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# GAPS ON THE CLASSES OF COMPOSITION OPERATORS

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ABSTRACT. There are many operator classes between quasinormal and spectraloid operators. These include quasinormal, $\infty$-hyponormal, $p$-hyponormal, $p$-quasihyponormal, absolute $p$-paranormal, $p$-paranormal, normaloid, and spectraloid. In this article, we discuss measure theoretic composition operators in these classes.

**1. Introduction.** The main theorems and examples of this article will be appeared in some other journals (with coauthors, C. Burnap and A. Lambert). First we review some definitions of operators which will be discussed here. Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Let $A = U|A|$ be the canonical polar decomposition for $A \in \mathcal{L}(\mathcal{H})$ and let $p \in (0,\infty)$. An operator $A$ is *p-hyponormal* if $(A^*A)^p \geq (AA^*)^p$. And $A$ is $\infty$-*hyponormal* if $A$ is $p$-hyponormal for all $p \in (0,\infty)$ ([13]). An operator $A$ is *p-quasihyponormal* if $A^*(A^*A)^pA \geq A^*(AA^*)^pA$. For all unit vectors $x \in \mathcal{H}$, if $\| \, |A|^p U \, |A|^p x\| \geq \| \, |A|^p x\|^2$, then $A$ is called a *p-paranormal operator*. In particular, 1-paranormal is referred to as paranormal. And also an operator $A$ is of *A(p)-class* if $(A^* \, |A|^{2p} A)^{1/(p+1)} \geq |A|^2$, and *absolute-p-paranormal operator* if $\| |A|^p Ax\| \geq \|Ax\|^{p+1}$ for all unit vectors $x$ in $\mathcal{H}$. Note that absolute-1-paranormal is the same as 1-paranormal. Let $\widetilde{A} := |A|^{1/2} \, U \, |A|^{1/2}$ be the *Aluthge transform* of $A$. Then $A$ is *w-hyponormal* if $|\widetilde{A}| \geq |A|$ ([3], [4]). An operator $A$ is *normaloid* if $\|A\| = r(A)$, where $r(A)$ is the spectral radius of $A$, which is equivalent to the condition $\|A^n\| = \|A\|^n$ for all natural numbers $n$ (see [8], p.100). An operator $A$ is *spectraloid* if $w(A) = r(A)$, where $w(A)$ is the numerical radius of $A$.

There are several well known relationships among these classes ([8]). The ones of concern in this article are as follows: $p$-hyponormal $\Rightarrow$ $p$-quasihyponormal $\Rightarrow$ $A(p)$-class operator $\Rightarrow$ absolute-$p$-paranormal $\Rightarrow$ normaloid $\Rightarrow$ spectraloid, $(p > 0)$; absolute-$p$-paranormal $\Rightarrow$ $p$-paranormal $(p \geq 1)$; $p$-paranormal $\Rightarrow$ absolute-$p$-paranormal $(0 < p < 1)$; $w$-hyponormal $\Rightarrow$ $\frac{1}{2}$-paranormal (Example 2.11 shows this implication cannot be strengthened). For $0 < p < q$, if $T$ is $p$-paranormal, then $A$ is $q$-paranormal. All the other $p$-properties except $p$-hyponormality share this type of implication. For $p$-hyponormality, the implication is reversed: if $A$ is $q$-hyponormal, then $A$ is $p$-hyponormal.

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In particular, we show that composition operators can separate the $p$-hyponormal, $w$-hyponormal, $p$-paranormal, normaloid, and spectraloid classes. They cannot, however, be used to separate the $p$-quasihyponormal, $A(p)$-class, absolute $p$-paranormal, or $p$-paranormal classes (see Theorem 3.1).

Before giving our results, we briefly review some essential notation and background information on composition operators. Let $(X, \mathcal{F}, \mu)$ be a $\sigma$ finite measure space and let $T : X \to X$ be a transformation such that $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. We assume that the Radon-Nikodym derivative $h = d\mu \circ T^{-1}/d\mu$ is in $L^\infty$ and we define $h_n = d\mu \circ T^{-n}/d\mu$. The composition operator $C$ acting on $L^2 := L^2(X, \mathcal{F}, \mu)$ is defined by $Cf = f \circ T$. The condition $h \in L^\infty$ assures that $C$ is bounded. We denote the conditional expectation of $f$ with respect to $T^{-1}\mathcal{F}$ by $Ef = E(f|T^{-1}\mathcal{F})$. We recall some known results from [11], [12], and [9], which will be used frequently through this article. Every $T^{-1}\mathcal{F}$ measurable function has the form $F \circ T$ (hence $Ef$ is of this form). Note that $F \circ T = G \circ T$ if and only if $hF = hG$; in fact, $F \circ T \geq G \circ T$ if and only if $F\chi_S \geq G\chi_S$ where $S =$ support $h$ and $\chi_S$ is the characteristic function of $S$ ([5]). It is known that $C^*f = h(Ef) \circ T^{-1}$ (the previous two properties show that this expression is well-defined) and $h \circ T > 0$ a.e. In this work, we used certain properties of the conditional expectation operator $E$: $E = E(\cdot|T^{-1}\mathcal{F})$ is the self adjoint projection onto $L^2(X, T^{-1}\mathcal{F}, \mu)$. For any $T^{-1}\mathcal{F}$ set $A$ and $L^2$ function $f$, $\int_A f d\mu = \int_A E(f) d\mu$. For $T^{-1}\mathcal{F}$ measurable $a$ and $\mathcal{F}$ measurable $f$, $E(af) = aE(f)$. The interested reader can find a more extensive list of properties for conditional expectations in [14].

