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Logarithmic trace inequalities

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Abstract We shall extend logarithmic trace inequalities shown by Bebiano, Lemos and Providencia and also by Hiai and Petz, by applying log majorization equivalent to an order preserving operator inequality. We shall consider the convergence of certain logarithmic trace inequalities, as some extensions of Bebiano, Lemos and Providencia and Hiai-Petz. As an appendix, we state the following result. *Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold:*

$$\log S(1)\mathrm{Tr}[A] + S(A,B) \geq -\mathrm{Tr}[\hat{S}(A|B)] \geq S(A,B).$$

where $S(A,B) = \mathrm{Tr}[A(\log A - \log B)]$, $\hat{S}(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}} B A^{\frac{-1}{2}})A^{\frac{1}{2}}$ and $S(1) = \dfrac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ $(h > 1)$. The first inequality is *the reverse one* of the well known second one.

§1. Introduction

In this paper a capital letter means $n \times n$ matrix. Following Ando and Hiai [1], let us define the *log majorization* for positive semidefinite matrices $A, B \geq 0$, denoted by $A \underset{(\log)}{\succ} B$ if

$$\prod_{i=1}^{k} \lambda_i(A) \geq \prod_{i=1}^{k} \lambda_i(B), \qquad k = 1, 2, \ldots, n-1,$$

and

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B), \quad \text{i.e.,} \quad \det A = \det B,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq \ldots \geq \lambda_n(B)$ are the eigenvalues of A and B, respectively, arranged in decreasing order. When $0 \leq \alpha \leq 1$, the α-power mean of positive invertible matrices $A, B > 0$ is defined by

$$A \#_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}.$$

Further, $A \#_\alpha B$ for $A, B \geq 0$ is defined by $A \#_\alpha B = \lim_{\epsilon \downarrow 0} (A + \epsilon I) \#_\alpha (B + \epsilon I)$.

For the sake of convenience for symbolic expression, we define $A \natural_s B$, for any real number $s \geq 0$ and for $A > 0$ and $B \geq 0$, by the following

$$A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}}.$$

$A \natural_\alpha B$ in the case $0 \leq \alpha \leq 1$ just coincides with the usual α-power mean.

The following excellent and useful log majorization is shown in Ando and Hiai [1, Theorem 2.1].

Theorem A. *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,*

$$(A\#_\alpha B)^r \underset{(\log)}{\succ} A^r\#_\alpha B^r \qquad \textit{for } r \geq 1. \tag{1.1}$$

Also, (1.1) can be transformed into the following matrix inequality (1.2) of Theorem B in Ando and Hiai [1, Theorem 3.5]:

Theorem B. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}} \qquad \textit{for } r, p \geq 1. \tag{1.2}$$

We obtained the following extension of Theorem A in Furuta [11, Therorem 2.1] applying the method in Ando and Hiai [1] to Theorem G (see §3).

Theorem C. *For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and for each $t \in [0, 1]$,*

$$(A\#_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r}\#_\beta\,(A^{1-t}\natural_s B)$$

holds for $s \geq 1$, and $r \geq t \geq 0$, where $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}$ and $h = \dfrac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

Next, we state the following result which is shown in Hiai and Petz [13, Theorem 3.5] and, recently, a new proof is given in Bebiano, Lemos and Providencia [2, Theorem 2.2].

Theorem D. *If $A, B \geq 0$, then for every $p \geq 0$*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] \geq \mathrm{Tr}[A(\log A + \log B)] \tag{1.3}$$

holds and the left hand side of (1.3) *converges the right hand side as $p \downarrow 0$.*

Theorem E. *If $A \geq 0$, $B > 0$, $0 \leq \alpha \leq 1$ and $p > 0$, then*

$$\frac{1}{p}\mathrm{Tr}[A\log(A^p\sharp_\alpha B^p)] + \frac{\alpha}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A] \tag{1.4}$$

holds and the left hand side of (1.4) *converges the the right hand side as $p \downarrow 0$.*

The inequality (1.4) is shown in Ando and Hiai [1, Theorem 5.3], and the convergence of (1.4) is shown in Bebiano, Lemos and Providencia [2, Corollary 2.2].

