数理解析研究所講究録 1427 巻 2005 年 137-142

Principal functions of operators

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Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. About the trace formula, we have the following:

Theorem 1 (M. Krein, 1953). *Let A be a self-adjoint operator on $\mathcal{H}$ and K be a trace class self-adjoint operator on $\mathcal{H}$. Then there exists a unique function $\delta(t)$ such that*

$$\mathrm{Tr}\Big(p(A+K)-p(A)\Big)=\int p'(t)\delta(t)dt,$$

where p is a polynomial.

Theorem 2 (Carey-Pincus [5], Helton-Howe [11]). *Let $T=X+iY$ be an operator on $\mathcal{H}$ with trace class self-commutator ($[T^*,T]\in\mathcal{C}_1$). Then there exists a function $g(x,y)$ such that*

$$\mathrm{Tr}\Big([p(X,Y),q(X,Y)]\Big)=\frac{1}{2\pi i}\int\int J(p,q)(x,y)g(x,y)dx\,dy,$$

where p and q are polynomials of two variables.

Functions $\delta(t)$ and $g(x,y)$ in Theorems 1 and 2 are called the phase shift of the perturbation problem $A\to A+K$, and the (Cartesian) principal function of T, respectively. Let T be hyponormal and satisfy $[T^*,T]\in\mathcal{C}_1$. For operators A and K of Theorem 1, let $A=TT^*$ and $K=T^*T-TT^*$. Then

$$\delta(t)=\frac{1}{2\pi}\int_0^{2\pi}g(\sqrt{t}\cos\theta,\sqrt{t}\sin\theta)d\theta \quad \text{a.e. } t>0.$$

For the polar decomposition $T=U|T|$, we have the following:

Theorem 3 ([5],[8],[17]). *Let $T=U|T|$ be semi-hyponormal operator satisfying $[|T|,U]\in\mathcal{C}_1$ with unitary U. Then there exists a function g_T such that, for $\mathcal{P},\mathcal{Q}\in\mathcal{A}$,*

$$\mathrm{Tr}([\mathcal{P}(|T|,U),\mathcal{Q}(|T|,U)])=\frac{1}{2\pi}\int\int J(\mathcal{P},\mathcal{Q})(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)drd\theta.$$

Let $\mathcal{C}_1$ be the trace class and $\mathcal{A}$ be Laurent polynomials; $\mathcal{P}(r,z)=\sum_{k=-N}^{N}p_k(r)z^k$. Let $J(\mathcal{P},\mathcal{Q})$ be the Jacobian of $\mathcal{P},\mathcal{Q}$.

Functions g and g_T of Theorems 2 and 3 are called *the principal functions* of T related to the Cartesian decomposition $T = X + iY$ and the polar decomposition $T = U|T|$, respectively. We have two ways of the principal functions of T: One is by the Cartesian decomposition $T = X + iY$. In the case of a hyponormal operator T, by the mosaic $0 \leq B(x,y) \leq I$ we define

$$g(x,y) = \mathrm{Tr}(B(x,y)).$$

Others is by the determining function:

$$\det\Big((X-z)(Y-w)(X-z)^{-1}(Y-w)^{-1}\Big) = \exp\left(\frac{1}{2\pi i}\int\int g(x,y)\frac{dx}{x-z}\frac{dy}{y-w}\right).$$

Principal function gives many information of T.

If T is hyponormal, then $g \geq 0$ and for sufficiently high n the operator T^n has a non-trivial invariant subspace (Berger [3], Martin-Putinar [14]). Other properties are

(1) $z \notin \sigma_e(T) \implies g(x,y) = -\,\mathrm{ind}(T-z) \quad (z = x+iy)$.

(2) If T has a cyclic vector, then $g \leq 1$.

(3) If T has a finite rational cyclic multiplicity m, then $g \leq m$.

Carey-Pincus [5, Th.7.1] showed a relation between $g(x,y)$ and $g_T(e^{i\theta}, r)$ as follows: For $x + iy = re^{i\theta}$,

$$g(x,y) = g_T(e^{i\theta}, r^2).$$

Definition 1. T is p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. Especially, T is called hyponormal and semi-hyponormal if $p = 1$ and $p = 1/2$, respectively. Since

$$\text{hyponormal} \implies \text{semi-hyponormal} \implies p\text{-hyponormal},$$

if we have the principal functions of p-hyponormal operators and more weak operators, then we can study similar properties.
For an operator $T = U|T|$ let $T_t = |T|^t U |T|^{1-t}$ $(0 < t < 1)$ be the Aluthge transformation of T. Let g_T and g_{T_t} be the principal functions of T and $T_t = |T|^t U|T|^{1-t}$ $(0 < t < 1)$, respectively. Then we have following problems:
(1) $g_T = g_{T_t}$
(2) If T is hyponormal, then $g(x,y) = g_T(e^{i\theta}, r)$ for $x + iy = re^{i\theta}$.