**2. p-hyponormality.** To establish a characterization of $p$-hyponormality for $p \in (0, \infty)$, we first examine the operators $(C^*C)^p$ and $(CC^*)^p$.

**Lemma 2.1.** $(C^*C)^p f = h^p f$ and $(CC^*)^p f = (h^p \circ T)Ef$.

It was shown in [9] that if $C$ is hyponormal, then $h > 0$ a.e.. Below, we show that this remains valid for $p$-hyponormal composition operators. We will show here that this conclusion is not justified if $C$ is assumed to be only weakly hyponormal. It will be convenient to establish two general (not composition operator specific) results. The results in the following proposition were proved by J. Herron as part of his doctoral dissertation, currently under preparation. As noted earlier, for any nonnegative function $f$, support $f \subset$ support $Ef^r$ for any $r > 0$. For this reason we adopt the notational convention of writing expressions such as $f/(Ef^r)$ for $[f/(Ef^r)]\chi_{\text{supp} f}$. In some of the more involved calculations we will display the appropriate characteristic functions where there would otherwise be zero division problems.

**Proposition 2.2 ([10]).** *Let $E = E(\cdot|\mathcal{A})$ and let $\phi$ be a nonnegative $\mathcal{F}$-measurable function.*

(i) *Define the positive operator* $P_\phi$ *by* $P_\phi f = \phi E(\phi f)$. *Let* $\widehat{\phi} = \phi/(E(\phi^2))^{1/4}$. *Then* $P_\phi^{1/2} = P_{\widehat{\phi}}$.

(ii) *Define the operator* $R_\phi$ *by* $R_\phi f = E(\phi f)$. *Then* $\|R_\phi\| = \|\sqrt{E(\phi^2)}\|_\infty$.

**Lemma 2.3.** *Let* $\alpha$ *and* $\beta$ *be nonnegative functions, with* $S =$ support $\alpha$. *Then the following are equivalent.*

(i) *for any* $f \in L^2(X, \mathcal{F}, \mu)$, $\int_X \alpha|f|^2 d\mu \geq \int_X |E(\beta f|\mathcal{A})|^2 d\mu$.

(ii) supp$\beta \subset S$ *and* $E(\frac{\beta^2}{\alpha}\chi_S|\mathcal{A}) \leq 1$ a.e..

Here is our classification of $p$-hyponormal composition operators:

**Theorem 2.4.** *$C$ is $p$-hyponormal if and only if* $h > 0$ *and* $E(1/h^p) \leq 1/(h^p \circ T)$.

As mentioned earlier, D. Harrington and R. Whitley showed in [9] that $C$ is quasinormal if and only if $h \circ T = h$. But, quasinormality $\Rightarrow$ $\infty$-hyponormality $\Rightarrow$ $p$-hyponormality for all $p \in (0, \infty)$. It is not too surprising then, that the measure-theoretic characterization of this class clearly exhibits this lineage.

**Theorem 2.5.** *$C$ is $\infty$-hyponormal if and only if* $h \circ T \leq h$.

Our final general classification regards weak hyponormality and some of its generalizations. For a function $w$, define the linear transformation $W$ by $Wf = w(f \circ T)$. The transformation $W$ is called a *weighted composition operator*. We will make use of several properties of such operators. Detailed analysis of these operators is found in [5], [6], and [7].

**Proposition 2.6 ([5]).** *For* $w \geq 0$,

(i) $W^*Wf = h \cdot [E(w^2)] \circ T^{-1} f$.

(ii) $WW^*f = w \cdot h \circ T \; E(wf)$.

It follows from the preceding proposition that $|W|f = \sqrt{h \cdot [E(w^2)] \circ T^{-1}} f$. As for $|W^*|$, note that $WW^*f = w \cdot h \circ T \; E(wf) = w \cdot \sqrt{h \circ T} \; E(w\sqrt{h \circ T} \; f)$; i.e., with the notation from Herron's proposition, $WW^* = P_{w\sqrt{h \circ T}}$. We then have $|W^*| = P_v$, where $v = \frac{w\sqrt{h \circ T}}{[E(w\sqrt{h \circ T})^2]^{1/4}}$.

