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Principal functions of operators

Department of Mathematics, Kanagawa University
Faculty of Education and Human Sciences, Niigata University

Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. About the trace formula, we have the following:

Theorem 1 (M. Krein, 1953). Let $A$ be a self-adjoint operator on $\mathcal{H}$ and $K$ be a trace class self-adjoint operator on $\mathcal{H}$. Then there exists a unique function $\delta(t)$ such that

$$
\text{Tr} \left( p(A + K) - p(A) \right) = \int p'(t)\delta(t)dt,
$$

where $p$ is a polynomial.

Theorem 2 (Carey-Pincus [5], Helton-Howe [11]). Let $T = X + iY$ be an operator on $\mathcal{H}$ with trace class self-commutator $([T^*, T] \in \mathcal{C}_1)$. Then there exists a function $g(x, y)$ such that

$$
\text{Tr} \left( [p(X, Y), q(X, Y)] \right) = \frac{1}{2\pi i} \int \int J(p, q)(x, y)g(x, y)dx dy,
$$

where $p$ and $q$ are polynomials of two variables.

Functions $\delta(t)$ and $g(x, y)$ in Theorems 1 and 2 are called the phase shift of the perturbation problem $A \rightarrow A + K$, and the (Cartesian) principal function of $T$, respectively. Let $T$ be hyponormal and satisfy $[T^*, T] \in \mathcal{C}_1$. For operators $A$ and $K$ of Theorem 1, let $A = TT^*$ and $K = T^*T - TT^*$. Then

$$
\delta(t) = \frac{1}{2\pi} \int_0^{2\pi} g(\sqrt{t}\cos\theta, \sqrt{t}\sin\theta)d\theta \quad \text{a.e. } t > 0.
$$

For the polar decomposition $T = U|T|$, we have the following:

Theorem 3 ([5],[8],[17]). Let $T = U|T|$ be semi-hyponormal operator satisfying $[|T|, U] \in \mathcal{C}_1$ with unitary $U$. Then there exists a function $g_T$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,

$$
\text{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drd\theta.
$$

Let $\mathcal{C}_1$ be the trace class and $\mathcal{A}$ be Laurent polynomials; $\mathcal{P}(r, z) = \sum_{k=-N}^N p_k(r)z^k$. Let $J(\mathcal{P}, \mathcal{Q})$ be the Jacobian of $\mathcal{P}, \mathcal{Q}$. 
Functions $g$ and $g_T$ of Theorems 2 and 3 are called the principal functions of $T$ related to the Cartesian decomposition $T = X + iY$ and the polar decomposition $T = U|T|$, respectively. We have two ways of the principal functions of $T$: One is by the Cartesian decomposition $T = X + iY$. In the case of a hyponormal operator $T$, by the mosaic $0 \leq B(x, y) \leq I$ we define

$$g(x, y) = \text{Tr}(B(x, y)).$$

Others is by the determining function:

$$\det \left( (X - z)(Y - w)(X - z)^{-1}(Y - w)^{-1} \right) = \exp \left( \frac{1}{2\pi i} \int \int g(x, y) \frac{dx}{x - z} \frac{dy}{y - w} \right).$$

Principal function gives many information of $T$.

If $T$ is hyponormal, then $g \geq 0$ and for sufficiently high $n$ the operator $T^n$ has a non-trivial invariant subspace (Berger [3], Martin-Putinar [14]). Other properties are

1. $z \not\in \sigma_e(T) \implies g(x, y) = -\text{ind}(T - z)$ ($z = x + iy$).

2. If $T$ has a cyclic vector, then $g \leq 1$.

3. If $T$ has a finite rational cyclic multiplicity $m$, then $g \leq m$.

Carey-Pincus [5, Th.7.1] showed a relation between $g(x, y)$ and $g_T(e^{i\theta}, r)$ as follows: For $x + iy = re^{i\theta}$,

$$g(x, y) = g_T(e^{i\theta}, r^2).$$

**Definition 1.** $T$ is $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$. Especially, $T$ is called hyponormal and semi-hyponormal if $p = 1$ and $p = 1/2$, respectively. Since

$$\text{hyponormal} \implies \text{semi-hyponormal} \implies p\text{-hyponormal},$$

if we have the principal functions of $p$-hyponormal operators and more weak operators, then we can study similar properties.