We shall extend Theorem D and Theorem E by applying the trace inequality derived from log majorization equivalent to an order preserving inequality, and also by applying the generalized Lie-Trotter formulae of Lemma 6.1 in §6 and Lemma 7.1 in §7.

§2. Log majorization equivalent to an order preserving operator inequality

We shall show a log majorization equivalent to an order preserving operator inequality.

Theorem 2.1. *The following* (i) *and* (ii) *hold and are equivalent:*

(i) *If $A, B \geq 0$, then for each $t \in [0,1]$ and $r \geq t$*

$$A^{\frac{1}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{(p-tq)s+rq}{2ps}}\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^rB^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{ps}}A^{\frac{(p-tq)s+rq}{2ps}}$$

holds for any $s \geq 1$ and $p \geq q > 0$.

(ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $r \geq t$*

$$A^{\frac{(p-tq)s+rq}{ps}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{\frac{p}{q}}A^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q}{ps}}.$$

holds for any $s \geq 1$ and $p \geq q > 0$.

Corollary 2.2. *The following* (i) *and* (ii) *hold and are equivalent:*

(i) *If $A, B \geq 0$, then for each $r \geq 0$*

$$A^{\frac{1}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{q}{p}}A^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1}{2}(1+\frac{r}{p}q)}B^qA^{\frac{1}{2}(1+\frac{r}{p}q)}.$$

holds for any $p \geq q > 0$.

(ii) *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$A^{1+\frac{r}{p}q} \geq (A^{\frac{r}{2}}B^{\frac{p}{q}}A^{\frac{r}{2}})^{\frac{q}{p}}$$

holds for any $p \geq q > 0$.

Corollary 2.3. *The following* (i) *and* (ii) *hold and are equivalent:*

(i) *If $A, B \geq 0$, then for each $r \geq 1$*

$$A^{\frac{1}{2}}(A^{\frac{r-1}{2}}BA^{\frac{r-1}{2}})^qA^{\frac{1}{2}} \underset{(\log)}{\succ} A^{\frac{1}{2}}\{B^{\frac{1}{2}}(B^{\frac{1}{2}}A^rB^{\frac{1}{2}})^{r-1}B^{\frac{1}{2}}\}^{\frac{q}{r}}A^{\frac{1}{2}}$$

holds for any $1 \geq q > 0$.

(ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $r \geq 1$*

$$A \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^{\frac{1}{q}}A^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{q}{r}}.$$

holds for any $1 \geq q > 0$.

§3. Results needed to give proofs of the results in §2

Throughout this section, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if (Tx, x) for all $x \in H$. Also, an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible. We state the following celebrated Löwner-Heinz inequality in operator theory.

Theorem L-H (Löwner-Heinz inequality).
If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.

Lemma A. [11, Lemma 1]. *Let $A > 0$ and also let B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Theorem F (Furuta inequality).

If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

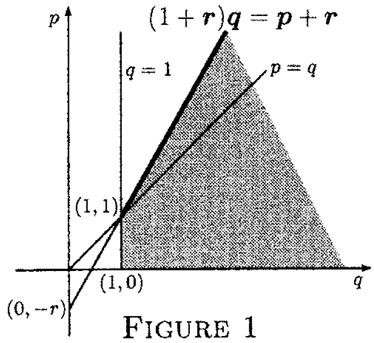

FIGURE 1

It is shown in Tanahashi [16], that the domain drawn for p,q and r in Figure 1 is the best possible one for Theorem F. Theorem F yields Löwner-Heinz inequality asserting that $A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$, when we put $r = 0$ in (i) or (ii) of Theorem F. The original proof is in Furuta [9], alternative proofs can be found in Fujii [4], Kamei [14] and one page proof in Furuta [10].

