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Hajós Calculus on Planar Graphs

Abstract: The Hajós calculus is a nondeterministic procedure which generates the class of non-3-colorable graphs [3]. If all non-3-colorable graphs can be constructed in polynomial steps by the calculus, NP = co-NP holds. Up to date, however, it remains open whether there exists a family of graphs that can be generated in polynomial steps. To attack this problem, we propose two graph calculi PHC and PHC* that generate non-3-colorable planar graphs, where intermediate graphs in the calculi are also restricted to be planar. Then we prove that PHC and PHC* are sound and complete. We also show that PHC* can polynomially simulate PHC.

Keywords: Hajós calculus, Planar graph, Coloring, Proof systems

1 Introduction

Graph k-coloring problem is the problem to decide whether we can assign one of k colors to each vertex so that adjacent pairs of vertices are assigned different colors [15]. This problem is one of the most fundamental NP-complete problems [5, 9]. Even when k ≥ 3, it is NP-complete. When k ≤ 2, we can solve the problem in polynomial time. If graphs are restricted to be planar, it is believed for a long time that every graph is 4-colorable [10]. Appel and Haken finally proved the Four-Color Theorem, i.e., every planar graph is 4-colorable [7, 8, 12, 19]. Therefore, when k ≥ 4, we can decide whether given planar graph is k-colorable in polynomial time. When k = 3, the problem is still NP-complete.

In order to characterize k-colorable graphs, many approaches have been attempted. The most typical one is Hadwiger’s conjecture to relate the non-k-colorability and the (k + 1)-cliques [1]. Let k be the fewest number of colors necessary to color vertices in a given graph. Then, we can obtain a k-clique by contracting adjacent vertices. This conjecture is true for k ≤ 5 [1, 2, 4].

Another approach is the Hajós calculus. The calculus is a nondeterministic procedure that constructs all non-k-colorable graphs from a (k + 1)-clique [3]. A graph calculus is a collection of initial graphs, together with a finite set of rules which allows us to derive new graphs. A construction of a graph G is a sequence of graphs (G₁, G₂, ..., Gᵢ) such that the sequence ends with G (i.e., Gᵢ = G) and every graph in the sequence is one of the initial graphs, or follows from its previous graphs by applying one of the rules.

The complexity of the Hajós calculus was first studied by Mansfield and Welsh [11]: If all non-3-colorable graphs have polynomial size Hajós constructions then, NP = co-NP holds, thus there may exist graphs that cannot be constructed in polynomial steps. A construction of a graph in the Hajós calculus gives the proof of the non-k-colorability of the graph.

Our Contribution: Our motivation is to give intermediate subsystems that are more powerful than bounded-depth Frege system and yet we can prove super-polynomial lower bounds. For this purpose, we consider the calculus on planar graphs, more precisely, the calculus that generates the class of non-3-colorable planar graphs, where intermediate graphs in the calculus are also restricted to be planar. Although the Hajós calculus can generate all non-3-colorable planar graphs, intermediate graphs are not guaranteed to be planar. When restricting the intermediate graphs to be planar, by adding only one new rule, we can obtain a sound and complete calculus PHC. By modifying the second rule (edge elimination rule) in the Hajós calculus, we can obtain another sound and complete calculus PHC*. We compare the powers of the two calculi.

Previous work: It is known that the Hajós calculus is polynomially bounded if and only if Extended Frege proof systems are polynomially bounded [16]. This result links an open problem in graph theory to an important open problem in the complexity of propositional proof systems: Is there a strong system to
produce a short proof of any tautology? As formalized by Cook and Reckhow [6], there exists a propositional proof system giving rise to short (polynomial-size) proofs of all tautologies if and only if NP equals co-NP. Since Extended Frege system is powerful enough that obtaining super-polynomial lower bounds is beyond our current technique, research interests shift into subsystems of the calculus. For example, Ait Jai and others showed exponential lower bounds for bounded-depth Frege proofs [13, 14, 18], which lead exponential lower bounds on the subsystems of the Hajó's calculus [16, 17].