For an operator $T = U|T|$ let $T_t = |T|^tU|T|^{1-t} (0 < t < 1)$ be the Aluthge transformation of $T$. Let $g_T$ and $g_{T_t}$ be the principal functions of $T$ and $T_t = |T|^tU|T|^{1-t} (0 < t < 1)$, respectively. Then we have following problems:

1. $g_T = g_{T_t}$

2. If $T$ is hyponormal, then $g(x, y) = g_T(e^{i\theta}, r) = g_T(e^{i\theta}, r) = g_T(x + iy = re^{i\theta})$.

In this talk, we study relations above.
2. Relations with principal functions associated with polar decompositions

**Definition 2.** Let $T = U|T|$ be a $p$-hyponormal operator with unitary $U$ such that $[|T|^{2p}, U] \in \mathcal{C}_1$. Put $S = U|T|^{2p}$. Then $S$ is semi-hyponormal. By Theorem 3, there exists the principal function $g_S$ of $S$ and we define the principal function $g_T$ of $T$ by

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r^{\frac{1}{2p}})$$

(see [8, Definition 3]).

We prepare some lemmas.

**Lemma 4.** If operators $A, B, C$ satisfy $[A, C], [B, C] \in \mathcal{C}_1$, then we have $[AB, C] \in \mathcal{C}_1$.

Let $||A||_1 = \text{Tr}(|A|)$ for $A \in \mathcal{C}_1$, that is, $||A||_1$ is the trace norm of $A$.

**Lemma 5.** If a positive invertible operator $A$ and an operator $D$ satisfy $[A, D] \in \mathcal{C}_1$, then, for any real number $\alpha$, we have $[A^\alpha, D] \in \mathcal{C}_1$.

**Lemma 6.** Let $T = U|T|$ be an invertible operator. Put $T_t = |T|^{t}U|T|^{1-t}$. If $[|T|, U] \in \mathcal{C}_1$, then $[T_t^*, T_t] \in \mathcal{C}_1$.

**Lemma 7.** Let $T = U|T|$ be an invertible operator and $T_t = |T|^{t}U|T|^{1-t}$. For the polar decomposition $T_t = V|T_t|$ of $T_t$, if $[|T|, U] \in \mathcal{C}_1$, then, for every real number $\alpha$,

$$[|T_t|^\alpha, V] \in \mathcal{C}_1.$$ 

**Lemma 8.** Let $T = U|T|$ be an invertible operator and $T_t = |T|^{t}U|T|^{1-t}$. For the polar decomposition $T_t = V|T_t|$ of $T_t$, if $[|T|, U] \in \mathcal{C}_1$, then, for a positive integer $n$, it holds that

$$\text{Tr}([U^n|T|^n, |T|^2]) = \text{Tr}([V^n|T_t|^n, |T_t|^2]),$$

$$\text{Tr}([U^{-n}|T|^n, |T|^2]) = \text{Tr}([V^{-n}|T_t|^n, |T_t|^2])$$

and

$$\text{Tr}([U^*|T|, U|T|]) = \text{Tr}([V^*|T_t|, V|T_t|]).$$

Therefore, we have the following.

**Theorem 9.** Let $T = U|T|$ be an invertible semi-hyponormal operator such that $[|T|, U] \in \mathcal{C}_1$. For $T_t = |T|^{t}U|T|^{1-t}$, let $g_T$ and $g_{T_t}$ be the principal functions of $T$ and $T_t$, respectively. Then we have

$$g_T = g_{T_t}$$

almost everywhere on $\mathbb{C}$.

Next we recall the principal functions for log-hyponormal operators.
Definition 3. Let $T = U|T|$ be log-hyponormal with $\log |T| \geq 0$ such that $[\log |T|, U] \in \mathcal{C}_1$. Put $S = U \log |T|$. Then $S$ is semi-hyponormal with unitary $U$. Hence there exists the principal function $g_S$ of $S$ and we define the principal function $g_T$ of $T$ by

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, \log r)$$

(see [6, Definition 4]).

It is known that, if $T = U|T|$ is log-hyponormal, then the Aluthge transformation $T_{1/2} = |T|^{1/2}U|T|^{1/2}$ is semi-hyponormal (see [16]). Hence there exists the principal function $g_{T_{1/2}}$ of $T_{1/2}$.

Then we have the following.

Theorem 10. Let $T = U|T|$ be a log-hyponormal operator such that $\log |T| \geq 0$ and $[\log |T|, U] \in \mathcal{C}_1$. For $T_{1/2} = |T|^{1/2}U|T|^{1/2} = V|T_{1/2}|$, let $g_T$ and $g_{T_{1/2}}$ be the principal functions of $T$ and $T_{1/2}$, respectively. Then we have

$$g_T = g_{T_{1/2}}$$

almost everywhere on $\mathbb{C}$.

Now we generalize Theorem 9 as follows.

Theorem 11. Let $T = U|T|$ be an invertible $p$-hyponormal operator such that $[|T|, U] \in \mathcal{C}_1$. For $T_t = |T|^tU|T|^{1-t}$, let $g_T$ and $g_{T_t}$ be the principal functions of $T$ and $T_t$, respectively. Then we have $g_T = g_{T_t}$ almost everywhere on $\mathbb{C}$.

3. Relation with principal functions associated with two decompositions

Next, we show the following theorem (cf. [5, Theorem 7.1]).