In this talk, we study relations above.

2. Relations with principal functions associated with polar decompositions

Definition 2. Let $T = U|T|$ be a p-hyponormal operator with unitary U such that $[|T|^{2p}, U] \in \mathcal{C}_1$. Put $S = U|T|^{2p}$. Then S is semi-hyporomal. By Theorem 3, there exists the principal function g_S of S and we define the principal function g_T of T by

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r^{\frac{1}{2p}})$$

(see [8, Definition 3]).

We prepare some lemmas.

Lemma 4. *If operators A, B, C satisfy $[A, C], [B, C] \in \mathcal{C}_1$, then we have* $[AB, C] \in \mathcal{C}_1$.

Let $||A||_1 = \mathrm{Tr}(|A|)$ for $A \in \mathcal{C}_1$, that is, $||A||_1$ is the trace norm of A.

Lemma 5. *If a positive invertible operator A and an operator D satisfy $[A, D] \in \mathcal{C}_1$, then, for any real number α, we have*

$$[A^\alpha, D] \in \mathcal{C}_1.$$

Lemma 6. *Let $T = U|T|$ be an invertible operator. Put $T_t = |T|^t U |T|^{1-t}$. If $[|T|, U] \in \mathcal{C}_1$, then $[T_t^*, T_t] \in \mathcal{C}_1$.*

Lemma 7. *Let $T = U|T|$ be an invertible operator and $T_t = |T|^t U |T|^{1-t}$. For the polar decomposition $T_t = V|T_t|$ of T_t, if $[|T|, U] \in \mathcal{C}_1$, then, for every real number α,*

$$[|T_t|^\alpha, V] \in \mathcal{C}_1.$$

Lemma 8. *Let $T = U|T|$ be an invertible operator and $T_t = |T|^t U |T|^{1-t}$. For the polar decomposition $T_t = V|T_t|$ of T_t, if $[|T|, U] \in \mathcal{C}_1$, then, for a positive integer n, it holds that*

$$\mathrm{Tr}([U^n|T|^n, |T|^2]) = \mathrm{Tr}([V^n|T_t|^n, |T_t|^2]),$$

$$\mathrm{Tr}([U^{-n}|T|^n, |T|^2]) = \mathrm{Tr}([V^{-n}|T_t|^n, |T_t|^2])$$

and

$$\mathrm{Tr}([U^*|T|, U|T|]) = \mathrm{Tr}([V^*|T_t|, V|T_t|]).$$

Therefore, we have the following.

Theorem 9. *Let $T = U|T|$ be an invertible semi-hyponormal operator such that $[|T|, U] \in \mathcal{C}_1$. For $T_t = |T|^t U |T|^{1-t}$, let g_T and g_{T_t} be the principal functions of T and T_t, respectively. Then we have*

$$g_T = g_{T_t}$$

almost everywhere on $\mathbf{C}$.

Next we recall the principal functions for log-hyponormal operators.

Definition 3. Let $T = U|T|$ be log-hyponormal with $\log|T| \geq 0$ such that $[\log|T|, U] \in \mathcal{C}_1$. Put $S = U\log|T|$. Then S is semi-hyponormal with unitary U. Hence there exists the principal function g_S of S and we define the principal function g_T of T by

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, \log r)$$

(see [6, Definition 4]).

It is known that, if $T = U|T|$ is log-hyponormal, then the Aluthge transformation $T_{1/2} = |T|^{1/2}U|T|^{1/2}$ is semi-hyponormal (see [16]). Hence there exists the principal function $g_{T_{1/2}}$ of $T_{1/2}$.

Then we have the following.

Theorem 10. *Let $T = U|T|$ be a log-hyponormal operator such that $\log|T| \geq 0$ and $[\log|T|, U] \in \mathcal{C}_1$. For $T_{1/2} = |T|^{1/2}U|T|^{1/2} = V|T_{1/2}|$, let g_T and $g_{T_{1/2}}$ be the principal functions of T and $T_{1/2}$, respectively. Then we have*

$$g_T = g_{T_{1/2}}$$

almost everywhere on $\mathbf{C}$.

Now we generalize Theorem 9 as follows.

Theorem 11. *Let $T = U|T|$ be an invertible p-hyponormal operator such that $[|T|, U] \in \mathcal{C}_1$. For $T_t = |T|^t U|T|^{1-t}$, let g_T and g_{T_t} be the principal functions of T and T_t, respectively. Then we have $g_T = g_{T_t}$ almost everywhere on* $\mathbf{C}$.