**Theorem 2.7.** *Let* $W$ *be a weighted composition operator with weight* $w \geq 0$, *and let* $S$ *be the support of* $h$.

(i) $|W| \geq |C|$ *if and only if* $E(w^2) \geq 1$.

(ii) $|C| \geq |W^*|$ *if and only if* support $w \subset$ support $h$ *and* $E(\frac{v^2}{\sqrt{h}}\chi_S) \leq 1$.

A tool which has been of considerable use in operator theory in recent years is the *Aluthge transform* ([1], [2]): For any operator $A$, let $A = U|A|$ be the canonical polar decomposition for A. The Aluthge transform of $A$ is the operator $\widetilde{A}$ given by $\widetilde{A} = |A|^{1/2}U|A|^{1/2}$. More generally, we may form the family of operators $\{A_r : 0 < r \leq 1\}$, where $A_r = |A|^r U|A|^{1-r}$ ([2]). Our first task in this context is to calculate these entities for a composition operator $C$. One may easily verify that the parts of the polar decomposition $U$, $|C|$ for $C$ are given by

$$|C|f = \sqrt{h}f \ ; \ \ Uf = \frac{1}{\sqrt{h\circ T}} f\circ T.$$

This is valid for all composition operators, even if $h$ vanishes on a set of positive measure. We then have

$$C_r f = h^{r/2}U(h^{(1-r)/2}f) = \frac{h^{r/2}(h^{(1-r)/2}\circ T)}{\sqrt{h\circ T}} f\circ T = \left(\frac{h}{h\circ T}\right)^{r/2} f\circ T.$$

We see then that $C_r$ is a weighted composition operator with weight $w_r = (h/h\circ T)^{r/2}$. We now catalog the pieces needed for our analysis:

$$w_r = (\frac{h}{h\circ T})^{r/2} \ ; \ \ |C_r|f = \left(\sqrt{h\cdot [E(w_r^2)]\circ T^{-1}}\right)\cdot f.$$

$$v_r = \frac{w_r\sqrt{h\circ T}}{[E(w_r\sqrt{h\circ T})^2]^{1/4}} \ ; \ \ |(C_r)^*|f = P_{v_r}f = v_r E(v_r f).$$

Note that support $v_r$ = support $w_r$ = support $h$, which we denote by $S$.

Our immediate goal is to characterize *weakly hyponormal* composition operators. An operator $A$ is defined to be weakly hyponormal if $|\widetilde{A}| \geq |A| \geq |(\widetilde{A})^*|$ ([3], [4]). We will actually obtain characterizations for the more general situation $|C_r| \geq |C| \geq |C_r^*|$. If these inequalities hold, we say that $C$ is *r-weakly hyponormal.* (Note that $\widetilde{C} = C_{1/2}$. so that weak hyponormality coincides with (1/2)-weak hyponormality.)

**Theorem 2.8.**
(i) $|C_r| \geq |C| \Longleftrightarrow [Eh^r] \geq h^r\circ T$.
(ii) $|C| \geq |C_r^*| \Longleftrightarrow [E(h^{r-1/2}\chi_S)]^2\ h^{1-r}\circ T \leq Eh^r$.

*Remark.* In the expression $[E(h^{r-1/2}\chi_S)]^2$ from (ii) of the preceding theorem, the appearance of $\chi_S$ is only needed if $r \leq 1/2$. This apparent split in the theory at weak $(r = 1/2)$ hyponormality might be a point for further development.

Of special interest is the case $r = 1/2$, that is to say, the weakly hyponormal case:

**Corollary 2.9.** *$C$ is weakly hyponormal if and only if $\sqrt{h\circ T} \leq E\sqrt{h}$.*

To our knowledge, the invariant subspace problem remains open for composition operators. Of course, if we are dealing with a finite measure space then the constant function 1 is an eigenvector for the composition operator. If $T^{-1}\mathcal{F} \neq \mathcal{F}$, then the closure of the range of the composition operator $C$ is a nontrivial invariant subspace for $C$. Also, if $S = \text{support } h \neq X$, then for any set $A \subset X \sim S$ with $0 < \mu(T^{-1}A) < \infty$, $\chi_{T^{-1}A}$ is a nonzero member of the kernel of $C^*$. These considerations allow us to make a small contribution to the sought general solution of the invariant subspace problem for composition operators:

**Corollary 2.10.** *Suppose that the composition operator $C$ is $r$-weakly hyponormal for some $r \in (0,1)$. Then $C$ has a nontrivial invariant subspace.*

Now we show that composition operators provide examples precisely marking the distinctions between the different partial normality classes. This is especially noteworthy because composition operators are often viewed as somewhat generalized weighted shifts, and weighted shifts have long been used to concretely illustrate various operator traits, from compact and quasinilpotent to hyponormal and subnormal. Shifts, however prove to be essentially useless in the exploration of $p$-hyponormality; indeed, all levels of hyponormality (but not subnormality) for a weighted shift hold together or not at all. In fact, even the square of a weighted shift is not a good candidate for this type of analysis because the square of a shift is (unitarily equivalent to) the orthogonal direct sum of two weighted shifts, and $p$-hyponormality is easily seen to be inherited by such direct summands. Thus it may be somewhat surprising that the class of composition operators we shall use to distinguish the respective $p$-hyponormal classes are unitarily equivalent to rank one perturbations of the direct sum of two weighted shifts.