As an extension of Theorem F, we obtain the following Theorem G which interpolates Theorem F and Theorem B. Theorem G is used to prove Theorem C.

Theorem G. *If* $A \geq B \geq 0$ *with* $A > 0$, *then for each* $t \in [0,1]$ *and* $p \geq 1$

$$\begin{aligned} A^{1-t+r} &= \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}A^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \\ &\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \end{aligned}$$

for any $s \geq 1$ *and* $r \geq t$.

The original proof of Theorem G is in Furuta [11, Theorem 1.1], alternative proofs can be found in Fujii and Kamei [5] and one page proof in Furuta [12]. The original proof of the best possible exponent $\dfrac{1-t+r}{(p-t)s+r}$ in Theorem G is obtained in Tanahashi [17], and alternative proofs can be found in M.Fujii, Matsumoto and Nakamoto [6], and also in Yamazaki [19].

§4 Proofs of the results in §2

Applying Theorem G and [Theorem 2.1, Ando-Hiai [1]], we can give a proof of Theorem 2.1 and we omit it. Corollary 2.2 and Corollary 2.3 are immediate consequence of Theorem 2.1.

§5. Logarithmic trace inequalities as an application of Theorem 2.1

For $A, B > 0$, the relative operator entropy $\hat{S}(A|B)$ is defined by

$$\hat{S}(A|B) = A^{\frac{1}{2}} \log(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

in J.I.Fujii and Kamei [3], and $\hat{S}(A|I) = -A \log A$ is the usual operator entropy, (see [15]). The Umegaki operator entropy $S(A, B)$ is defined by

$$S(A, B) = \mathrm{Tr}[A(\log A - \log B)]$$

(see Umegaki [18]). For $A, B > 0$, let $\Delta(A|B)$ are defined by

$$\Delta(A|B) = -\mathrm{Tr}[\hat{S}(A|B)] - S(A, B).$$

We shall discuss the lower bound of $\Delta(A|B)$ in terms of the trace of A and B and a parameter, and this result implies the well known inequality $\Delta(A|B) \geq 0$ (for example, [13],[2]).

Theorem 5.1. *If $A, B \geq 0$, then, for every $t \in [0, 1]$ and $p \geq 0$,*

$$\begin{aligned} &\mathrm{Tr}[A \log(A^{\frac{r-t}{2}} B^p A^{\frac{r-t}{2}})^s] \\ &\geq (r - ts)\mathrm{Tr}[A \log A] + \mathrm{Tr}[A \log\{B^{\frac{p}{2}}(B^{\frac{p}{2}} A^r B^{\frac{p}{2}})^{s-1} B^{\frac{p}{2}}\}] \end{aligned} \tag{5.1}$$

holds for any $r \geq t$ and $s \geq 1$.

Corollary 5.2. *If $A, B \geq 0$, then, for every $p \geq 0$ and $r \geq 0$,*

$$\mathrm{Tr}[A \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^s] \geq \mathrm{Tr}[A \log A^r] + \mathrm{Tr}[A \log\{B^{\frac{p}{2}}(B^{\frac{p}{2}} A^r B^{\frac{p}{2}})^{s-1} B^{\frac{p}{2}}\}] \tag{5.2}$$

holds for any $s \geq 1$. In particular,

$$\mathrm{Tr}[A \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})] \geq \mathrm{Tr}[A \log A^r + A \log B^p] \tag{5.3}$$

and

$$\mathrm{Tr}[A \log(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^2] \geq \mathrm{Tr}[A \log A^r] + \mathrm{Tr}[A \log(B^p A^r B^p)]. \tag{5.4}$$

The inequality (5.3) of Corollary 5.2 may be considered as the two variable version of (1.3) in Theorem D. In fact, (5.3) of Corollary 5.2 is equivalent to (1.3) in Theorem D (see Remark 5.1).

