数理解析研究所講究録 1426 巻 2005 年 140-146

Hausdorff Dimension and the Stochastic Traveling Salesman Problem

東京工業大学大学院情報理工学研究科数理・計算科学専攻特別研究員　高橋勇人
Hayato Takahashi
Tokyo Institute of Technology
Department of Mathematical and Computing Sciences
2-12-1-W8-25, O-okayama, Meguro-ku, Tokyo 152-8552, Japan.
E-mail: Hayato.Takahashi@is.titech.ac.jp,
tel/fax: 81-3-5734-3872.

Abstract

The traveling salesman problem is a problem of finding the shortest tour through given points. We characterize the asymptotic order of the optimal tour length with Hausdorff dimension.

1 Introduction

The traveling salesman problem (TSP) is a problem of finding the shortest tour through given points. We study asymptotic length of the shortest tour through points on Euclidean space.

Though TSP is an NP-hard problem, Karp [5] showed that if points $X_1, \cdots, X_n$ are uniformly distributed on the unit square then there is a polynomial time algorithm that generate a tour of length $L(X_1, \cdots, X_n)$ such that

$$\lim_{n\to\infty} L(X_1, \cdots, X_n)/L_{opt}(X_1, \cdots, X_n) = 1, a.s.,$$

where L_{opt} is the length of the shortest tour. Karp's algorithm is based on the following theorem by Beardwood, Halton, and Hammersley (BHH theorem):

Theorem 1.1 (BHH[2]) *If points $X_1, \cdots, X_n$ are i.i.d. random variables with respect to distribution μ on $[0,1]^d$ then*

$$\lim_{n\to\infty} L_{opt}(X_1, \cdots, X_n)/n^{1-\frac{1}{d}} = \beta(d) \int_{[0,1]^d} f(x)^{1-\frac{1}{d}} dx, \ \mu - a.s.,$$

where $\beta(d)$ is a constant that depend on the dimension d, and $f(x)$ is the density of μ with respect to Lebesgue measure.

We show that an analogous result holds for the case that the points are distributed over a positive Hausdorff dimensional set. To state the result we introduce some notations and results shown in [3]. Let $x \in [0,1]^d$. Let $B_r(x)$ be the d-dimensional ball with center x and radius r. Let μ_h be a probability distribution on $[0,1]^d$ such that

$$\lim_{r\to 0} \log \mu_h(B_r(x) \cap [0,1]^d)/\log r = h, \ \mu_h - a.s. \tag{1}$$

Let $H(\mu_h)$ be the support set of μ_h, i.e.,

$$H(\mu_h) = \{x | \lim_{r\to 0} \log \mu_h(B_r(x) \cap [0,1]^d)/\log r = h\}. \tag{2}$$

Then it is known that

$$\dim H(\mu_h) = h, \tag{3}$$

where $\dim H$ is the Hausdorff dimension of H. For a proof of (3) see [3]. Note that many of sets having positive Hausdorff dimension (including fractal sets, e.g., Cantor set) are described by such a manner [3].

We prove that:

Theorem 1.2 (Main result) *If points $X_1, \cdots, X_n$ are i.i.d. random variables with respect to μ_h, then under conditions on μ_h, there exist two constants c_1 and c_2 $(0 < c_1 \le c_2 < \infty)$ such that for $h > 1$*

$$c_1 \le \liminf_n L_{opt}(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \le \limsup_n L_{opt}(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \le c_2, \ \mu_h - a.s.,$$

and for $0 < h \le 1$, $L_{opt}(X_1, \cdots, X_n) = O(\sqrt{\log n})$, $\mu_h - a.s.$

Note that if $h < d$, the measure μ_h is singular with respect to Lebesgue measure on $[0,1]^d$; and therefore BHH theorem cannot be applied to the measure μ_h since the density of the absolutely continuous part is 0.