2 Hajó's Calculus

We describe the Hajó's calculus for $k = 3$. The set of initial graphs in Hajó's calculus contains all graphs isomorphic to complete graph $K_4$. There are three rules for generating new graphs:

1. **Vertex/edge introduction rule:** Add (any number of) vertices and edges.

2. **Join rule:** Let $G_1$ and $G_2$ be disjoint graphs, $a_1$ and $b_1$ adjacent vertices in $G_1$, and $a_2$ and $b_2$ adjacent vertices in $G_2$. Construct a graph $G_3$ from $G_1 \cup G_2$ as follows. First, remove edges $(a_1, b_1)$ and $(a_2, b_2)$; then add an edge $(b_1, b_2)$; lastly, contract vertices $a_1$ and $a_2$ into a single vertex, named $a_1$.

3. **Contraction rule:** Contract two nonadjacent vertices into a single vertex, and remove any resulting duplicated edges.

Vertex/edge introduction rule implies that if a subgraph of $G$ has a construction, $G$ also has a construction. Rules 1 and 2 increase vertices and/or edges, but Rule 3 reduces vertices and edges, thus the construction may not be polynomially bounded.

We consider a minor revision of the Hajó's calculus, $\mathcal{HC}$. The system $\mathcal{HC}$ has the same set of initial graphs, as well as Rules 1 and 3 of the Hajó's calculus, but now Rule 2 of the Hajó's calculus is replaced by the following rule:

2. **Edge elimination rule:** Let $G_1$ and $G_2$ be two graphs with common vertex set $\{v_1, \ldots, v_n\}$ which are identical except that $G_1$ contains edges $(v_1, v_2)$ and $(v_2, v_3)$ and not $(v_1, v_3)$, whereas $G_2$ contains edges $(v_1, v_2)$ and $(v_1, v_3)$ and not $(v_2, v_3)$. Then from $G_1$ and $G_2$, we can construct a graph $G_3$ that is identical to $G_1$ but does not contain $(v_2, v_3)$.

To associate two calculi, Hajó's calculus and $\mathcal{HC}$, we define a binary relation: Let $C$ and $C'$ be two graph calculus systems, then $C$ $p$-simulates $C'$ if there is a polynomial-time computable function $f$ so that for all graphs $G$, if $s$ is a graph construction of $G$ in $C'$, then $f(s)$ is a graph construction of $G$ in $C$. $C$ and $C'$ are $p$-equivalent if $C$ $p$-simulates $C'$ and $C'$ $p$-simulates $C$.

Fact 1 $\mathcal{HC}$ is $p$-equivalent to the Hajó's calculus.

3 Planar Calculus $\mathcal{PHC}$

First, we propose planar calculus $\mathcal{PHC}$. The set of initial graphs in $\mathcal{PHC}$ contains all graphs isomorphic to $K_4$. There are four rules, where Rules 1 to 3 are same as the system $\mathcal{HC}$, but edge addition and vertex contraction are restricted so that the resulting graphs are planar. Rule 4 is as follows:

4. **Quadrilateral rule:** Let $G_3$ be a graph with vertex set $\{v_1, \ldots, v_n\}$ that contains a face $v_1, v_2, v_3, v_4$. Let $G_1$ be a graph obtained by contracting vertices $v_1$ and $v_2$ of $G_3$. Let $G_2$ be a graph obtained by contracting vertices $v_2$ and $v_4$ of $G_3$. Then from $G_1$ and $G_2$, we can construct the graph $G_3$ (See Figure 1).

Rule 4 is important when given graph consists of only triangle and quadrilateral faces.

For example, we show that the graph $G_3$ of Figure 2 has a construction in $\mathcal{PHC}$. Let $G_1$ and $G_2$ be the graphs shown in Figure 2. $G_1$ contains $K_4$ as a subgraph induced by $v_1, v_2, v_3, v_4$, $G_2$ also contains $K_4$ as a subgraph induced by $v_1, v_3, v_4, v_6$. Therefore $G_1$ and $G_2$ can be constructed in $\mathcal{PHC}$. $G_1$ can be
constructed from $G_1$ and $G_2$ by Rule 4, since $v_1, v_2, v_7, v_6$ is a quadrilateral face and $G_1$ is identical to $G_3$ with $v_2$ and $v_6$ contracted and $G_2$ is identical to $G_3$ with $v_1$ and $v_7$ contracted.