Corollary 5.3. *If $A, B \geq 0$, then*

$$\mathrm{Tr}[A \log(A^{\frac{r-1}{2}} B A^{\frac{r-1}{2}})^r] \geq \mathrm{Tr}[A \log\{B^{\frac{1}{2}}(B^{\frac{1}{2}} A^r B^{\frac{1}{2}})^{r-1} B^{\frac{1}{2}}\}] \tag{5.5}$$

holds for every real number $r \geq 1$. In particular,

$$\operatorname{Tr}[A \log(A^{\frac{1}{2}} B A^{\frac{1}{2}})^2] \geq \operatorname{Tr}[A \log(BA^2B)]. \tag{5.6}$$

Corollary 5.4. *If $A, B > 0$, then*

$$\Delta(A|B) \geq \operatorname{Tr}[A \log B] - \frac{2(s-1)}{s} \operatorname{Tr}[A \log A] + \frac{1}{s} \operatorname{Tr}[A \log\{A^{-1}(AB^{-1}A)^s A^{-1}\}] \tag{5.7}$$

holds for every real number $s \geq 1$. In particular, $\Delta(A|B) \geq 0$ holds.

We remark that the right hand side of (5.7) is zero when $s = 1$, or when A commutes with B.

Proof of Theorem 5.1 is obtained by Theorem 2.1 and we omit it and Corollaries in this section are shown by Theorem 5.1.

§6 Generalized Lie-Trotter formulae, I

We adopt the usual convention $X^0 = I$ for $X > 0$. We obtain a convenient generalization of the Lie-Trotter formulae to prove the results in §8. The famous Lie-Trotter formula states

$$e^{A+B} = \lim_{p \downarrow 0} (e^{\frac{pA}{2}} e^{pB} e^{\frac{pA}{2}})^{\frac{1}{p}} \text{ for any Hermitian } A \text{ and } B.$$

The following lemma is an α-mean variant of the Lie-Trotter formula.

Lemma H [13, Lemma 3.3]. *If A and B are Hermitian and $\alpha \in [0, 1]$, then*

$$e^{(1-\alpha)A+\alpha B} = \lim_{p \downarrow 0} (e^{pA} \sharp_\alpha e^{pB})^{\frac{1}{p}}.$$

We remark that the Lie-Trotter formula and the α-mean variant of the Lie-Trotter formula are both quite useful in operator theory.

By retracing the proof of Lemma H, we shall obtain the following lemma.

Lemma 6.1. *If A, B, C and D are Hermitian, then, for any positive numbers α and β*

$$e^{A+\alpha B+\alpha\beta(C+D)} = \lim_{p \downarrow 0} \left\{ e^{\frac{pA}{2}} \left(e^{\frac{pB}{2}} (e^{\frac{pC}{2}} e^{pD} e^{\frac{pC}{2}})^\beta e^{\frac{pB}{2}} \right)^\alpha e^{\frac{pA}{2}} \right\}^{\frac{1}{p}}, \tag{6.1}$$

in particular,

$$e^{A+\alpha(B+C)} = \lim_{p \downarrow 0} \{ e^{\frac{pA}{2}} (e^{\frac{pB}{2}} e^{pC} e^{\frac{pB}{2}})^\alpha e^{\frac{pA}{2}} \}^{\frac{1}{p}}. \tag{6.2}$$

To prove the results in §8, we rewrite Lemma 6.1 in the following convenient form.

Lemma 6.1'. *If A, B, C and D are positive, then, for any positive numbers α and β,* (6.1') $$\log A + \alpha \log B + \alpha\beta(\log C + \log D) = \lim_{p\downarrow 0} \log\Big\{A^{\frac{p}{2}}\Big(B^{\frac{p}{2}}(C^{\frac{p}{2}}D^pC^{\frac{p}{2}})^{\beta}B^{\frac{p}{2}}\Big)^{\alpha}A^{\frac{p}{2}}\Big\}^{\frac{1}{p}}.$$

In particular,

(6.2') $$\log A + \alpha(\log B + \log C) = \lim_{p\downarrow 0} \log\{A^{\frac{p}{2}}(B^{\frac{p}{2}}C^pB^{\frac{p}{2}})^{\alpha}A^{\frac{p}{2}}\}^{\frac{1}{p}}.$$

Following analogous steps to those in the proof of Lemma 6.1, we can easily prove (6.7).