**Example 2.11.** We let $X$ be the set of nonnegative integers, let $\mathcal{F}$ be the $\sigma$ algebra of all subsets of $X$, and take $\mu$ to be the measure determined by the strictly positive sequence $\{m_k\}_{k\geq 0}$. Our point transformation $T$ is defined as follows

$$T(k) = \begin{cases} 0 & k = 0,1,2, \\ k-2 & k \geq 3. \end{cases}$$

The action of $T$ may be viewed as two paths leading back to 0, with 0 tied to itself. We specify our point mass measure $m$ as follows (initializing the sequence at $m_0$):

$$m = 1, \ \ 1, \ \ 1, \ \ c, \ \ d, \ \ c^2, \ \ d^2, \ \ c^3, \ \ d^3, \ \ \cdots;$$

where $c$ and $d$ are fixed positive numbers. The powers of $c$ occur for odd integers and those of $d$ for even integers. The precise formula for power *vs* position will not be of consequence in our calculations. It follows that the $\sigma$ algebra $T^{-1}\mathcal{F}$ is generated by the atoms

$$\{0,1,2\}, \ \ \{3\}, \ \ \{4\}, \ \ \cdots.$$

We now calculate the Radon-Nikodym derivative $h$ :

$$\mu\circ T^{-1}(0)=\mu(\{0,1,2\})=3\ ;\ h(0)=\frac{\mu\circ T^{-1}(0)}{m_0}=3,$$

$$T^{-1}\{k\}=\begin{cases}\{k+2\}\ ;\ h(k)=\frac{m_{k+2}}{m_k}=c, & \text{for odd } k\geq 1,\\ \{k+2\}\ ;\ h(k)=\frac{m_{k+2}}{m_k}=d, & \text{for even } k\geq 2.\end{cases}$$

In sequence form

$$h\ =\ 3,\ \ c,\ d,\ c,\ d,\cdots$$

and consequently

$$h\circ T\ =3,\ 3,\ 3,\ c,\ d,\ c,\cdots.$$

In order to compute the necessary conditional expectations, recall the model for conditioning with respect to a partition $\{A_k\}_{k\geq 0}$ listed earlier:

$$E(f|\mathcal{A})=\sum_{k=0}^{\infty}\frac{1}{\mu(A_k)}\left(\int_{A_k}fd\mu\right)\chi_{A_k}.$$

So with respect to our current situation we have

$$Ef=\frac{f_0+f_1+f_2}{3}\chi_{\{0,1,2\}}+\sum_{k\geq 3}f_k\chi_{\{k\}}.$$

In sequence form:

$$Ef=\frac{f_0+f_1+f_2}{3},\quad \frac{f_0+f_1+f_2}{3},\quad \frac{f_0+f_1+f_2}{3},\quad f_3,\quad f_4,\quad \cdots.$$

Now fix a number $p>0$, and let us consider $E(1/h^p)$ and $1/(h^p\circ T)$.

$$E\left(\frac{1}{h^p}\right)=\frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3},\quad \frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3},\quad \frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3},\quad \frac{1}{c^p},\ \frac{1}{d^p},\ \frac{1}{c^p},\ \cdots. \tag{2.1}$$

In particular, $1/h^p\circ T$ and $E(1/h^p)$ agree for $k\geq 3$, so we need only compare their values for $k=0$, or, to the same ends, consider $(h^p\circ T(0))E(1/h^p)(0)$. This product is

$$3^p\cdot\frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3}=\frac{1+(\frac{3}{c})^p+(\frac{3}{d})^p}{3}.$$