Since $G_3$ is edge minimal with respect to the 3-colorability, $G_3$ cannot be constructed directly by Rule 1. Each face of $G_3$ is triangle or quadrilateral, thus there is not a triplet of vertices $v, v', v''$ of satisfying the condition of Rule 2. This means that $G_3$ cannot be constructed directly by Rule 2. Contraction rule cannot break the structure of non-3-colorability. Therefore, probably $G_3$ is an example of graphs that essentially need Rule 4 in $\mathcal{PHC}$.

In the rest of this section, we prove the soundness and the completeness of $\mathcal{PHC}$.

**Theorem 2** $\mathcal{PHC}$ is sound.

**Proof:** We only need to show that Rule 4 is sound since other rules also appear in $\mathcal{HC}$ and are shown to be sound [3]. Assume that there exists a 3-colorable graph $G_3$ generated by Rule 4. Then, its face $v_1, v_2, v_3, v_4$ has a coloring satisfying one of the following cases:

Case 1: $\text{color}(v_1) = \text{color}(v_3)$.
Case 2: $\text{color}(v_2) = \text{color}(v_4)$.

Note that, if neither of the cases are satisfied, we have $\text{color}(v_1) \neq \text{color}(v_3)$ and $\text{color}(v_2) \neq \text{color}(v_4)$. In this case, we need more than four colors to the face, which contradicts the 3-colorability of $G_3$. Cases 1 (Case 2, respectively) implies that $G_1$ ($G_2$, respectively) is 3-colorable. Therefore, only non-3-colorable graphs are generated. \qed

**Theorem 3** $\mathcal{PHC}$ is complete.

**Proof:** We prove this theorem by induction on the size $n$ of the graph. In case $n < 4$, all graphs are colorable by at most 3 colors. In case $n = 4$, $K_4$ is the initial graph of $\mathcal{PHC}$. Other graphs of size 4 are all 3-colorable, so we do not care them.

Assume that all non-3-colorable graphs of size $n - 1$ can be constructed in $\mathcal{PHC}$. We assume that there exists a nonempty set $\mathcal{G}$ of non-3-colorable graphs of size $n$ that cannot be constructed in $\mathcal{PHC}$. Then we lead a contradiction that edge maximal graph $G \in \mathcal{G}$ can be constructed. By considering the size of the faces in $G$, we have the following three cases.

Case 1: All faces are triangle. According to Theorem 5 (we prove this theorem later), $G$ can be constructed in $\mathcal{PHC}$.

Case 2: There is a face $f$ of size $k \geq 5$. Let $v_1, v_2, v_3, v_4, \ldots, v_k$ be the vertices of face $f$. $G' = G + (v_1, v_3) \text{ and } G'' = G + (v_1, v_4)$ can be constructed, since $G$ is a edge maximal graph in $\mathcal{G}$. Therefore we can construct $G$ from $G'$ and $G''$ applying by Rule 2.
Case 3: $G$ is composed of triangle or quadrilateral faces. Let $f = v_1, v_2, v_3, v_4$ be a quadrilateral face of $G$. Let $G'$ be a graph obtained by contracting vertices $v_1$ and $v_3$ of $G$. Let $G''$ be a graph obtained by contracting vertices $v_2$ and $v_4$ of $G$. $G'$ and $G''$ can be constructed in $\mathcal{PHC}$ because of the assumption. Therefore we can construct $G$ from $G'$ and $G''$ applying the Rule 4.

In any case, $G \subseteq G$ can be constructed in $\mathcal{PHC}$, which contradict to the definition of $G$. Thus, any non-3-colorable graph can be constructed in $\mathcal{PHC}$. $\square$

Now, we prove that any triangulate non-3-colorable planar graph can be constructed in polynomial number of steps. First we prove the following lemma, which construct an essential component of triangulate planar graphs.

