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<td>著者</td>
<td>Yanagida, Masahiro</td>
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<tr>
<td>引用</td>
<td>数理解析研究所講究録 2005年 1427号 21-30</td>
</tr>
<tr>
<td>発行年月日</td>
<td>2005-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47309">http://hdl.handle.net/2433/47309</a></td>
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<td>テキスト版</td>
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<td>取扱校</td>
<td>Kyoto University</td>
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Class $A-f$ and $A-f$-paranormal operators

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1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

Furuta-Ito-Yamazaki [12] introduced the following class of non-normal operators.

Definition ([12]). $T \in$ class $A$ $\iff |T^2| \geq |T|^2$.

An operator $T$ is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$ ([9][14]). Ando [3] showed that $T$ is paranormal if and only if

$$T^2*T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$$

for all $\lambda > 0$, and that if $T$ is $p$-hyponormal (i.e., $(T^*T)^p \geq (TT^*)^p$) for some $p > 0$ or log-hyponormal (i.e., $T$ is invertible and $\log T^*T \geq \log TT^*$), then $T$ is paranormal. It was shown in [12] that class $A$ includes the class of $p$-hyponormal and log-hyponormal operators, and is included in that of paranormal operators.

M. Fujii-D. Jung-S. H. Lee-M. Y. Lee-Nakamoto [8] introduced a generalization of class $A$. In fact, class $A$ coincides with class $A(1,1)$ ([24]).

Definition ([8]). For $s, t > 0$,

$$T \in \text{class } A(s, t) \iff (|T^*|^s|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}.$$  

On the other hand, Aluthge-Wang [1][2] introduced $w$-hyponormality. An operator $T$ is said to be $w$-hyponormal if $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $T = U|T|$ is the polar decomposition and $\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$ (Aluthge transformation), or equivalently,

$$\left(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \geq |T^*|$$

and $|T| \geq (|T|^\frac{1}{2}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}}$.

Ito-Yamazaki [17] showed that

$$(B^5 AP B^5)^{\frac{1}{r+t}} \geq B^r \implies A^p \geq (A^5 B^r A^5)^{\frac{1}{r+t}}$$
for $A, B \geq 0$ and $p, r \geq 0$, so that the class of $w$-hyponormal operators coincides with class $A(\frac{1}{2}, \frac{1}{2})$.

As parallel concept to class $A(s, t)$, we introduced a generalization of paranormality in [26]. In fact, paranormality coincides with absolute-$(1, 1)$-paranormality.

**Definition ([26]).** For $s, t > 0$,

$T$ is absolute-$(s, t)$-paranormal
\[ \iff \| |T|^s |T^*|^t x||^t \geq \| |T^*|^t x||^{s+t} \text{ for every unit vector } x \in H \]
\[ \iff t|T^*|^t |T|^{2s} |T^*|^t - (s + t) \lambda^s |T^*|^{2t} + s \lambda^{s+t} I \geq 0 \text{ for all } \lambda > 0. \]

We remark that class $A(k)$ and absolute-$k$-paranormality introduced in [12] coincide with class $A(k, 1)$ and absolute-$(k, 1)$-paranormality for each $k > 0$, respectively, and $p$-paranormality introduced in [7] coincides with absolute-$(p, p)$-paranormality for each $p > 0$.

## 2 Generalizations of class A and paranormality

We introduce further generalizations of class A and paranormality.

**Definition 2.1.** Let $f$ be a non-negative continuous function on $[0, \infty)$.

(i) $T \in \text{class } (s, t, f)$ $\iff$ $f(|T|^s |T^*|^t) \geq |T^*|^t$.

(ii) $T$ is A-$f$-paranormal $\iff \lambda T \in \text{class } (s, t, f)$ for all $\lambda > 0$.

When $f$ is a representing function of an operator connection $\sigma$ (see [20]), we also call class $A-f$ and A-$f$-paranormal class $A-\sigma$ and A-$\sigma$-paranormal, respectively.

