), \text{dec}_{21}(s), \text{dec}_{22}(s)\} \) where \( \text{dec}_{11}(s) = \{s, \neg X, X\} \), \( \text{dec}_{21}(s) = \{s, Y \supset Z, Y\} \) and \( \text{dec}_{22}(s) = \{s, Y \supset Z, Z\} \).

Then \( \text{dec} \{t\} = \{\text{dec}_{1}(t), \text{dec}_{21}(t), \text{dec}_{22}(t)\} \) where \( \text{dec}_{1}(t) = \{t, Y\} \), \( \text{dec}_{21}(t) = \{t, X \lor Z, X\} \) and \( \text{dec}_{22}(t) = \{t, X \lor Z, Z\} \).

By the following cases (1), (2) and (3), \( \text{dec} \{s\} >_{\text{EL}} \text{dec} \{t\} \) holds. Then \( s >_{\text{IRDS}} t \) holds by definition of IRDS.
(1) $dec_{11}(s) \gg_{EL} dec_{21}(t)$ holds. $s >_{EL} t$ and $-X >_{EL} X \lor Z$ since $sub(dec_{11}(s), s) \gg_{EL} sub(dec_{21}(t), t)$ and $\neg >_{F} \lor$.

(2) $dec_{21}(s) \gg_{EL} dec_{1}(t)$ holds. $s >_{EL} t$ since $sub(dec_{21}(s), s) \gg_{EL} sub(dec_{1}(t), t)$.

(3) $dec_{22}(s) \gg_{EL} dec_{22}(t)$ holds. $s >_{EL} t$ and $Y \supset Z >_{EL} X \lor Z$ since $sub(dec_{22}(s), s) \gg_{EL} sub(dec_{22}(t), t)$ and $\supset >_{F} \lor$.

Figure 1: $\neg X \supset (Y \supset Z) >_{IRDS} Y \supset (X \lor Z)$.

We review that IRDS is a simplification ordering, i.e., IRDS is a partial ordering has the subterm property on $\mathcal{T}(F, \mathcal{V})$ that is closed under substitutions and is closed under contexts. These properties are essential for applying IRDS to termination proof of TRS.

Lemma 3.6 The IRDS is partial ordering on $\mathcal{T}(F, \mathcal{V})$.

Proof. Let $s' \in dec_p(s)$, $t' \in dec_q(t)$ and $u' \in dec_r(u)$. We can show that $dec_p(s) \ni s' >_{EL} t' \in dec_q(t)$ and $dec_q(t) \ni t' >_{EL} u' \in dec_r(u)$ imply $dec_p(s) \ni s' >_{EL} t' \in dec_q(t)$ by induction on $|s'| + |t'| + |u'|$. For any term $s$ and $s'$ in $dec_p(s)$, we can prove that $dec_p(s) \ni s' >_{EL} s' \in dec_p(s)$ by induction on $|s|$.

Lemma 3.7 The IRDS on $\mathcal{T}(F, \mathcal{V})$ has the subterm property.

Proof. Let $s$ and $t$ be terms such that $s > t$. It is shown by induction on $|s|$ that $s >_{IRDS} t$.

The following lemma is the key to prove the main result in this paper that IRDS is closed under substitutions.

Lemma 3.8 Let $dec_p(s) \gg_{EL} dec_q(t)$ where $s$ and $t$ are terms and $p \in \mathcal{P}_{\tau}(s)$ and $q \in \mathcal{P}_{\tau}(t)$. Then for any substitution $\theta$, the following two claims hold.

(1) If $s_{|p} = t_{|q} \in \mathcal{V}$ then $dec_{p,i}(s\theta) \gg_{EL} dec_{q,i}(t\theta)$, for any $i \in N^*$ such that $p.i \in \mathcal{P}_{\tau}(s\theta)$ and $q.i \in \mathcal{P}_{\tau}(t\theta)$.

(2) If $t_{|q} \notin \mathcal{V}$ then $dec_{p,i}(s\theta) \gg_{EL} dec_{q,i}(t\theta)$, for any $i \in N^*$ such that $p.i \in \mathcal{P}_{\tau}(s\theta)$.

Proof. See appendix A.

Lemma 3.9 Let $s$ and $t$ be terms. Then $dec(\{s\}) \gg_{EL} dec(\{t\})$ implies $dec(\{s\}) \cap dec(\{t\}) = \emptyset$.

The following lemma is the main result in this paper. We completely show that IRDS is closed under substitutions.