**Lemma 4** Let $G_n = (V, E)$ be a graph of $3n + 1$ vertices, where $n \geq 1$,

$V = \{a_0\} \cup \{a_i, b_i, c_i \mid i \in \{1, \ldots, n\}\},$

$E = \{(a_0, a_n)\} \cup \{(a_{i-1}, b_i), (a_{i-1}, c_i), (a_i, b_i), (a_i, c_i), (b_i, c_i) \mid i \in \{1, \ldots, n\}\}$

then, $G$ has a linear size construction in $\mathcal{PHC}$.

**PROOF:** We prove this lemma by induction on $n$. In case $n = 1$, the lemma obviously holds since $G_1$ is isomorphic to an initial graph $K_4$.

We prove that $G_n$ can be constructed by the assumption that $G_{n-1}$ can be constructed. $G' = G_n + (a_0, a_{n-1})$ can be constructed in $\mathcal{PHC}$ since $G_{n-1}$ is subgraph of $G'$. $G'' = G_n + (a_{n-1}, a_n)$ can be constructed in $\mathcal{PHC}$ since subgraph of $G''$ induced by $\{a_{n-1}, a_n, b_n, c_n\}$ is isomorphic to $K_4$. Therefore we can construct $G_n$ by applying Rule 2 to $G'$ and $G''$. Since we apply Rule 1 twice and Rule 2 once at each induction step, the whole construction is linearly bounded. $\square$

**Theorem 5** Triangulate non-3-colorable planar graphs have a polynomial size construction in $\mathcal{PHC}$.

**PROOF:** Our goal is to find a structure $G_n$ of Lemma 4 as a subgraph of a given graph $G$. We try to assign colors to vertices of $G$. Initially, we choose a triangle face $v_1, v_2, v_3$ and assign different color to each vertex. color($v_1$) = $R$, color($v_2$) = $G$ and color($v_3$) = $B$. We introduce three trees $T_B, T_G, T_R$. The root node of each tree is one of the vertices $v_1, v_2, v_3$. The face that its vertices are already assigned a color is called a colored face. Next, we repeat the following procedure until all vertices are assigned a color or adjacent vertices are assigned the same color. Choose a non-colored triangle face $f'$ adjacent to colored face $f$. We need not think about the case that non-colored faces exist but are not adjacent to colored face because the given graph is connected and triangulate. Let $v$ be a vertex that belongs to $f$ and does not belong to $f'$. Let $v'$ be a vertex that belongs to $f'$ and does not belong to $f$. Vertices $v$ and $v'$ are uniquely determined. Then we assign $c = \text{color}(v)$ to $v'$ and add the vertex $v'$ to the tree $T_c$ as a child node of $v$. This replication stops before all vertices assigned a color because $G$ is non-3-colorable. When the repetition stops, we find adjacent vertices $v'$ and $v''$ on $G$ that are assigned the same color $c$. The tree $T_c$ includes $v'$ and $v''$ so that there is a path $p$ from $v'$ to $v''$ in $T_c$. An Edge $(v_{i}, v_{j}) \in T_c$ corresponds to a subgraph of $G$ as the Figure 4.

Let $G'$ be a subgraph of $G$ that corresponds to path $p$ ($G'$ of Figure 5 is an example) that corresponds to path $p$ (dotted line of the figure). Let $G''$ be a graph $G_{|p}$ of Lemma 4. $G''$ can be constructed because of Lemma 4. $G'$ can be constructed from $G''$ with some vertices contracted. $G$ can be constructed from $G'$ by Rule 1. Therefore $G$ has a construction in $\mathcal{PHC}$. $\square$
4 Planar Calculus $\mathcal{PHC}^*$

In this section, we propose another planar calculus $\mathcal{PHC}^*$. The set of initial graphs in $\mathcal{PHC}^*$ contains all graphs isomorphic to $K_4$. There are three rules for generating new graphs. Rule 1 and Rule 3 are same as the system $\mathcal{PHC}$. Our new Rule 2 is as follows:

2. Vertex division/edge elimination rule: Let $G_1$ be a graph with $n$ vertices $\{v_1, \ldots, v_n\}$ that contains an edge $(v_1, v_2)$, and $G_2$ be the graph obtained by contracting $v_1$ and $v_2$ of $G_1$. Then from $G_1$ and $G_2$, we can construct a graph $G_3$ graph that are identical to $G_1$ but does not contain $(v_1, v_2)$.