The theorem above shows that if points are distributed over a set $H(\mu_h)$ of Hausdorff dimension h $(< d)$, then the optimal tour length is much shorter than that of the case for uniform distribution for large number of points. Roughly speaking, this is because if $h < d$, the points $X_1, \cdots, X_n$ are distributed over the d-dimensional volume 0 set and therefore the average distance from a given point $X \in H(\mu_h)$ to the nearest point of $X_1, \cdots, X_n$ is much smaller than that of the case for uniform distribution.

Finally we note that our results are a generalization of those of Stadje [7] and Steel [8].

2 Average optimal tour length

In this paper we consider the class of distributions that satisfy the following condition:

Condition 1 *Let μ_h be a distribution on $[0,1]^d$ that satisfies the following property: There exist a subset $H(\mu_h)$ of $[0,1]^d$ such that*

$$\mu_h(H(\mu_h)) = 1,$$

and for $x \in H(\mu_h)$

$$\mu_h(B_r(x) \cap [0,1]^d) = f(x) r^{h+g(r,x)}, \tag{4}$$

where

$$h > 0, \ f(x) > 0, \ \lim_{r \to 0} g(r,x) = 0,$$

and f is the density. Let $\tilde{\mu}$ be the measure defined by $\tilde{\mu}_h(B_r(x)) = r^{h+g(r,x)}$. We assume that $\tilde{\mu}_h([0,1]^d) < \infty$.

Note that μ_h and $H(\mu_h)$ that satisfy the condition above satisfy (1) and that $\dim H(\mu_h) = h > 0$. Conversely if μ_h satisfies (1) and $h > 0$, then there exists $H(\mu_h)$, g, and f that satisfy the condition above such that $\mu_h(H(\mu_h)) = 1$ and $\dim H(\mu_h) = h > 0$.

Let

$$q_n(x) = E(\min_{1 \le i \le n} |X_i - x|). \tag{5}$$

In [7], Stadje showed that if $X_1, \cdots, X_n$ are i.i.d. random variables with respect to an absolutely continuous distribution with respect to Lebesgue measure on $[0,1]^d$ then

$$\lim_{n\to\infty} n^{\frac{1}{d}} q_n(x) = f(x)^{-\frac{1}{d}} d^{-1} \pi^{-\frac{1}{2}} \Gamma(\frac{1}{d}) \Gamma(1 + \frac{d}{2})^{\frac{1}{d}}, \tag{6}$$

where f is the density and $f(x) > 0$.

In the following let $X_1, \cdots, X_n$ be i.i.d. random variables with respect to μ_h in (5). We show that an analogous result of (6) holds for the distribution μ_h.

Lemma 2.1 *Let $h(n)$ be function of n such that $\lim_{n\to\infty} h(n) = h > 0$. For any positive constant a, b, and c, we have*

$$\lim_{n\to\infty} (cn)^{\frac{1}{h(n)}} \int_0^a (1 - cr^{h(n)})^n dr = \lim_{n\to\infty} (cn)^{\frac{1}{h(n)}} \int_0^{n^{-\frac{1}{(1+b)h}}} (1 - cr^{h(n)})^n dr = \frac{\Gamma(\frac{1}{h})}{h}. \tag{7}$$

Proof) We prove (7) by Laplace method. Let $cr^{h(n)} = \frac{1}{n}\tilde{r}^{h(n)}$, i.e., $\tilde{r} = (cn)^{\frac{1}{h(n)}} r$. Then we have

$$\int_0^{n^{-\frac{1}{(1+b)h}}} (1 - cr^{h(n)})^n dr = (cn)^{-\frac{1}{h(n)}} \int_0^\infty I_{[0, c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h}+\frac{1}{h(n)}}]} \exp\{n \log(1 - \frac{1}{n}\tilde{r}^{h(n)})\} d\tilde{r},$$