§7 Generalized Lie-Trotter formula, II

In this section, we present generalizations of the Lie-Trotter formulae different from those in Lemma 6.1 in §6 in order to prove the results in §8.

Lemma 7.1. *If A, B and C are Hermitian, then, for any $\alpha \in [0,1]$ and $r \geq 0$,*

(7.1) $$e^{r(1-\alpha)A+r\alpha B+C} = \lim_{p\downarrow 0}\Big(e^{\frac{pC}{2}}\big((1-\alpha)e^{pA} + \alpha e^{pB}\big)^r e^{\frac{pC}{2}}\Big)^{\frac{1}{p}}.$$

In particular,

(7.2) $$e^{(1-\alpha)A+\alpha B} = \lim_{p\downarrow 0}\big((1-\alpha)e^{pA} + \alpha e^{pB}\big)^{\frac{1}{p}}.$$

Lemma 7.1 can be rewritten as follows.

Lemma 7.1'. *If A, B and C are positive definite, then, for any $\alpha \in [0,1]$ and $r \geq 0$,*

(7.1') $$e^{r(1-\alpha)\log A + r\alpha \log B + \log C} = \lim_{p\downarrow 0}\Big(C^{\frac{p}{2}}\big((1-\alpha)A^p + \alpha B^p\big)^r C^{\frac{p}{2}}\Big)^{\frac{1}{p}}.$$

In particular,

(7.2') $$e^{(1-\alpha)\log A + \alpha \log B} = \lim_{p\downarrow 0}\big((1-\alpha)A^p + \alpha B^p\big)^{\frac{1}{p}}.$$

Next, we shall state an application of Lemma 7.1'. M.Fujii and R.Nakamoto [7] defined the chaotically α-geometric mean $A\Diamond_\alpha B$ which is different from the usual α-geometric mean $A\sharp_\alpha B$:

$$A\Diamond_\alpha B = e^{(1-\alpha)\log A + \alpha \log B}, \quad \text{for } A, B > 0 \text{ and } \alpha \in [0,1].$$

Among others, M.Fujii and R.Nakamoto [7] proved the following result.

Theorem I. *If A and B are strictly positive operators on a Hilbert space and $\alpha \in [0,1]$, then $(A^p\nabla_\alpha B^p)^{\frac{1}{p}}$,$(A^p\sharp_\alpha B^p)^{\frac{1}{p}}$ and $(A^p!_\alpha B^p)^{\frac{1}{p}}$ strongly converge to the chaotically α-geometric mean $A\Diamond_\alpha B$ as $p \downarrow 0$, where $S\nabla_\alpha T = (1-\alpha)S + \alpha T$ and $S!_\alpha T = ((1-\alpha)S^{-1} + \alpha T^{-1})^{-1}$ for strictly positive operators S and T.*

Two proofs to Theorem I are given in (Theorem 4, [7]) and (§4, [8]). We shall extend Theorem I as an application of Lemma 7.1, that is, we shall show that:

The chaotically $\alpha-$geometric mean $A\Diamond_\alpha B$ is the uniform limit of $(A^p\nabla_\alpha B^p)^{\frac{1}{p}}$.

Proposition 7.2. *If A and B are Hermitian and $\alpha \in [0,1]$, then $(e^{pA}\nabla_\alpha e^{pB})^{\frac{1}{p}}$,$(e^{pA}\sharp_\alpha e^{pB})^{\frac{1}{p}}$ and $(e^{pA}!_\alpha e^{pB})^{\frac{1}{p}}$ uniformly converge to $e^A\Diamond_\alpha e^B$ as $p \downarrow 0$.*

Proposition 7.2 can be rewritten as follows.