Rule 2 is simple but powerful to generate non-3-colorable graphs. This rule means that none adjacent vertices $v_1$ and $v_2$ can be assigned the same color or different colors.

In the rest of this section, we prove the soundness and the completeness of $\mathcal{PHC}^*$.

**Theorem 6** $\mathcal{PHC}^*$ is sound.

**Proof:** We only need to show the soundness of Rule 2 since other rules also appear in $\mathcal{HC}$ and are shown to be sound [3]. Assume that there exists a 3-colorable graph $G_3$ generated by Rule 2. Then, its vertices $v_1$ and $v_2$ has a coloring satisfying one of the following two cases:

Case 1: $\text{color}(v_1) \neq \text{color}(v_2)$.

Case 2: $\text{color}(v_1) = \text{color}(v_2)$.

In Case 1, the coloring is also valid for $G_1$, i.e., $G_1$ is 3-colorable. In Case 2, we can contract vertices $v_1$ and $v_2$ in $G_3$, i.e., $G_2$ is also 3-colorable. Therefore, in $\mathcal{PHC}^*$, all graphs generated by Rule 2 are non-3-colorable. $\square$

**Theorem 7** $\mathcal{PHC}^*$ is complete.

**Proof:** All non-3-colorable planar graphs can be constructed in $\mathcal{PHC}$. We can simulate $\mathcal{PHC}$ by $\mathcal{PHC}^*$, so that $\mathcal{PHC}^*$ is complete. $\square$

5 Polynomial-Time Simulation

We show the relationship between $\mathcal{PHC}$ and $\mathcal{PHC}^*$. First direction is that we simulate $\mathcal{PHC}$ by $\mathcal{PHC}^*$.

**Theorem 8** $\mathcal{PHC}^*$ p-simulates $\mathcal{PHC}$.

**Proof:** Rules 1 and 3 are common in $\mathcal{PHC}^*$ and $\mathcal{PHC}$. We only need to simulate Rule 2 and Rule 4 in $\mathcal{PHC}$ by $\mathcal{PHC}^*$. According to Lemma 9, Rule 2 can be simulated. According to Lemma 10, Rule 4 can be simulated. In each case the series of simulating steps can be constructed in polynomial time. Therefore, $\mathcal{PHC}^*$ p-simulates $\mathcal{PHC}$. $\square$

**Lemma 9** $\mathcal{PHC}^*$ p-simulates Rule 2 of $\mathcal{PHC}$.

**Proof:** We prove that a graph $G_3$ can be constructed from $G_1$ and $G_2$ in $\mathcal{PHC}^*$. Let $G_1$ and $G_2$ be two graphs with common vertex set $\{v_1, \ldots, v_n\}$ which are identical except that $G_1$ contains edges $(v_1, v_2)$ and $(v_2, v_3)$ and not $(v_1, v_3)$, whereas $G_2$ contains edges $(v_1, v_2)$ and $(v_1, v_3)$ and not $(v_2, v_3)$. $G_3$ is identical to $G_1$ but does not contain $(v_2, v_3)$. Let $G'_1$ be a graph identical to $G_1$ with vertices $v_1$ and $v_3$ are contracted. $(G_1, G'_1, G_2, G_3)$ is a subsequence of a construction in $\mathcal{PHC}^*$ since $G'_1$ can be constructed from $G_1$ by Rule 3 and $G_3$ can be constructed from $G'_1$ and $G_2$ by Rule 2 with particular vertices $v_1$ and $v_3$. $\square$
Lemma 10 \( \mathcal{HC}^* \) p-simulates Rule 4 of PHC.