where I_A is the characteristic function of a set A. Since $c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h}+\frac{1}{h(n)}} \to \infty$ as $n \to \infty$, we have for sufficiently large n,
$I_{[0, c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h}+\frac{1}{h(n)}}]} \exp\{n \log(1 - \frac{1}{n}\tilde{r}^{h(n)})\} \le \exp\{-\tilde{r}^{\frac{h}{2}}\}$, $\int_0^\infty \exp\{-\tilde{r}^{\frac{h}{2}}\} d\tilde{r} < \infty$, and
$\lim_{n\to\infty} I_{[0, c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h}+\frac{1}{h(n)}}]} \exp\{n \log(1 - \frac{1}{n}\tilde{r}^{h(n)})\} = \exp\{-\tilde{r}^h\}$ for $\tilde{r} > 0$; and therefore by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int I_{[0, c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h}+\frac{1}{h(n)}}]} \exp\{n \log(1 - \frac{1}{n}\tilde{r}^{h(n)})\} d\tilde{r} = \int_0^\infty \exp\{-\tilde{r}^h\} d\tilde{r} = \frac{\Gamma(\frac{1}{h})}{h},$$

which proves the second equality of (7).

For the first equality, observe that

$$\int_{n^{-\frac{1}{(1+b)h}}}^a (1 - cr^{h(n)})^n dr \le a(1 - cn^{-\frac{h(n)}{(1+b)h}})^n \le a \exp(-cn^{1-\frac{h(n)}{(1+b)h}}). \tag{8}$$

Since $1 - \frac{h(n)}{(1+b)h} > 0$ for sufficiently large n, by (8), and the second equality of (7), we have the first equality of (7). ■

In the following, let b be a positive constant, and let

$$\delta(n, x) = \sup_{0 \le r \le n^{-\frac{1}{(1+b)h}}} |g(r, x)|, \tag{9}$$

and

$$\delta(n) = \sup_{x \in H(\mu_h)} \delta(n, x).$$

Lemma 2.2 *Let μ_h and $H(\mu_h)$ be a distribution on $[0,1]^d$ and its support set that satisfy Condition 1. Let $C_1^h(x) = f(x)^{-\frac{1}{h}} \frac{\Gamma(\frac{1}{h})}{h}$. Then for $x \in H(\mu_h)$, we have*

$$\limsup_n q_n(x) n^{\frac{1}{h+\delta(n,x)}} \le C_1^h(x) \le \liminf_n q_n(x) n^{\frac{1}{h-\delta(n,x)}}. \tag{10}$$

In particular if $\delta(n, x) = o((\log n)^{-1})$, we have for $x \in H(\mu_h)$,

$$\lim_{n\to\infty} q_n(x) n^{\frac{1}{h}} = C_1^h(x). \tag{11}$$

Proof) Let $x \in H(\mu_h)$. We have

$$\mu_h(\min_{1\le i\le n} |X_i - x| \ge r) = (1 - \mu_h(B_r(x) \cap [0,1]^d))^n,$$

and hence

$$\begin{aligned} q_n(x) = E(\min_{1\le i\le m} |X_i - x|) &= \int_0^{\sqrt{d}} \mu_h(\min_{1\le i\le m} |X_i - x| \ge r) dr \\ &= \int_0^{\sqrt{d}} (1 - \mu_h(B_r(x) \cap [0,1]^d))^n dr \\ &= \int_0^{a(n)} A_n(r) dr + \int_{a(n)}^{\sqrt{d}} A_n(r) dr \end{aligned} \tag{12}$$

where $A_n(r) = (1-\mu_h(B_r(x)\cap[0,1]^d))^n$, $a(n) = n^{-\frac{1}{(1+b)h}}$, and b is a positive constant.

We have

$$\begin{aligned}\int_0^{a(n)} A_n(r)dr &= \int_0^{a(n)} (1-f(x)r^{h+g(r,x)})^n dr \\ &\le \int_0^{a(n)} (1-f(x)r^{h+\delta(n,x)})^n dr \qquad (13)\\ &= (f(x)n)^{-\frac{1}{h+\delta(n,x)}}\frac{\Gamma(\frac{1}{h})}{h}(1+o(1)), \qquad (14)\end{aligned}$$

where the first equality and the first inequality follow from (4) and (9); for the last equality observe that $\lim_{n\to\infty}\delta(n,x)=0$, and hence (14) follows from Lemma 2.1.