In fact, class A and paranormality coincide with class A-$\|\|$ and A-$\nabla$-paranormality, where $\nabla$ and $\|$ are the arithmetic and geometric means, that is,

\[
A \nabla B = \frac{1}{2}(A + B) \quad \text{and} \quad A \| B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\frac{1}{2}A^{\frac{1}{2}} \quad \text{for } A, B > 0,
\]

and their representing functions are $f_{\nabla}(t) = \frac{1}{2}(1 + t)$ and $f_{\|}(t) = t^\frac{1}{2}$, respectively. We remark that "$T \in \text{class } A \implies T$ is paranormal" can be shown as follows:

$T \in \text{class } A-\| \iff T$ is A-$\|$-paranormal
\[ \implies T$ is A-$\nabla$-paranormal since $f_{\|}(\lambda^\frac{1}{2} t) = \lambda^\frac{1}{2} f_{\|}(t) \]

Moreover, we introduce further generalizations of class $A(s, t)$ and absolute-$(s, t)$-paranormality.
Definition 2.2. Let \( f \) be a non-negative continuous function on \([0, \infty)\), and \( s, t > 0 \).

(i) \( T \in \text{class } A(s, t) \Leftrightarrow f(|T^{*}|^{t}|T|^{2s}|T^{*}|^{t}) \geq |T^{*}|^{2t} \).

(ii) \( T \) is \( A(s, t) \)-paranormal \( \Leftrightarrow \lambda T \in \text{class } A(s, t) \) for all \( \lambda > 0 \).

When \( f \) is a representing function of an operator connection \( \sigma \), we also call class \( A(s, t) \)-\( f \) and \( A(s, t) \)-\( f \)-paranormal class \( A(s, t) \)-\( \sigma \) and \( A(s, t) \)-\( \sigma \)-paranormal, respectively.

In fact, for each \( s, t > 0 \), class \( A(s, t) \) and absolute-(\( s, t \))-paranormality coincide with class \( A(s, t)\nabla_{\frac{s}{s+t}} \) and \( A(s, t)\nabla_{\frac{s}{s+t}} \)-paranormality, where \( \nabla_{\alpha} \) and \( \mathfrak{g}_{\alpha} \) are generalized arithmetic and geometric means for \( \alpha \in [0, 1] \), that is,

\[
A \nabla_{\alpha} B = (1 - \alpha)A + \alpha B \quad \text{and} \quad A \mathfrak{g}_{\alpha} B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \text{for } A, B > 0,
\]

and their representing functions are \( f_{\nabla_{\alpha}}(t) = (1 - \alpha) + \alpha t \) and \( f_{\mathfrak{g}_{\alpha}}(t) = t^{\alpha} \), respectively.

3 Properties of class \( A-f \) and \( A-f \)-paranormality

The following results have been shown on class \( A(s, t) \) and absolute-(\( s, t \))-paranormal operators.

Theorem 3.A ([8][15][17][25][26]).

(i) \( T \) is \( p \)-hyponormal for some \( p > 0 \) or log-hyponormal \( \Rightarrow T \in \text{class } A(s, t) \) for all \( s, t > 0 \).

(ii) For each \( s, t > 0 \), \( T \in \text{class } A(s, t) \Rightarrow T \) is absolute-(\( s, t \))-paranormal.

(iii) \( T \) is absolute-(\( s, t \))-paranormal for some \( s, t > 0 \) \( \Rightarrow T \) is normaloid (i.e., \( \|T\| = r(T) \)), where \( r(T) \) is the spectral radius of \( T \).

(iv) For each \( 0 < s_{1} \leq s_{2} \) and \( 0 < t_{1} \leq t_{2} \),

\[
T \in \text{class } A(s_{1}, t_{1}) \Rightarrow T \in \text{class } A(s_{2}, t_{2}),
\]

\( T \) is absolute-(\( s_{1}, t_{1} \))-paranormal \( \Rightarrow T \) is absolute-(\( s_{2}, t_{2} \))-paranormal.