Using Theorem 2.4, this shows that we have $p$-hyponormality if and only if

$$\left(\frac{3}{c}\right)^p+\left(\frac{3}{d}\right)^p\leq 2. \tag{2.2}$$

First consider the extreme case, $\left(\frac{3}{c}\right)^p + \left(\frac{3}{d}\right)^p = 2$. We must have $2 - (3/d)^p > 0$; equivalently, $d > 3 \cdot 2^{-1/p}$. Choose any such $d$, and let $c = 3 \cdot [2 - (\frac{3}{d})^p]^{-1/p}$. Then the corresponding composition operator is $p$-hyponormal. In fact, it satisfies the equality $h^p \circ TE(\frac{1}{h^p}) = 1$. With $c$ and $d$ chosen as above with regard to a fixed $p > 0$, we show that for any $q > p$ our composition operator is not $q$-hyponormal. To this end we must show that for $q > p$, $(\frac{3}{c})^q + (\frac{3}{d})^q > 2$. For positive numbers $A$ and $B$, consider the following functions of the nonnegative variable $x$ : $u(x) = A^x$ and $v(x) = 2 - B^x$. Their graphs cross when $x = 0$, and they may cross at no more than one other point (unless $A = B = 1$). In our case ($A = 3/c$ and $B = 3/d$) we have found that point of intersection; namely $x = p$. For all larger $x$ their difference (in the order presented) is positive.

According to Theorem 2.5, Corollary 2.9, (2.1), and (2.2), we have that

(i) $C$ is quasinormal if and only if $c = d = 3$;

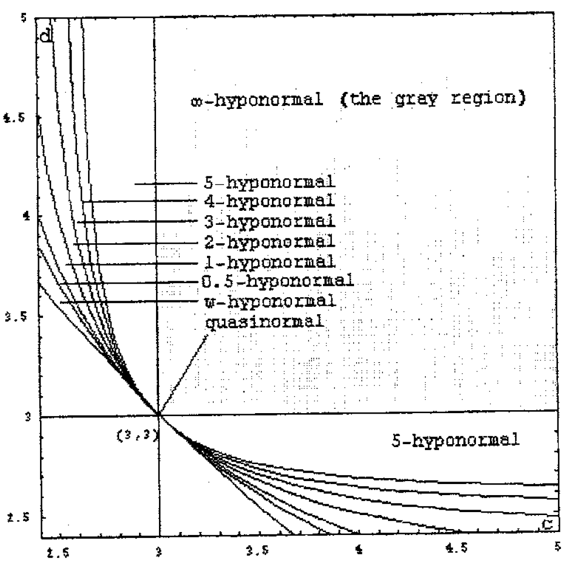
(ii) $C$ is $\infty$-hyponormal if and only if $c \geq 3$ and $d \geq 3$;

(iii) $C$ is $p$-hyponormal if and only if $\left(\frac{3}{c}\right)^p + \left(\frac{3}{d}\right)^p \leq 2$;

(iv) $C$ is $w$-hyponormal if and only if $2\sqrt{\frac{c}{3}} + \sqrt{\frac{d}{3}} \geq 2$.

Hence we have Figure 2.1, which shows clearly the distinction for the classes of $p$-hyponormal operators:

FIGURE 2.1




We asserted earlier that the specific type of composition operator used above to separate the $p$-hyponormal classes is a rank one perturbation of the direct sum of two weighted shifts. To see this, let $\chi_k$ be the characteristic function of the singleton $\{k\}$. Then $\{e_k = \frac{1}{\sqrt{m_k}}\chi_k : k \geq 0\}$ is an orthonormal basis for our weighted $l^2$ space. Now our construction for $T$ may be rephrased as

$$\chi_0 \circ T = \chi_0 + \chi_1 + \chi_2 \; ; \quad \chi_k \circ T = \chi_{k+2} \text{ for } k \geq 1.$$

In terms of the given orthonormal basis, these take the forms

$$Ce_k = \begin{cases} e_0 + \sqrt{\frac{m_1}{m_0}}e_1 + \sqrt{\frac{m_2}{m_0}}e_2, & \text{for } k = 0, \\ \sqrt{\frac{m_{k+2}}{m_k}}e_{k+2}, & \text{for } k \geq 1. \end{cases}$$

Suppose that $\{\alpha_k\}_{k\geq 0}$ is a bounded sequence of nonzero complex numbers. Let

$$\mathcal{H} = \bigvee\{e_{2k} : k \geq 0\} \text{ and } \mathcal{K} = \bigvee\{e_{2k+1} : k \geq 0\}.$$

We then define shifts $A$ and $B$ on $\mathcal{H}$ and $\mathcal{K}$ respectively by $Ae_{2k} = \alpha_{2k}e_{2k+2}$ ; $Be_{2k+1} = \alpha_{2k+1}e_{2k+3}$. Then the operator $W$ given by $We_k = \alpha_k e_{k+2}$; $k \geq 0$ is unitarily equivalent to $A \oplus B$. Returning to our composition operator setting, let $\alpha_k = \sqrt{\frac{m_{k+2}}{m_k}}$ ; $k \geq 0$, and let $F$ be the rank one operator $e_0 \otimes \left(e_0 + \sqrt{\frac{m_1}{m_0}}e_1\right)$. Then $C$ is $F + W$.