Since $A_n(r)$ is decreasing as r grows, we have

$$\begin{aligned}\int_{a(n)}^{\sqrt{d}} A_n(r)dr &\le \sqrt{d}A_n(a(n)) \\ &= \sqrt{d}(1-f(x)a(n)^{h+g(a(n),x)})^n \\ &\le \sqrt{d}\exp(-f(x)n^{1-\frac{h+g(a(n),x)}{(1+b)h}}). \qquad (15)\end{aligned}$$

Since $\lim_{n\to\infty} g(a(n),x)=0$, we see $\int_{a(n)}^{\sqrt{d}} A_n(r)dr = o(n^{-\frac{1}{h+\delta(n,x)}})$; hence we have the first inequality of (10). In a similar way, we can prove the other inequality of (10). If $\delta(n,x)=o((\log n)^{-1})$, we have (11). ■

Remark 2.1 *If μ_d is an absolutely continuous distribution with respect to Lebesgue measure on $[0,1]^d$ and if x is a interior point of $[0,1]^d$, we see $\mu_d(B_r(x)) = f(r,x)c_d r^d$, where c_d $(=\pi^{d/2}/\Gamma((d+2)/2))$ is the volume of the d-dimensional unit ball, and $f(r,x)$ converges to the density $f(x)$ as r goes to 0. By applying Lemma 2.2 to $\mu_d(B_r(x))$, we have (6).*

Lemma 2.3 *Let μ_h be a distribution that satisfy Condition 1.*
Let $C_2^h = E(C_1^h(x)) = E(f(x)^{-\frac{1}{h}})\frac{\Gamma(\frac{1}{h})}{h} \le \infty$. We have

$$\limsup_n E(q_n(x)n^{\frac{1}{h+\delta(n,x)}}) \le C_2^h \le \liminf_n E(q_n(x)n^{\frac{1}{h-\delta(n,x)}}). \qquad (16)$$

In particular if $\delta(n,x)=o((\log n)^{-1})$, we have

$$\lim_{n\to\infty} E(q_n(x))n^{\frac{1}{h}} = C_2^h. \qquad (17)$$

Proof) First we show the lemma when $C_2^h<\infty$. Since $C_2^h = E(C_1^h(x))<\infty$ and $\mu_h(H_\mu)=1$, by Fatou lemma and (10), we have (16). If $\delta(n,x)=o((\log n)^{-1})$, we have (17).

Note that by Fatou lemma, $\liminf_n E(q_n(x)n^{\frac{1}{h-\delta(n,x)}}) \ge E(\liminf_n q_n(x)n^{\frac{1}{h-\delta(n,x)}})$ holds without assuming that $q_n(x)n^{\frac{1}{h-\delta(n,x)}}$ is bounded by integrable function; hence the lemma holds for $C_2^h=\infty$. ■

Remark 2.2 *If $h\ge 1$, $E(f(x)^{-\frac{1}{h}})$ always exists and have a finite value, because by Jensen's inequality we have $E((\frac{1}{f(x)})^{\frac{1}{h}}) \le E(1/f(x))^{\frac{1}{h}} = (\int_{H(\mu_h)} d\tilde{\mu}_h)^{\frac{1}{h}} < \infty$ where $\tilde{\mu}_h$ is the finite measure defined by $\tilde{\mu}_h(B_r(x)) = r^{h+g(r,x)}$.*

In the following for simplicity, L denote L_{opt}. Then it is known that

$$nE(q_{n-1}(X)) \le E(L(X_1,\cdots,X_n)) \le 2\sum_{i=1}^n E(q_i(X)). \qquad (18)$$

For a proof, see [7, 8].

From (18) and Lemma 2.3, we have:

Theorem 2.1 *Assume that $C_2^h < \infty$ and $\delta(n) = o((\log n)^{-1})$. Under Condition 1, for $1 < h$*

$$c_1 \leq \liminf_n E(L(X_1, \cdots, X_n))/n^{1-\frac{1}{h}} \leq \limsup_n E(L(X_1, \cdots, X_n))/n^{1-\frac{1}{h}} \leq c_2, \tag{19}$$

and for $0 < h \leq 1$, $\sup_n E(X_1, \cdots, X_n) < \infty$, where c_1 and c_2 are constants dependent on h such that $0 < c_1 \leq c_2 < \infty$.