(v) \( T \) is invertible and absolute-(\( p, p \))-paranormal for all \( p > 0 \) \( \Rightarrow T \) is log-hyponormal.

Theorem 3.B ([17][27]). Let \( s, t \in (0, 1] \). Then

\[
T \in \text{class } A(s, t) \Rightarrow T^{n} \in \text{class } A(s_{n}, t_{n}) \quad \text{for every positive integer } n.
\]

These were obtained as applications of the following result.
Theorem F (Furuta inequality [10]).
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $(B^\frac{r}{2}A^pB^\frac{r}{2})^{\frac{1}{q}} \geq (B^\frac{r}{2}B^pB^\frac{r}{2})^{\frac{1}{q}}$

and

(ii) $(A^\frac{r}{2}A^pA^\frac{r}{2})^{\frac{1}{q}} \geq (A^\frac{r}{2}B^pA^\frac{r}{2})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

We remark that Theorem F yields Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$” when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [5][19] and also an elementary one-page proof in [11]. It is shown in [22] that the domain of $p, q$ and $r$ drawn in Figure is the best possible for Theorem F.

First, we show monotonicity of class $A(s, t)-f_{s,t}$ for $s$ and $t$ as a generalization of (iv) in Theorem 3.A.

Theorem 3.1. Let $s_0, t_0 > 0$ and $\{f_{s,t} | s \geq s_0, t \geq t_0\}$ be a family of non-negative operator monotone functions on $[0, \infty)$ satisfying $f_{s,t}(x^r g(x)^s) = x^r$, where $g$ is a continuous increasing function. Then

$T$ is invertible and $T \in$ class $A(s_0, t_0)-f_{s_0, t_0}$

$\implies T \in$ class $A(s, t)-f_{s, t}$ for all $s \geq s_0$ and $t \geq t_0$.

We use the following result in order to give a proof of Theorem 3.1.

Theorem 3.C ([23]). Let $A$ and $B$ be positive operators, and let $\{\psi_r | r \geq a\}$ $(a > 0)$ be a family of non-negative operator monotone functions on $[0, \infty)$ satisfying $\psi_r(x^r g(x)^s) = x^s$, i.e., $x^{-r} \sigma_{\psi_r} g(x) = 1$, where $g$ is a continuous increasing function. Then the following hold:

(i) If $A^a \sigma_{\psi_r} B \geq I$, then $A^r \sigma_{\psi_r} B$ is increasing for $r \geq a$.

(ii) If $A$ and $B$ are invertible and if $A^a \sigma_{\psi_r} B \leq I$, then $A^r \sigma_{\psi_r} B$ is decreasing for $r \geq a$.

We also use the following result which is an extension of a result in [17].

Theorem 3.D ([16]). Let $A$ and $B$ be positive operators, and let $f$ and $g$ be non-negative continuous functions on $[0, \infty)$ satisfying $f(x)g(x) = x$. Then the following hold:

(i) $f(B^\frac{1}{2}AB^\frac{1}{2}) \geq B$ ensures $A - g(A^\frac{1}{2}BA^\frac{1}{2}) \geq A^\frac{1}{2}E_B A^\frac{1}{2} - g(0)E_{A^\frac{1}{2}BA^\frac{1}{2}}$.

(ii) $B \geq f(B^\frac{1}{2}AB^\frac{1}{2})$ ensures $g(A^\frac{1}{2}BA^\frac{1}{2}) - A \geq g(0)E_{A^\frac{1}{2}BA^\frac{1}{2}} - A^\frac{1}{2}E_B A^\frac{1}{2}$.