*Remark.* It might seem that in the examples involving subnormal composition operators, one may choose the specific moment sequences rather arbitrarily. However, only certain measures can occur for specific examples. As an illustration, suppose we construct a subnormal operator with point masses. Specifically, suppose that,

$$h_n(x) = \int t^n d\delta_{a(x)} \text{ for a.e. } x.$$

It then follows that $a = h$. Recall that $h > 0$ in any case involving subnormality. Now

$$h \cdot [Eh] \circ T^{-1} = h_2 = h^2 \quad \Rightarrow \quad Eh = h \circ T,$$

and so

$$h \cdot [Eh_2] \circ T^{-1} = h_3 = h^3 \quad \Rightarrow \quad E(h^2) = Eh_2 = h^2 \circ T.$$

We then have a nonnegative function $h$ with conditional variance $E(h^2) = (Eh)^2$; and this happens if and only if $Eh = h$. But then $h = Eh = h \circ T$, so $C$ must be quasinormal. In fact, the converse is true: suppose $C$ is quasinormal. Then $h = h \circ T$. Because $h_{n+1} = hE(h_n) \circ T^{-1}$, we see that

$$h_n(x) = h^n(x) = \int t^n d\delta_{h(x)}.$$

So $C$ is quasinormal if and only if $\{h_n(x)\}_{n\geq 0}$ is almost everywhere a moment sequence corresponding to a point mass measure. These measures could not be used to separate subnormal and $\infty$-hyponormal operators.

**3. p-paranormality, normaloid, and spectraloid.** In this section we determine necessary and sufficient conditions for a composition operator to be $p$-quasihyponormal, an $A(p)$-class operator, absolute $p$-paranormal, or $p$-paranormal. A characterization of normaloid operators in terms of the Radon Nikodym derivatives $h_n, n = 1, 2, \ldots$ is given in the remark at the end of this section. The characterizations for the classes above are transparent: they only depend on $h$ (or $h_n$), T, and the conditional expectation $E$. We are unable to characterize spectraloid composition operators in this same fashion. However, in Example 3.4 we show that there are spectraloid composition operators which are not normaloid.

**Theorem 3.1.** *Let $C$ be a composition operator on $L^2$. Then the following are equivalent:*

(i) *$C$ is $p$-quasihyponormal;*
(ii) *$C$ is an $A(p)$-class operator;*
(iii) *$C$ is absolute-$p$-paranormal;*
(iv) *$C$ is $p$-paranormal.*
(v) $E(h^p) \geq h^p \circ T$.

Notice that, since $\|C\| = \|h\|_\infty^{1/2}$ and $\|C^n\| = \|h_n\|_\infty^{1/2}$ (recall that $h_n = d\mu \circ T^{-n}/d\mu$), $C$ is normaloid if and only if $\|h\|_\infty = \|h_n\|_\infty^{1/n}$ for all $n \in \mathbb{N}$.

**Example 3.2 (p-paranormality).** This example is continued from Example 2.11. Let $C$ be the composition operator in Example 2.11. By Theorem 3.1, $C$ is $p$-paranormal if and only if $\left(\frac{c}{3}\right)^p + \left(\frac{d}{3}\right)^p \geq 2$. After some computation, one can show

$$\bigcap_{p>0} \{(c,d)|C \text{ is } p\text{-paranormal}\} = \{(c,d)|cd \geq 9\}$$

and

$$\bigcup_{p>0} \{(c,d)|C \text{ is } p\text{-paranormal}\} = \{(c,d)|c > 3 \text{ or } d > 3\} \cup \{(3,3)\}.$$

Using the characterization of $p$-hyponormality, we also have

$$\bigcup_{p>0} \{(c,d)|C \text{ is } p\text{-hyponormal}\} = \{(c,d)|cd > 9\} \cup \{(3,3)\}.$$

We now show that composition operators can separate all $p$-paranormality classes. Fix $p > 0$ and choose any $d$ such that $3 < d < 3(2^{1/p})$. Then find $c > 0$ such that

$(c/3)^p + (d/3)^p = 2$. Then $C$ is $p$-paranormal. Let $0 < q < p$. We will show that $C$ is not $q$-paranormal. With $A = c/3$, $B = d/3$, and $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = A^x + B^x$, $C$ is $q$-paranormal if and only if $f(q) \geq 2$. But $f(0) = f(p) = 2$, and $f''(x) > 0$ for all $x$, implies $f(q) < 2$. Thus, $C$ is not $q$-paranormal.

Recall that the composition operator in this example is $w$-hyponormal if and only if $(c/3)^{1/2} + (d/3)^{1/2} \geq 2$, i.e. if and only if $C$ is 1/2-paranormal. The discussion above shows that there are $w$-hyponormal composition operators which are not $q$-hyponormal for any $q \in (0, 1/2)$. This proves that the general implication $w$-hyponormal $\Rightarrow$ $(1/2)$-paranormal given in the introduction cannot be improved.