3 Concentration

Let F_n be the σ-algebra generated by $X_1, \cdots, X_n$. Let f be a measurable function with respect to F_n. Let $d_i = E(f|F_i) - E(f|F_{i-1})$. We see $f - E(f) = \sum_{i=1}^n d_i$, and $\{d_i\}$ is a martingale sequence with respect to F_i, $1 \leq i \leq n$. For a random variable X, let $\operatorname{ess\,sup}_X f(X) = \inf\{a \,|P(f(X) > a) = 0\}$, and $\operatorname{ess\,inf}_X f(X) = \sup\{a \,|P(f(X) < a) = 0\}$. Let $\tilde{d}_i = \operatorname{ess\,sup} d_i - \operatorname{ess\,inf} d_i$. Then the following Azuma-Hoeffding inequality holds.

Theorem 3.1 (Azuma-Hoeffding[1, 4]) *For any $t > 0$,*
$P(|f - E(f)| \geq t) \leq 2\exp(-2t^2/\sum_{i=1}^n \tilde{d}_i^2)$.

For some applications of the theorem to combinatorics, see [6, 8] and for Markov processes see [9]. In [6], Rhee and Talagrand applied Azuma-Hoeffding inequality to TSP for the case that points are distributed uniformly over the unit square. In this section we apply Azuma-Hoeffding inequality to our model.

In Theorem 3.1, let $f = L(X_1, \cdots, X_n)$. In order to obtain $\tilde{d}_i$, observe that [7, 8]

$$L(X_1, \cdots, \hat{X}_i, \cdots, X_n) \leq L(X_1, \cdots, X_n) \quad \leq \quad L(X_1, \cdots, \hat{X}_i, \cdots, X_n) + 2 \min_{1 \leq j \leq n, j \neq i} |X_i - X_j|,$$

where $(X_1, \cdots, \hat{X}_i, \cdots, X_n)$ is the random vector obtained by deleting X_i from $(X_1, \cdots, X_n)$. Thus we have

$$\begin{aligned}\tilde{d}_i &\leq 2 \operatorname*{ess\,sup}_{X_1, \cdots, X_i} E(\min_{1 \leq j \leq n, j \neq i} |X_i - X_j| \; |F_i) \\ &\leq 2 \operatorname*{ess\,sup}_{X_1, \cdots, X_i} E(\min_{i < j \leq n} |X_i - X_j| \; |F_i) \\ &= 2 \operatorname*{ess\,sup}_{X_i} E(\min_{i < j \leq n} |X_i - X_j| \; |X_i) = 2 \operatorname*{ess\,sup}_{X_i} q_{n-i}(X_i), \end{aligned} \tag{20}$$

where the first equality follows from that $X_1, \cdots, X_n$ are i.i.d. random variables.

To prove the following theorem we need a condition.

Condition 2 *Assume that there exists a positive constant m such that $\inf_{x \in H(\mu_h)} f(x) > m > 0$. Assume that $\lim_{n \to \infty} \delta(n) = 0$.*

Lemma 3.1 *Under Condition 1 and 2, there exists a constant M such that*

$$\sup_{x \in H(\mu_h)} q_n(x) \leq M n^{-\frac{1}{h+\delta(n)}}. \tag{21}$$

Proof) Let $A_n(r)$ and $a(n)$ be the same as in the proof of Lemma 2.2. From (13), Condition 2, and Lemma 2.1, we have for sufficiently large n,

$$\begin{aligned}\int_0^{a(n)} A_n(r)dr &\leq \int_0^{a(n)} (1 - f(x)r^{h+\delta(n,x)})^n dr \\ &\leq \int_0^{a(n)} (1 - mr^{h+\delta(n)})^n dr \\ &\leq mn^{-\frac{1}{h+\delta(n)}}, \end{aligned} \tag{22}$$

where m is a constant. Note that $a(n) \to 0$ as $n \to \infty$.