Here $E_B$ and $E_{A^\frac{1}{2}BA^\frac{1}{2}}$ are the orthoprojections to $\mathcal{N}(B)$ and $\mathcal{N}(A^\frac{1}{2}BA^\frac{1}{2})$, respectively.
Proof of Theorem 3.1. $T$ belongs to class $\mathrm{A}(s_0, t_0)- f_{s_0, t_0}$ if and only if

$$f_{s_0, t_0}(|T^*|^{t_0} |T|^{2s_0} |T^*|^{t_0}) \geq |T^*|^{2t_0}.$$ 

Since $f_{s_0, t}(x^t g(x)^{s_0}) = x^t$ and $g(x)^{s_0}$ is a continuous increasing function,

$$|T^*|^{-t} f_{s_0, t}(|T^*|^{t} |T|^{2s_0} |T^*|^{t}) |T^*|^{-t} \geq |T^*|^{-t_0} f_{s_0, t_0}(|T^*|^{t_0} |T|^{2s_0} |T^*|^{t_0}) |T^*|^{-t_0}$$

holds for $t \geq t_0$ by (i) of Theorem 3.C. Hence

$$f_{s_0, t}(|T^*|^{t} |T|^{2s_0} |T^*|^{t}) \geq |T^*|^{2t}.$$ 

By (i) of Theorem 3.D, this implies

$$|T|^{2s_0} \geq f_{s_0, t}(|T|^{s_0} |T^*|^{2t_0} |T^*|^{t_0}),$$

where $f_{s_0, t}(x) = \frac{x}{f_{s_0, t}(x)}$. Since

$$f_{s_0, t}(x^s g^{-1}(x)^{t}) = \frac{x^s g^{-1}(x)^{t}}{f_{s_0, t}(x^s g^{-1}(x)^{t})} = x^s$$

and $g^{-1}(x)^{t}$ is a continuous increasing function,

$$|T|^{-s} f_{s_0, t}(|T|^s |T^*|^{2t} |T^*|^s) |T|^{-s} \leq |T|^{-s_0} f_{s_0, t}(|T|^{s_0} |T^*|^{2t} |T^*|^{s_0}) |T|^{-s_0}$$

holds for $s \geq s_0$ by (ii) of Theorem 3.C. Hence

$$|T|^{2s} \geq f_{s, t}(|T|^s |T^*|^{2t} |T|^s).$$

By (ii) of Theorem 3.D, this implies

$$f_{s, t}(|T|^s |T^*|^{2t} |T|^s) \geq |T^*|^{2t},$$

that is, $T$ belongs to class $\mathrm{A}(s, t)- f_{s, t}$. $\square$

Secondly, we show a sufficient condition for log-hyponormality in terms of class $\mathrm{A}(s, t)- f$ as a generalization of (v) in Theorem 3.A.

Theorem 3.2. Let $f$ be a non-negative, continuously differentiable and concave (or convex) function on $[0, \infty)$ satisfying $f(1) \leq 1$ and $0 < f'(1) < 1$, and $p_0 > 0$. Then

$T$ is invertible and $T \in \mathrm{A}(\theta'p, \theta p)- f$ for all $p \in (0, p_0)$

$\implies T$ is log-hyponormal,

where $\theta = f'(1)$ and $\theta + \theta' = 1$. 

Proof. There exists a continuous function \( g \) on \([0, \infty)\) such that \( f'(g(x)) = \frac{f(x) - f(1)}{x - 1} \) for \( x \neq 1 \) by the mean value theorem and concavity (or convexity) of \( f \). Then we have
\[
\frac{|T^*|^{2\theta p} - I}{p} \leq \frac{f(|T^*|^{\theta p} |T|^{2\theta' p} |T^*|^{\theta p}) - f(1)I}{p} = f'(g(|T^*|^{\theta p} |T|^{2\theta' p} |T^*|^{\theta p})) \frac{|T^*|^{\theta p} |T|^{2\theta' p} |T^*|^{\theta p} - I}{p}
\]
for \( 0 < p < p_0 \). By tending \( p \to +0 \), we have
\[
\log |T^*|^{2\theta} \leq \theta (\log |T|^{2\theta'} + \log |T^*|^{2\theta})
\]
hence \( T \) is log-hyponormal.