**Proof:** Let \( G_1, G_2 \) and \( G_3 \) be graphs as Figure 6. \( G_3 \) is a graph with vertex set \( \{v_1, \ldots, v_n\} \) that contains a face \( v_1, v_2, v_3, v_4 \). \( G_1 \) is a graph obtained by contracting vertices \( v_1 \) and \( v_3 \) of \( G_3 \). \( G_2 \) is a graph obtained by contracting vertices \( v_2 \) and \( v_4 \) of \( G_3 \). We prove that a graph \( G_3 \) can be constructed from \( G_1 \) and \( G_2 \) in \( \mathcal{PHC}^* \). Let \( G'_1 \) be the graph as Figure 7. \( G'_1 \) is identical to \( G_1 \), but has two additional vertices \( u \) and \( w \) and three additional edges \((v_1, u), (v_1, w), (u, w)\). \( G'_1 \) can be constructed from \( G_1 \) by Rule 1, since \( G'_1 \) is a subgraph of \( G_3 \). Let \( G'_2, G''_3 \) and \( G'''_3 \) be graphs as Figure 7 that are identical to \( G_3 \) but some vertices and edges in the figure is different from \( G_3 \). \( G''_3 \) can be constructed from \( K_4 \) by Rule 1, since a subgraph induced by \( \{v_1, v_3, u, w\} \) is isomorphic to \( K_4 \). \( G''_3 \) can be constructed from \( G'_1 \) and \( G'_2 \) by Rule 2 in \( \mathcal{PHC}^* \) with particular vertices \( v_1 \) and \( v_3 \) (\( G''_3 \) has an edge \((v_1, v_3)\) and \( G'_1 \) is identical to \( G''_3 \) with \( v_1 \) and \( v_3 \) contracted). \( G'''_3 \) can be constructed from \( G''_3 \) by contracting two pairs of vertices \((v_2, u)\) and \((v_4, w)\). This construction needs twice of applying Rule 3 in \( \mathcal{PHC}^* \). Then \( G_3 \) can be constructed from \( G'''_3 \) and \( G_2 \) by Rule 2 in \( \mathcal{PHC}^* \) with particular vertices \( v_2 \) and \( v_4 \). Thus Rule 4 in \( \mathcal{PHC} \) can be p-simulated by \( \mathcal{PHC}^* \). \( \square \)

Theorem 8 implies that the modified rule (Rule 2) is at least as powerful as the original one in \( \mathcal{HC} \).

Corollary 11 \( \mathcal{HC} \) can be p-simulated by \( \mathcal{PHC}^* \) without planarity restriction on the intermediate graphs.

It is difficult to show that \( \mathcal{PHC} \) p-simulates \( \mathcal{PHC}^* \). For example, as shown in Figure 8, Rule 2 in \( \mathcal{PHC}^* \) can generate a new quadrilateral of \( G_3 \) from \( G_1 \) and \( G_2 \). To simulate this construction, we must remove an edge \((v_2, v_4)\) of \( G_1 \) by rules in \( \mathcal{PHC} \), but edge elimination rule cannot be applied since \((v_2, v_4)\) are sandwiched between triangle faces and the other rules cannot eliminate edges.
6 Concluding Remarks

We show that there exist a system of generating non-3-colorable planar graphs, where intermediate graphs in the system are restricted to be planar. Two calculi $PHC$ and $PHC^*$ are sound and complete graph construction system that generates the class of non-3-colorable planar graphs. $PHC^*$ is simple but powerful calculus, since $PHC^*$ $p$-simulation $PHC$.

Relationship between construction in planar graph calculus and general graph calculus may be interesting. There is a structure that can replace crossing edges keeping the colorability condition, so that non-planar graphs can be mapped to planar graphs. Thus a class of graphs that have super-polynomial lower bound in $HC$ may be associated with a class of graphs in a planar calculus. For future discussion, we would like to consider polynomial-time simulation of $PHC^*$ by $PHC$. Also lower bound of planar graph calculus is an interesting work.

References