From (15), we have

$$\int_{a(n)}^{\sqrt{d}} A_n(r)dr \le \sqrt{d}\exp(-f(a(n),x)n^{1-\frac{h+g(a(n),x)}{(1+b)h}}) \le \sqrt{d}\exp(-mn^{1-\frac{h+\delta(n)}{(1+b)h}}). \tag{23}$$

Since $\lim_{n\to\infty} \delta(n) = 0$ (Condition 2), from (22), (23), and (12), we have (21). ■

Theorem 3.2 *Under Condition 1, and 2, if $\delta(n) = o((\log n)^{-1})$, there exist constants M_1, M_2, and M_3 such that*

$$\sum_{i=1}^{n} \tilde{d}_i^2 \le \begin{cases} M_1, & \text{if } h < 2, \\ M_2 \log n, & \text{if } h = 2, \\ M_3 n^{1-\frac{2}{h}}, & \text{if } h > 2, \end{cases}$$

and for any $t > 0$,

$$\mu_h(|f - E(f)| \ge t) \le 2\exp(-2t^2 / \sum_{i=1}^{n} \tilde{d}_i^2),$$

where $f = L(X_1, \cdots, X_n)$.

Proof) Since $\mu_h(H(\mu_h)) = 1$, by (20) and Lemma 3.1, we have

$$\tilde{d}_i \le M(n-i)^{-\frac{1}{h}},$$

where M is a positive constant. Theorem 3.2 follows from Theorem 3.1. ■

Theorem 3.3 *Assume that $\delta(n) = o((\log n)^{-1})$. Under Condition 1, and 2, for $1 < h$,*

$$c_1 \le \liminf_n L(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \le \limsup_n L(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \le c_2, \ \mu_h - a.e., \tag{24}$$

where c_1 and c_2 are constants that depend on h. For $0 < h \le 1$, we have $L(X_1, \cdots, X_n) = O(\sqrt{\log n})$, $\mu_h - a.s.$

Proof) By Borel-Cantelli's lemma and Theorem 3.2, we have

$$\limsup_n \frac{|f - E(f)|}{g(n)} \le 1, \ \mu_h - a.s.,$$

where $f = L(X_1, \cdots, X_n)$, and

$$g(n) = \begin{cases} O(\sqrt{\log n}), & \text{if } h < 2, \\ O(\log n), & \text{if } h = 2, \\ O(n^{\frac{1}{2}-\frac{1}{h}}\sqrt{\log n}), & \text{if } h > 2. \end{cases}$$

By Theorem 2.1, we have the theorem. ■

Acknowledgment.

The author thanks Prof. Osamu Watanabe (Tokyo Institute of Technology) for a discussion and comments.

References

[1] K. Azuma. Weighted sums of certain dependent random variables. *Tohoku Math. J.*, 19(3):357–367, 1967.

[2] Jillian Beardwood, J. H. Halton, and J. M. Hammersley. The shortest path through many points. *Proc. Cambridge Philos. Soc.*, 55:299–327, 1959.

[3] K. J. Falconer. *Fractal Geometry.* John Wiley, Chichester, 1990.

[4] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 53:13–30, 1963.

[5] R. M. Karp. The probabilistic analysis of some combinatorial search algorithms. In J. F. Traub, editor, *Algorithms and Complexity: New Directions and Recent Results*, pages 1–19. Academic Press, New York, 1976.

[6] W. T. Rhee and M. Talagrand. Martingale inequalities and NP-complete problems. *Math. Oper. Res.*, 12(1):177–181, 1987.

[7] W. Stadje. Two asymptotic inequalities for the stochastic traveling salesman problem. *Sankhyā Ser. A*, 57:33–40, 1995.

[8] J. Michael Steel. *Probability Theory and Combinatorial Optimization.* SIAM, Philadelphia, 1997.

[9] Hayato Takahashi and Yasuaki Niikura. An extension of Azuma-Hoeffding inequalities and its application to an analysis for randomized local search algorithms. In *Proceedings of the 26th Symposium on Information Theory and Its Applications (SITA2003)*, pages 541–544, 2003.