Thirdly, we show a result on powers of class \( A(s, t) - f \) operators as a generalization of Theorem 3.B.

**Theorem 3.3.** Let \( f \) be a non-negative operator monotone function on \([0, \infty)\) satisfying \( f(0) = 0 \), and \( s, t \in (0, 1] \). Then
\[
T \in \text{class } A(s, t) - f \text{ and } T \in \text{class } A \implies T^n \in \text{class } A\left(\frac{s}{n}, \frac{t}{n}\right) - f \text{ for every positive integer } n.
\]

We use the following result in order to give a proof of Theorem 3.3.

**Theorem 3.E ([17][27]).** If \( T \) belongs to class \( A \), then
\[
|T^n|^{\frac{2}{n}} \geq |T|^2 \quad \text{and} \quad |T^*|^2 \geq |T^{n*}|^{\frac{2}{n}}
\]
for every positive integer \( n \).

We also use the following which is an extension of results in [18][27].

**Lemma 3.4.** Let \( A, B \) and \( C \) be positive operators, and \( f \) be an operator monotone function on \([0, \infty)\) satisfying \( f(0) \leq 0 \). Then
\[
f(B^{\frac{1}{2}} AB^{\frac{1}{2}}) \geq B \quad \text{and} \quad B \geq C \implies f(C^{\frac{1}{2}} AC^{\frac{1}{2}}) \geq C.
\]

**Proof.** There exists an operator \( X \) such that
\[
B^{\frac{1}{2}} X = X^* B^{\frac{1}{2}} = C^{\frac{1}{2}} \quad \text{and} \quad \|X\| \leq 1
\]
by Douglas’ theorem [4]. Then we have
\[
f(C^{\frac{1}{2}} AC^{\frac{1}{2}}) = f(X^* B^{\frac{1}{2}} AB^{\frac{1}{2}} X) \geq X^* f(B^{\frac{1}{2}} AB^{\frac{1}{2}}) X \geq X^* BX = C
\]
by Hansen’s inequality [13].
Proof of Theorem 3.3. By Löwner-Heinz theorem and Theorem 3.E,
\[ |T^n|^{\frac{2s}{n}} \geq |T|^{2s} \quad \text{and} \quad |T^*|^{2t} \geq |T^{n*}|^{\frac{2t}{n}}. \]
Hence \( f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t} \) implies
\[ f(|T^{n*}|^{\frac{t}{n}}|T|^{2s}|T^{n*}|^{\frac{t}{n}}) \geq |T^{n*}|^{\frac{2t}{n}} \]
by Lemma 3.4, and
\[ f(|T^{n*}|^{\frac{t}{n}}|T|^{2s}|T^{n*}|^{\frac{t}{n}}) \geq |T^{n*}|^{\frac{2t}{n}} \]
since \( f \) is operator monotone. \( \square \)

The following can be obtained similarly by using Lemma 3.4, which is an extension of a result on class \( A(s, t) \) operators in [21].

**Proposition 3.5.** Let \( f \) be a non-negative operator monotone function on \( [0, \infty) \) satisfying \( f(0) = 0 \), and \( s, t \in (0, 1] \). Then
\[
T \in \text{class } A(s, t)-f \implies T|_M \in \text{class } A(s, t)-f,
\]
where \( M \) is an invariant subspace of \( T \) and \( T|_M \) is the restriction of \( T \) onto \( M \).