Theorem 12. Let $T = X + iY = U|T|$ be hyponormal with unitary $U$. Suppose that $[|T|, U] \in \mathcal{C}_1$. Let $g$ and $g_T$ be the principal functions corresponding to the Cartesian and the polar decompositions of $T$, respectively. For $x + iy = re^{i\theta}$, let $g_T(x, y) = g_T(e^{i\theta}, r)$. Then $g = g_T$ almost everywhere on $\mathbb{C}$.

Though the results above can be generalized to operators with trace-class self-commutator, we confine ourselves to deal only with the $p$-hyponormal case (cf. [5]).

4. Application: Berger’s Theorem and index

In this section, we apply previous results to Berger’s Theorem [3] and an index property [11]. First we show the following:

Lemma 13. Let operators $S = V|S|$ and $T = U|T|$ be invertible. Assume that $|S| - |S^*|$, $|T| - |T^*| \in \mathcal{C}_1$ and there exists a trace class operator $A$ such that $SA = AT$ and $\ker(A) = \ker(A^*) = \{0\}$. Then, for $S_{1/2} = |S|^{1/2}V|S|^{1/2}$ and $T_{1/2} = |T|^{1/2}U|T|^{1/2}$, there exists $B \in \mathcal{C}_1$ such that $S_{1/2}B = BT_{1/2}$ and $\ker(B) = \ker(B^*) = \{0\}$.
Lemma 14. Let $S$ and $T$ be invertible semi-hyponormal operators. Assume that $|S| - |S^*|, |T| - |T^*| \in \mathcal{C}_1$ and there exists a trace class operator $A$ such that $SA = AT$ and $\ker(A) = \ker(A^*) = \{0\}$. Then $g_S \leq g_T$ almost everywhere on $\mathbb{C}$.

Corollary 15. Let $S$ and $T$ be invertible $p$-hyponormal or log-hyponormal operators. Assume that $|S| - |S^*|, |T| - |T^*| \in \mathcal{C}_1$ and there exists a trace class operator $A$ such that $SA = AT$ and $\ker(A) = \ker(A^*) = \{0\}$. Then $g_S \leq g_T$ almost everywhere on $\mathbb{C}$.

Theorem 16. Let $T = U|T|$ be an invertible cyclic $p$-hyponormal operator. Assume $|[T], U| \in \mathcal{C}_1$. Then $g_T \leq 1$ almost everywhere on $\mathbb{C}$.

Let $\text{Rat}(\sigma)$ be the set of all rational functions with poles off $\sigma$.

Definition 4. The rational multiplicity of $T \in \mathcal{B}(\mathcal{H})$ is the smallest cardinal number $m$ with the property which there exists a set $\{x_n\}_{n=1}^m$ of $m$-vectors in $\mathcal{H}$ such that
$$\bigvee\{f(T)x_i : f \in \text{Rat}(\sigma(T)), 1 \leq i \leq m\} = \mathcal{H}.$$ 

Also, we have

Theorem 17. Let $T = U|T|$ be an invertible cyclic $p$-hyponormal operator with finite rational cyclic multiplicity $m$. Assume $|[T], U| \in \mathcal{C}_1$. Then $g_T \leq m$ almost everywhere on $\mathbb{C}$.

Finally, we show index properties. Let $\sigma_e(T)$ be the essential spectrum of $T$ and $\text{ind}(T)$ the index of $T$; i.e.,
$$\text{ind}(T) = \dim \ker(T) - \dim \ker(T^*).$$

Then it is known the following result. Let $T$ be a pure hyponormal operator and $g(z)$ be the principal function of $T$. Then it holds that, for $z \notin \sigma_e(T)$,
$$g(z) = -\text{ind}(T - z)$$
[11, Theorem] (see also [4, Theorem 4]).

An operator $T$ is called pure if it has no nontrivial reducing subspace on which it is normal. Then we need the following

Lemma 18 (Lemma 4 of [7]). For an operator $T = U|T|$, let $T_{1/2} = |T|^{1/2}U|T|^{1/2}$. Assume that $T$ is an invertible $p$-hyponormal operator. If $T$ is pure, then $T_{1/2}$ is also pure.

Finally, we have the following.
Theorem 19. Let $T = U|T|$ be a pure invertible $p$-hyponormal operator. If $0 \neq z \notin \sigma_e(T)$, then $g_T(e^{i\theta}, r) = -\text{ind}(T - z)$, where $z = re^{i\theta}$.

REFERENCES


Muneo Chô
Department of Mathematics
Kanagawa University
Yokohama 221-8686, JAPAN
E-mail: chiyom01@kanagawa-u.ac.jp

Tadasi HURUYA
Faculty of Education and Human Sciences
Niigata University
Niigata 950-2181, JAPAN
E-mail: huruya@ed.niigata-u.ac.jp