Proposition 7.2'. *If A and B are positive definite and $\alpha \in [0,1]$, then $(A^p\nabla_\alpha B^p)^{\frac{1}{p}}$,$(A^p\sharp_\alpha B^p)^{\frac{1}{p}}$ and $(A^p!_\alpha B^p)^{\frac{1}{p}}$ uniformly converge to the chaotically α-geometric mean $A\Diamond_\alpha B$ as $p \downarrow 0$.*

We remark that Proposition 7.2' remains valid for Hilbert space operators because Lemma 6.1 still remains valid for operators, so that Proposition 7.2' may be considered to be a strong version of Theorem I.

§8. Convergence of logarithmic trace inequalities via generalized Lie-Trotter formulae

In this section, We shall discuss the convergence of the logarithmic trace inequalities obtained in §5 by applying generalized Lie-Trotter formulae of Lemma 6.1' in §6 and the Lemma 7.1' in §7.

Theorem 8.1. *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$\begin{aligned}(8.1)\qquad &\frac{s}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] - \frac{1}{p}\mathrm{Tr}[A\log\{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}]\\ &\geq \mathrm{Tr}[A\log A]\end{aligned}$$

holds for any $p \geq 0$ and $s \geq 1$, and the left hand side converges to the right hand side as $p \downarrow 0$.

Theorem 8.1 yields the following Corollary 8.2.

Corollary 8.2.

(i) *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$(8.2)\qquad \frac{1}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] \geq \mathrm{Tr}[A\log A + A\log B]$$

holds and the left hand side converges to the right hand side as $p \downarrow 0$.

(ii) *If $A, B \geq 0$, then, for every $p \geq 0$,*

$$(8.3)\qquad \frac{2}{p}\mathrm{Tr}[A\log(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})] - \frac{1}{p}\mathrm{Tr}[A\log(B^pA^pB^p)]$$

$$\geq \mathrm{Tr}[A \log A]$$

holds and the left hand side converges to the right hand side as $p \downarrow 0$.

We remark that (i) of Corollary 8.2 is Theorem D.

Theorem 8.3. *If* $A > 0$ *and* $B \geq 0$, *then, for every positive number* β,

$$\frac{s}{p}\mathrm{Tr}[A \log(A^p \natural_\beta B^p)] - \frac{1}{p}\mathrm{Tr}[A \log\{A^{\frac{-p}{2}}(A^p \natural_\beta B^p)^s A^{\frac{-p}{2}}\}] \tag{8.4}$$
$$\geq \mathrm{Tr}[A \log A]$$

holds for any $p \geq 0$, $s \geq 1$, *and the left hand side converges to the right hand side as* $p \downarrow 0$.

Theorem 8.3 implies the following Corollary 8.4.

Corollary 8.4.

(i) *If* $A, B > 0$, *then, for every positive number* β,

$$\frac{1}{p}\mathrm{Tr}[A \log(A^p \natural_\beta B^p)] + \frac{\beta}{p}\mathrm{Tr}[A \log(A^{\frac{p}{2}} B^{-p} A^{\frac{p}{2}})] \tag{8.5}$$
$$\geq \mathrm{Tr}[A \log A]$$

holds for any $p \geq 0$, *and the left hand side converges to the right hand side as* $p \downarrow 0$.

(ii) *If* $A, B > 0$, *then, for every positive number* β,

$$\frac{2}{p}\mathrm{Tr}[A \log(A^p \natural_\beta B^p)] - \frac{1}{p}\mathrm{Tr}[A \log(A^{\frac{-p}{2}} B^p A^{\frac{-p}{2}})^\beta A^p (A^{\frac{-p}{2}} B^p A^{\frac{-p}{2}})^\beta] \tag{8.6}$$
$$\geq \mathrm{Tr}[A \log A]$$

holds for any $p \geq 0$ *and the left hand side converges to the right hand side as* $p \downarrow 0$.