**Proof.** Let \( P \) be the orthogonal projection onto \( M \), and \( T_0 = TP \). Then
\[
|T_0|^{2s} = (P|T|^2P)^s \geq P|T|^{2s}P
\]
by Hansen's inequality [13], so that \( |T_0^*|^t|T_0|^{2s}|T_0^*|^t \geq |T^*|^t|T|^{2s}|T^*|^t \). And also,
\[
|T_0^*|^{2t} = (PPT^*)^t \leq (TT^*)^t = |T^*|^{2t}
\]
by Löwner-Heinz theorem. Hence \( f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t} \) implies
\[ f(|T_0^*|^t|T_0|^{2s}|T_0^*|^t) \geq |T_0|^2 \]
by Lemma 3.4, and
\[ f(|T_0^*|^t|T_0|^{2s}|T_0^*|^t) \geq |T_0|^2 \]
since \( f \) is operator monotone. \( \square \)

Lastly, we show a result on Aluthge transformation of class \( A(s, t)-f \) operators. We remark that for each \( s, t > 0 \), if \( T \) belongs to class \( A(s, t) \), then \( \tilde{T}_{s,t} = \frac{\min\{s,t\}}{s+t} \)-hyponormal ([15],[17]). An operator \( T \) is said to be \( f \)-hyponormal if \( f(T^*T) \geq f(TT^*) \) for a continuous function \( f \) ([6]).

**Theorem 3.6.** Let \( f \) and \( g \) be non-negative continuous increasing functions on \( [0, \infty) \) satisfying \( f(x)g(x) = x \) and \( g(0) = 0 \), and \( s, t > 0 \). If \( T \in \text{class } A(s, t)-f \), then the following hold, where \( T = U|T| \) is the polar decomposition and \( \tilde{T}_{s,t} = |T|^sU|T|^t \):
(i) $\tilde{T}_{s,t}$ is $f$-hyponormal if $f \circ g^{-1}$ is operator monotone and $x^t \geq (f \circ g^{-1})(x^s)$.

(ii) $\tilde{T}_{s,t}$ is $g$-hyponormal if $g \circ f^{-1}$ is operator monotone and $(g \circ f^{-1})(x^t) \geq x^s$.

We use the following result in order to give a proof of Theorem 3.6.

**Lemma 3.6** ([16]). Let $A$ be a positive operator and $U$ be a partial isometry such that $N(U) \subseteq N(A)$, and let $f$ be a continuous function on $[0, \infty)$. Then

$$Uf(A)U^* = f(UAU^*) - f(0)(I - UU^*).$$

**Proof of Theorem 3.6.** Noting that $N(U^*) = N(|T|) \subseteq N(|T|^s|T|^{2s}|T^*|)$,

$$f \left( (\tilde{T}_{s,t})^* \tilde{T}_{s,t} \right) = f(|T|^s|T|^{2s}|T^*|) = f(U^*|T^*|^s|T|^{2s}|T^*|^sU) + f(0)(I - U^*U) \text{ by Lemma 3.6}$$

$$\geq U^*|T^*|^s|T|^{2s}|T^*|^sU = |T|^{2t}.$$

On the other hand, $f(|T|^s|T|^{2s}|T^*|) \geq |T|^{2t}$ implies

$$|T|^{2s} \geq g(|T|^s|T|^{2t}|T^*|) = g(|T|^s|U|T|^{2s}|T^*|) = g\left( \tilde{T}_{s,t}(\tilde{T}_{s,t})^* \right)$$

by (i) of Theorem 3.D. If $f \circ g^{-1}$ is operator monotone and $x^t \geq (f \circ g^{-1})(x^s)$,

$$f \left( (\tilde{T}_{s,t})^* \tilde{T}_{s,t} \right) \geq (f \circ g^{-1})(|T|^{2s}) \geq f \left( \tilde{T}_{s,t}(\tilde{T}_{s,t})^* \right),$$

hence $\tilde{T}_{s,t}$ is $f$-hyponormal. If $g \circ f^{-1}$ is operator monotone and $(g \circ f^{-1})(x^t) \geq x^s$,

$$g \left( (\tilde{T}_{s,t})^* \tilde{T}_{s,t} \right) \geq (g \circ f^{-1})(|T|^{2s}) \geq |T|^{2s} \geq g \left( \tilde{T}_{s,t}(\tilde{T}_{s,t})^* \right),$$

hence $\tilde{T}_{s,t}$ is $g$-hyponormal.

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