**Example 3.3 (Normaloid).** Using the family of composition operators given in Example 3.2, we now determine when $C$ is normaloid. We have

$$
\begin{aligned}
&h_2 : \ 3 + (c+d)\,, \ c^2, \ d^2, \ c^2, \ d^2, \ldots \\
&h_3 : \ 3 + (c+d) + (c^2 + d^2)\,, \ c^3, \ d^3, \ c^3, \ d^3, \ldots \\
&\ \vdots \\
&h_n : \ 1 + \tfrac{c^n - 1}{c-1} + \tfrac{d^n - 1}{d-1}, \ c^n, \ d^n, \ c^n, \ d^n, \ldots \\
&\ \vdots \\
&\text{etc.}
\end{aligned}
$$

If $0 < c, d < 3$, then $\|h\|_\infty = 3$. Since $\|h_2\|_\infty < 9$, $C$ cannot be normaloid. We now assume that $c \geq d$ and $c \geq 3$. Then $\|h\|_\infty = c$. Because

$$h_n(0) = 1 + \frac{c^n - 1}{c-1} + \frac{d^n - 1}{d-1} \leq 1 + 2\frac{c^n - 1}{c-1} \leq c^n, \quad \text{for } n = 0, 1, 2, \ldots$$

we have $\|h_n\|_\infty^{1/n} = c = \|h\|_\infty$, for all $n \in \mathbb{N}$. Thus, $C$ is normaloid. Similarly, $C$ is normaloid if $d \geq c$ and $d \geq 3$. Consequently, $C$ is normaloid but not $p$-paranormal for any $p > 0$ if and only if $(c, d)$ is in the set $\{(3, d) | 0 \leq d < 3\} \cup \{(c, 3) | 0 \leq c < 3\}$. Thus, composition operators can separate the normaloid and $p$-paranormal classes. Of course, this also separates the normaloid and $w$-hyponormal classes ($w$-hyponormal $\Rightarrow$ 1/2-paranormal).

**Example 3.4 (Spectraloid).** Finally, we show that there is a region where our family of composition operators is spectraloid, but not normaloid. Because of the discussion above, we restrict our attention to the region $0 < c, d < 3$. Without loss of generality, we assume that $c \geq d$. We will show that $C$ is spectraloid when $c \geq (1 + \sqrt{5})^2/4 \approx 2.618$. We also show that when $d \leq c < 2.249$, $C$ is not spectraloid. The explicit formula for $h_n$ given in Example 3.2 can be used to show that $r(C) = \lim_{n\to\infty} \|h_n\|_\infty^{1/(2n)} = \max\{1, \sqrt{c}\}$. Since the inequality $w(C) \geq r(C)$

always holds, $C$ will be spectraloid if we can show $w(C) \leq \max\{1, \sqrt{c}\}$. To this end, suppose that, in vector form, $f = f_0, f_1, f_2, \ldots$. Then with

$$a_k = \begin{cases} |f_0|, & \text{if } k = 0 \\ c^{(k-1)/4}|f_k|, & \text{if } k = 1, 3, 5, \ldots \\ d^{(k-1)/4}|f_k|, & \text{if } k = 2, 4, 6, \ldots \end{cases}$$

we have $\|f\|^2 = \sum_{k=0}^{\infty} |a_k|^2$ and (with $A = a_0^2 + a_0a_1 + a_0a_2$)

$$|\langle Cf, f\rangle| \leq (1 - \sqrt{c})a_0^2 + a_0a_1 + a_0a_2 - (\sqrt{c}/2)(a_1^2 + a_2^2) + \sqrt{c}$$

whenever $\|f\| = 1$. We conclude that

$$w(C) \leq (1 - \sqrt{c})a_0^2 + a_0a_1 + a_0a_2 - (\sqrt{c}/2)(a_1^2 + a_2^2) + \sqrt{c}.$$

Assume $c \geq 1$. Set $a = \sqrt{c}$ and $k = \sqrt{a-1}$. Then, rewriting the right hand side of the inequality above, we have

$$w(C) \leq -\frac{1}{2}\left(ka_0 - \frac{a_1}{k}\right)^2 - \frac{1}{2}\left(ka_0 - \frac{a_2}{k}\right)^2 + \frac{1}{2}\left(\frac{1}{k^2} - a\right)\left(a_1^2 + a_2^2\right) + a.$$

We will have $w(C) \leq a = \sqrt{c}$ whenever $k^{-2} - a \leq 0$ i.e. when $a \geq (1 + \sqrt{5})/2$. This proves $C$ is spectraloid when $c \geq d$ and $\sqrt{c} \geq (1 + \sqrt{5})/2$, i.e. when $c > 2.618$. We conclude that if $c \geq d$ and $2.618 < c < 3$, then $C$ is spectraloid, but not normaloid.