Lemma 3.10 The IRDS is closed under substitutions, i.e., $s >_{IRDS} t$ implies $s\theta >_{IRDS} t\theta$ for any substitution $\theta$. 

Proof. Assume that \( s >_{IRDS} t \), i.e., \( \text{dec}(s) \gg_{EL} \text{dec}(t) \) where \( s \) and \( t \) are terms. We show that \( \text{dec}(\theta(t)) \gg_{EL} \text{dec}(t) \), i.e., \( \theta(t) \), holds for any substitution \( \theta \). Strictly speaking, we must prove: 

\[
(\forall q \in P_T(t), \exists p \in P_T(s) \text{ such that } \text{dec}_p(s) >_{EL} \text{dec}_q(t)) \implies (\forall q' \in P_T(t), \exists p' \in P_T(s) \text{ such that } \text{dec}_p(s) >_{EL} \text{dec}_q'(t)).
\]

Let \( q' = q.i \) and \( q \in P_T(t) \). Then there exists \( p \in P_T(s) \) such that \( \text{dec}_p(s) >_{EL} \text{dec}_q(t) \). To prove that \( \exists p' \in P_T(s) \text{ such that } \text{dec}_p(s) >_{EL} \text{dec}_q'(t) \), we have to distinguish two cases:

1. \( t \mid_q \in \mathcal{V} \), i.e. \( s \mid_p = t \mid_q \) (Otherwise \( \text{dec}_p(s) >_{EL} \text{dec}_q(t) \)).

   Since \( q' = q.i \) for some \( i \in N^* \), \( \text{dec}_p(s) >_{EL} \text{dec}_q(t) \), \( s \mid_p = t \mid_q \) and lemma 3.8, \( \forall i \in P_T(t|_q) \), \( \text{dec}_p(s\theta) >_{EL} \text{dec}_q(t\theta) \). Hence, \( p' = p.i \) with \( i \in P_T(t|_q) \).

2. \( t \mid_q \notin \mathcal{V} \), i.e. \( q' = q \).

   (1) \( s \mid_p \notin \mathcal{V} \). Since \( \text{dec}_p(s) >_{EL} \text{dec}_q(t) \) and lemma 3.8, \( \text{dec}_p(s\theta) >_{EL} \text{dec}_q(t\theta) \). Hence, \( p' = p \).

   (2) \( s \mid_p \in \mathcal{V} \). Since \( \text{dec}_p(s) >_{EL} \text{dec}_q(t) \) and lemma 3.8, \( \forall i \in P_T(t|_q) \), \( \text{dec}_p(s\theta) >_{EL} \text{dec}_q(t\theta) \).

   Hence, \( p' = p.i \).

\( \square \)

Lemma 3.11 The IRDS is closed under contexts.

Proof. Let \( s \) and \( t \) be terms. We have to show that \( s >_{IRDS} t \) implies \( C[s] >_{IRDS} C[t] \) for any context \( C[] \). It can be proved by induction on context \( C[] \). \( \square \)

Lemma 3.12 The IRDS is a simplification ordering on \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \).

Proof. By lemmas 3.6, 3.7, 3.10 and 3.11, the IRDS is partial ordering on \( \mathcal{T}(\mathcal{F}, \mathcal{V}) \) that is closed under substitutions, contexts and has the subterm property. \( \square \)

Example 3.13 ([18]) Given the following signature and TRS \( \mathcal{R} : \mathcal{F} = \{ \neg, \supset, \vee \} \),

\[
R = \{ \neg X \supset (Y \supset Z) \rightarrow Y \supset (X \vee Z) \}.
\]

We give the following precedence and status: \( \neg >_{\mathcal{F}} \supset >_{\mathcal{F}} \vee \) and \( \tau(f) = \text{mul} \) for any \( f \in \mathcal{F} \). Since \( \neg X \supset (Y \supset Z) >_{IRDS} Y \supset (X \vee Z) \) by example 3.5 and theorem 3.3 and lemma 3.12, \( \mathcal{R} \) is terminating.

4 Conclusion

We have investigated the improved recursive decomposition ordering to term rewriting systems for proving termination. We completely shown have that the improved recursive decomposition ordering is closed under substitutions as main result in this paper. Also we have reviewed the improved recursive decomposition ordering is a simplification ordering.

References

A Proof of lemma 3.8

Definition A.1 Let $\theta$ be a substitution. Let $\{s_1, \ldots, s_n\}$ be a subset of $T(F, V)$. $\{s_1, \ldots, s_n\} \theta$ denotes $\{s_1 \theta, \ldots, s_n \theta\}$.