3. Relation with principal functions associated with two decompositions

Next, we show the following theorem (cf. [5, Theorem 7.1]).

Theorem 12. *Let $T = X + iY = U|T|$ be hyponormal with unitary U. Suppose that $[|T|, U] \in \mathcal{C}_1$. Let g and g_T be the principal functions corresponding to the Cartesian and the polor decompositions of T, respectively. For $x + iy = re^{i\theta}$, let $g_T(x, y) = g_T(e^{i\theta}, r)$. Then $g = g_T$ almost everywhere on* $\mathbf{C}$.

Though the results above can be generalized to operators with trace-class self-commutator, we confine ourselves to deal only with the p-hyponormal case (cf. [5]).

4. Application: Berger's Theorem and index

In this section, we apply previous results to Berger's Theorem [3] and an index property [11]. First we show the following:

Lemma 13. *Let operators $S = V|S|$ and $T = U|T|$ be invertible. Assume that $|S| - |S^*|$, $|T| - |T^*| \in \mathcal{C}_1$ and there exists a trace class operator A such that $SA = AT$ and $\ker(A) = \ker(A^*) = \{0\}$. Then, for $S_{1/2} = |S|^{\frac{1}{2}}V|S|^{\frac{1}{2}}$ and $T_{1/2} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, there exists $B \in \mathcal{C}_1$ such that $S_{1/2}B = BT_{1/2}$ and $\ker(B) = \ker(B^*) = \{0\}$.*

Lemma 14. *Let S and T be invertible semi-hyponormal operators. Assume that $|S|-|S^*|, |T|-|T^*| \in \mathcal{C}_1$ and there exists a trace class operator A such that $SA = AT$ and $\ker(A) = \ker(A^*) = \{0\}$. Then $g_S \leq g_T$ almost everywhere on $\mathbf{C}$.*

Corollary 15. *Let S and T be invertible p-hyponormal or* log*-hyponormal operators. Assume that $|S|-|S^*|, |T|-|T^*| \in \mathcal{C}_1$ and there exists a trace class operator A such that $SA = AT$ and $\ker(A) = \ker(A^*) = \{0\}$. Then $g_S \leq g_T$ almost everywhere on $\mathbf{C}$.*

Theorem 16. *Let $T = U|T|$ be an invertible cyclic p-hyponormal operator. Assume $[|T|, U] \in \mathcal{C}_1$. Then*

$$g_T \leq 1 \quad \textit{almost everywhere on } \mathbf{C}.$$

Let $\mathbf{Rat}(\sigma)$ be the set of all rational functions with poles off σ.

Definition 4. The *rational multiplicity* of $T \in B(\mathcal{H})$ is the smallest cardinal number m with the property which there exists a set $\{x_n\}_{n=1}^m$ of m-vectors in $\mathcal{H}$ such that

$$\bigvee\{\, f(T)x_i \ ; \ f \in \mathbf{Rat}(\sigma(T)),\ 1 \leq i \leq m \,\} = \mathcal{H}.$$

Also, we have

Theorem 17. *Let $T = U|T|$ be an invertible cyclic p-hyponormal operator with finite rational cyclic multiplicity m. Assume $[|T|, U] \in \mathcal{C}_1$. Then*

$$g_T \leq m \quad \textit{almost everywhere on } \mathbf{C}.$$

Finally, we show index properties. Let $\sigma_e(T)$ be the essential spectrum of T and $\operatorname{ind}(T)$ the index of T; i.e.,

$$\operatorname{ind}(T) = \dim\ker(T) - \dim\ker(T^*).$$

Then it is known the following result. Let T be a pure hyponormal operator and $g(z)$ be the principal function of T. Then it holds that, for $z \notin \sigma_e(T)$,

$$g(z) = -\operatorname{ind}(T - z)$$

[11, Theorem] (see also [4, Theorem 4]).

An operator T is called *pure* if it has no nontrivial reducing subspace on which it is normal. Then we need the following

Lemma 18 (Lemma 4 of [7]). *For an operator $T = U|T|$, let $T_{1/2} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Assume that T is an invertible p-hyponormal operator. If T is pure, then $T_{1/2}$ is also pure.*

Finally, we have the following.

Theorem 19. *Let $T = U|T|$ be a pure invertible p-hyponormal operator. If $0 \neq z \notin \sigma_e(T)$, then $g_T(e^{i\theta}, r) = -\mathrm{ind}(T - z)$, where $z = re^{i\theta}$.*

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