We have already accomplished our goal of showing that composition operators can separate the spectraloid and normaloid classes, but unfortunately, we are currently unable to fully determine the region $\{(c,d) : \text{C is spectraloid}\}$. However, we are able to limit this region: We first prove that $C$ is not spectraloid when $0 < d \leq c < (1 + \sqrt{3})^2/4$. Let $0 \leq x \leq 1/\sqrt{2}$ and set $f_1 = f_2 = x, f_0 = \sqrt{1 - 2x^2}$, and $f_n = 0$ for $n \geq 3$. Then $f$ has norm 1 and $\langle Cf, f\rangle = 1 - 2x^2 + 2x\sqrt{1 - 2x^2}$. Maximizing this function over the region $0 \leq x \leq 1/\sqrt{2}$, we find that $x = (1 - 1/\sqrt{3})^{1/2}/2$ gives a maximum value of $(1 + \sqrt{3})/2$. This proves that $w(C) \geq (1 + \sqrt{3})/2$, but $r(C) = \max\{1, \sqrt{c}\}$ (see above) so that $C$ is not spectraloid if $(1 + \sqrt{3})^2/4 > c$.

We now improve the estimate for the region where $C$ is not spectraloid. The result obtained above allows us to restrict our attention to the case $c \geq d$ and $c > 1$. This assures that $r(C) = \sqrt{c}$. If we can demonstrate a $c_0 > 1$ such that $C = C_{c_0\,d}$ is not spectraloid, then $C_{c\,d}$ will not be spectraloid for $1 < c < c_0$: Define $g(c,d,f) := \langle C_{c\,d}|f|, |f|\rangle - r(C) = \langle C_{c\,d}|f|, |f|\rangle - \sqrt{c}$. The operator $C_{c_0\,d}$ is not spectraloid if and only if there is a unit vector $f$ such that $g(c_0, d, f) > 0$. Then, with this $f$ and $c_0 > c > 1$ (notation as above), $g(c_0, d, f) - g(c, d, f) \leq 0$. Thus, $0 < g(c_0, d, f) \leq g(c, d, f)$ and $C_{c\,d}$ is also non spectraloid.

We now demonstrate a $c_0$ and a unit vector $f$ such that $g(c_0, d, f) > 0$ i.e. $C_{c_0\ d}$ is not spectraloid. Let $c = c_0 = 2.249$ and let $0 < r < 1$. Setting $a_4 = a_6 = a_8 = \cdots = 0$ and $a_{2n+3} = r^n a_3$ for $n \in \mathbb{N}$, we have

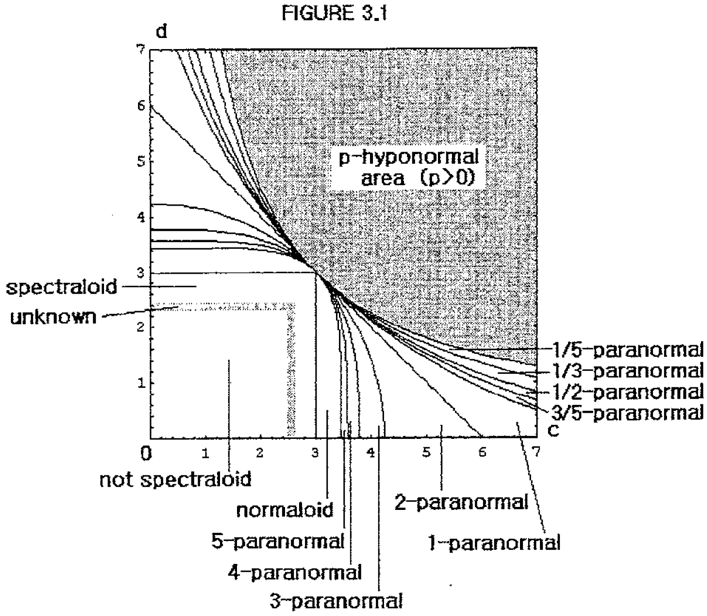
$$\langle Cf, f\rangle = a_0^2 + a_0a_1 + a_0a_2 + \sqrt{c}a_1a_3 + \frac{r\sqrt{c}a_3^2}{1-r^2}.$$

and $\|f\|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2/1 - r^2$. Defining $\widehat{a}_3 = a_3/\sqrt{1-r^2}$, fixing $r = 0.999$, and using the method of Lagrange multipliers, we find that $a_0 = 0.06780$, $a_1 = 0.04493$, $a_2 = 0.02263$, $\widehat{a}_3 = 0.9964297$ gives a unit vector with $g(c_0, d, f) \approx 8.44 \times 10^{-7} > 0$.

However, we do not know the exact function $f(c, d) = 0$ for the boundary of the region $\{(c, d) : C$ is not spectraloid$\}$.

Putting Examples 3.2, 3.3, and 3.4, together, we have Figure 3.1 which clearly illustrates the separation of the classes discussed above.

FIGURE 3.1

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