Lemma A.2 (Lemma 3.8) Let $\text{dec}_p(s) \gg_{EL} \text{dec}_q(t)$ where $s$ and $t$ be terms and $p \in \mathcal{P}_T(s)$ and $q \in \mathcal{P}_T(t)$. Then for any substitution $\theta$, the following two claims hold.

1. If $s|_p = t|_q \in V$ then $\text{dec}_p, i(s \theta) \gg_{EL} \text{dec}_q, i(t \theta)$, for any $i \in N^*$ such that $p.i \in \mathcal{P}_T(s \theta)$ and $q.i \in \mathcal{P}_T(t \theta)$.

2. If $t|_q \notin V$ then $\text{dec}_p, i(s \theta) \gg_{EL} \text{dec}_q(t \theta)$, for any $i \in N^*$ such that $p.i \in \mathcal{P}_T(s \theta)$.

Proof. We show that the claim (1) $\land$ (2) by induction on $|s| + |t|$. Assume that $\text{dec}_p(s) \gg_{EL} \text{dec}_q(t)$. 


Consider the case $s|_p = t|_q \in \mathcal{V}$.

By the assumption $dec_p(s) \gg_{EL} dec_q(t)$ and definition of multiset extension, consider the cases that $dec_p(s) = M \cup \{s_1, \ldots, s_m\}$, $dec_q(t) = M \cup \{t_1, \ldots, t_n\}$, and for any $k \in \{1, \ldots, n\}$, there exists $l \in \{1, \ldots, m\}$ such that $dec_p(s) \ni s_l >_{EL} t_k \in dec_q(t)$.

For any $i \in N^*$ ($q,i \in P_T(t\theta)$), we can show that $dec_{p,i}(s\theta) = M\theta \cup \{s_1\theta, \ldots, s_m\theta\} \cup L$, $dec_{q,i}(t\theta) = M\theta \cup \{t_1\theta, \ldots, t_n\theta\} \cup L$ where $L = \{v \mid v \in sub(dec_{i}(s|_p\theta), s|_p\theta)\}$. Hence we have to show that $dec_p(s) \ni s_l >_{EL} t_k \in dec_q(t)$ implies $dec_{p,i}(s\theta) \ni s_l\theta >_{EL} t_k\theta \in dec_{q,i}(t\theta)$. We distinguish the cases with respect to the definition of $>_{EL}$.

(a) If $root(s_l) > root(t_k)$ then $root(s_l\theta) > root(t_k\theta)$ holds.

(b) If $root(s_l) = root(t_k)$, $\tau(root(s_l)) = \text{mul}$ and $sub(dec_p(s), s_l) \gg_{EL} sub(dec_q(t), t_k)$ then we can show $sub(dec_{p,i}(s\theta), s_l\theta) \gg_{EL} sub(dec_{q,i}(t\theta), t_k\theta)$ by induction hypothesis.

(c) In the case that $root(s_l) = root(t_k)$, $\tau(root(s_l)) = \text{mul}$, $sub(dec_p(s), s_l) = sub(dec_q(t), t_k)$ and $dec(args(s_l)) \gg_{EL} dec(args(t_k))$, it follows that $dec(args(s_l\theta)) \gg_{EL} dec(args(t_k\theta))$ from induction hypothesis.

(d) Consider the case that $root(s_l) = root(t_k)$, $\tau(root(s_l)) \neq \text{mul}$, $args(s_l) \gg^\tau_{RDS} args(t_k)$ and $\{s_l\} \gg_{IDS} args(t_k)$. We can show $args(s_l\theta) \gg^\tau_{RDS} args(t_k\theta)$ and $\{s_l\theta\} \gg_{IDS} args(t_k\theta)$ by induction hypothesis.

In case of $t|_q \not\in \mathcal{V}$, for any $i \in N^*$ ($p,i \in P_T(s\theta)$), we can show that $dec_{p,i}(s\theta) = M\theta \cup \{s_1\theta, \ldots, s_m\theta\} \cup L$ where $L = \{v \mid v \in sub(dec_{i}(s|_p\theta), s|_p\theta)\}$, $dec_q(t\theta) = M\theta \cup \{t_1\theta, \ldots, t_n\theta\}$. Hence we can show that $dec_p(s) \ni s_l >_{EL} t_k \in dec_q(t)$ implies $dec_{p,i}(s\theta) \ni s_l\theta >_{EL} t_k\theta \in dec_{q,i}(t\theta)$, in similar to the proof of (1).