We remark that, when $A \geq 0$, $B > 0$ and $\beta \in [0, 1]$, (i) of Corollary 8.4 becomes Theorem E.

Theorem 8.5. *If* $A > 0$ *and* $B \geq 0$, *then for every* $\alpha \in [0, 1]$

$$\frac{s}{p}\mathrm{Tr}[A \log(A^p \nabla_\alpha B^p)] - \frac{1}{p}\mathrm{Tr}[A \log\{A^{\frac{-p}{2}}(A^p \nabla_\alpha B^p)^s A^{\frac{-p}{2}}\}] \tag{8.7}$$
$$\geq \mathrm{Tr}[A \log A]$$

holds for any $p \geq 0$, $s \geq 1$, *and the left hand side converges to the right hand side as* $p \downarrow 0$.

Theorem 8.5 implies the following Corollary 8.6.

Corollary 8.6. *If* $A > 0$ *and* $B \geq 0$, *then*

$$\frac{1}{p}\mathrm{Tr}[A\log((1-\alpha)A^p+\alpha B^p)] - \frac{1}{p}\mathrm{Tr}[A\log\{(1-\alpha)I+A^{\frac{-p}{2}}B^pA^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A] \tag{8.8}$$

holds for any $p \geq 0$, $\alpha \in [0,1]$, and the left hand side converges to the right hand side as $p \downarrow 0$. Moreover,

$$\frac{s}{p}\mathrm{Tr}[A\log\frac{A^p+B^p}{2}] - \frac{1}{p}\mathrm{Tr}[A\log\{A^{\frac{-p}{2}}(\frac{A^p+B^p}{2})^sA^{\frac{-p}{2}}\}] \geq \mathrm{Tr}[A\log A] \tag{8.9}$$

holds for any $p \geq 0$, $s \geq 1$, and the left hand side converges to the right hand side as $p \downarrow 0$.

§9 Proofs of the results in §8

The results in §8 are shown by Theorem 5.1 and Lemma 6.1' and Lemma 7.1' and we omit them.

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This paper except Appendix is early announcement of the following paper:

T.Furuta, Convergence of logarithmic trace inequalities via generalized Lie-Trotter formulae, to appear in LAA.

§10 Appendix.

Inequalities associated with Umegaki relative entropy $S(A,B) = \mathrm{Tr}[A\log A - A\log B]$ and the relative operator entropy $\hat{S}(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ by J.I.Fujii and E.Kamei

On December 2, 2004 I have spoken this appendix in my talk at the Mathematics Research Insitute of Kyoto University.

A capital letter means $n \times n$ complex matrix and $\mathrm{Tr}[X]$ means the trace on the matrix X. A matrix X is said to be *strictly positive definite* if X is positive definite and invertible (denoted by $X > 0$). Let A and B be strictly positive definite matrices. Umegaki relative entropy $S(A,B)$ in [8] is defined by

$$S(A,B) = \mathrm{Tr}[A(\log A - \log B)] \tag{1.1}$$

and the relative operator entropy $\hat{S}(A|B)$ in [3] is defined by

$$\hat{S}(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \tag{1.2}$$

as an extension of [7]. Let $h > 1$. $S(p)$ is defined by

$$S(p) = \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}} \tag{1.3}$$

for any real number p. In particular $S(1) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$ is said to be *the Specht ratio* and $S(1) > 1$ is well known. We shall show the following inequalities associated with $S(A,B)$ and $-\mathrm{Tr}[\hat{S}(A|B)]$.

Theorem 1. *Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1M_2}{m_1m_2} > 1$. Then the following inequalities hold:*

$$\begin{aligned} \log S(1)\mathrm{Tr}[A] &+ S(A,B) \\ &\geq \log S(1)\mathrm{Tr}[A] + \mathrm{Tr}[A(\log \mathrm{Tr}[A] - \log \mathrm{Tr}[B])] \\ &\geq -\mathrm{Tr}[\hat{S}(A|B)] \\ &\geq S(A,B) \end{aligned} \tag{1.4}$$

in particular,

$$\log S(1)\mathrm{Tr}[A] + S(A,B) \geq -\mathrm{Tr}[\hat{S}(A|B)] \geq S(A,B). \tag{1.5}$$

The first inequality of (1.5) is *the reverse one* of the second inequality which is well known in [5],[6] and [1]. We prepare the following results to prove Theorem 1.

Proposition 2. *Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$. Let φ be a normalized positive linear functional on $M_n(C)$. Then*

$$\log S(1)\varphi(A) + \varphi(\hat{S}(A|B)) \geq \varphi(A)(\log\varphi(B) - \log\varphi(A)) \geq \varphi(\hat{S}(A|B)). \tag{1.6}$$

Proof. Let A and B be two matrices satisfying the hypotheses in Proposition 2. By (iii) of of [Theorem 2.1, [4]], if Φ is a normalized positive linear map from $M_n(C)$ into itself, then

$$\log S(1)\Phi(A) + \Phi(\hat{S}(A|B)) \geq \hat{S}(\Phi(A)|\Phi(B)) \geq \Phi(\hat{S}(A|B)) \tag{1.7}$$

and (1.6) follows from (1.7) since φ be a normalized positive linear functional on $M_n(C)$ and

$$\hat{S}(\varphi(A)|\varphi(B)) = \varphi(A)(\log\varphi(B) - \log\varphi(A)).\square$$

Proposition 3 (Peierls-Bogoliubov inequality). *The following* (i) *and* (ii) *hold and are equivalent:*

(i) $$\mathrm{Tr}e^{A+B} \geq \mathrm{Tr}e^{A}\exp\left(\frac{\mathrm{Tr}e^{A}B}{\mathrm{Tr}e^{A}}\right) \quad \textit{for Hermitian } A \textit{ and } B$$

and

(ii) $$S(A,B) \geq \mathrm{Tr}[A(\log\mathrm{Tr}[A] - \log\mathrm{Tr}[B])] \quad \textit{for } A > 0 \textit{ and } B > 0.$$

Peierls-Bogoliubov inequality is well known in statistic dynamics and the equivalence relation between (i) and (ii) is stated in ([6] and [2]).

Proposition 4 ([5],[6] and [1]) *The following inequality holds:*

$$-\mathrm{Tr}[\hat{S}(A|B)] \geq S(A,B) \quad \textit{for } A > 0 \textit{ and } B > 0. \tag{1.8}$$

Proof of Theorem 1. Let A and B be two matrices satisfying the hypotheses in Theorem 1 and recall that these hypotheses are the same as ones in Proposition 2. Put $\varphi(X) = \frac{1}{n}\mathrm{Tr}[X]$ in Proposition 2. Then the first inequality of (1.6) implies

$$\log S(1)\mathrm{Tr}[A] + \mathrm{Tr}[A(\log\mathrm{Tr}[A] - \log\mathrm{Tr}[B])] \geq -\mathrm{Tr}[\hat{S}(A|B)]. \tag{1.9}$$

Therefore we have

$$\begin{aligned} \log S(1)\mathrm{Tr}[A] &+ S(A,B) \\ &\geq \log S(1)\mathrm{Tr}[A] + \mathrm{Tr}[A(\log\mathrm{Tr}[A] - \log\mathrm{Tr}[B])] \quad \text{by (ii) of Proposition 3} \\ &\geq -\mathrm{Tr}[\hat{S}(A|B)] \quad \text{by (1.9)} \\ &\geq S(A,B) \quad \text{by (1.8) of Proposition 4} \end{aligned} \tag{1.4}$$

so the proof is complete since (1.5) is an immediate consequence of (1.4). $\square$

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Further extension of this appendix will appear in the following:

T.Furuta, Reverse inequalities involving two relative operator entropies and two relative entropies, to appear in LAA.