<table>
<thead>
<tr>
<th>Title</th>
<th>Class A-$f$ and A-$f$-paranormal operators (Role of Operator Inequalities in Operator Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yanagida, Masahiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1427: 21-30</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47309">http://hdl.handle.net/2433/47309</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Class A-\(f\) and A-\(f\)-paranormal operators

東京理科大学・理 柳田昌宏 (Masahiro Yanagida)
Department of Mathematical Information Science,
Tokyo University of Science

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space \(H\). An operator \(T\) is said to be positive (denoted by \(T \geq 0\)) if \((Tx,x) \geq 0\) for all \(x \in H\), and also \(T\) is said to be strictly positive (denoted by \(T > 0\)) if \(T\) is positive and invertible.

Furuta-Ito-Yamazaki [12] introduced the following class of non-normal operators.

Definition ([12]). \(T \in \text{class A} \iff |T^2| \geq |T|^2\).

An operator \(T\) is said to be paranormal if \(||T^2x|| \geq ||Tx||^2\) for every unit vector \(x \in H\) ([9][14]). Ando [3] showed that \(T\) is paranormal if and only if

\[
T^2 - 2\lambda T^*T + \lambda^2 I \geq 0 \quad \text{for all } \lambda > 0,
\]

and that if \(T\) is \(p\)-hyponormal (i.e., \((T^*T)^p \geq (TT^*)^p\) for some \(p > 0\) or log-hyponormal (i.e., \(T\) is invertible and \(\log T^*T \geq \log TT^*\)), then \(T\) is paranormal. It was shown in [12] that class A includes the class of \(p\)-hyponormal and log-hyponormal operators, and is included in that of paranormal operators.

M. Fujii-D. Jung-S. H. Lee-M. Y. Lee-Nakamoto [8] introduced a generalization of class A. In fact, class A coincides with class \(A(1,1)\) ([24]).

Definition ([8]). For \(s, t > 0\),

\[
T \in \text{class A}(s,t) \iff (|T^*|^t |T|^2s |T^*|^t)^{\frac{1}{s+t}} \geq |T^*|^2t.
\]

On the other hand, Aluthge-Wang [1][2] introduced \(w\)-hyponormality. An operator \(T\) is said to be \(w\)-hyponormal if \(|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|\) where \(T = U|T|\) is the polar decomposition and \(\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\) (Aluthge transformation), or equivalently,

\[
(|T^*|^\frac{1}{2} |T||T^*|^\frac{1}{2})^\frac{1}{2} \geq |T^*| \quad \text{and} \quad |T| \geq (|T|^\frac{1}{2} |T^*||T|^\frac{1}{2})^\frac{1}{2}.
\]

Ito-Yamazaki [17] showed that

\[
(B^s A^p B^s)^{\frac{1}{s+t}} \geq B^r \implies A^p \geq (A^s B^r A^s)^{\frac{1}{s+t}}
\]
for $A, B \geq 0$ and $p, r \geq 0$, so that the class of $w$-hyponormal operators coincides with class $A(\frac{1}{2}, \frac{1}{2})$.

As parallel concept to class $A(s, t)$, we introduced a generalization of paranormality in [26]. In fact, paranormality coincides with absolute-$(1, 1)$-paranormality.

**Definition ([26]).** For $s, t > 0$,

$$T$$ is absolute-$(s, t)$-paranormal

$$\iff |||T|^s|T^*|^t x||^t \geq |||T^*|^t x||^{s+t}$$

for every unit vector $x \in H$

$$\iff |||T|^s|T^*|^t - (s + t)\lambda^s|T^*|^2t + s\lambda^{s+t}I \geq 0$$

for all $\lambda > 0$.

We remark that class $A(k)$ and absolute-$k$-paranormality introduced in [12] coincide with class $A(k, 1)$ and absolute-$(k, 1)$-paranormality for each $k > 0$, respectively, and $p$-paranormality introduced in [7] coincides with absolute-$(p, p)$-paranormality for each $p > 0$.

# 2 Generalizations of class A and paranormality

We introduce further generalizations of class A and paranormality.

**Definition 2.1.** Let $f$ be a non-negative continuous function on $[0, \infty)$.

(i) $T \in$ class $A-f$ $\iff f(||T^*||T^2||T^*||) \geq ||T^*||^2$.

(ii) $T$ is $A-f$-paranormal $\iff \lambda T \in$ class $A-f$ for all $\lambda > 0$.

When $f$ is a representing function of an operator connection $\sigma$ (see [20]), we also call class $A-f$ and $A-f$-paranormal class $A-\sigma$ and $A-\sigma$-paranormal, respectively.

In fact, class A and paranormality coincide with class $A-\#$ and $A-\nabla$-paranormality, where $\nabla$ and $\#$ are the arithmetic and geometric means, that is,

$$A \nabla B = \frac{1}{2}(A + B) \quad \text{and} \quad A \# B = A^\frac{1}{2}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\frac{1}{2}A^\frac{1}{2}$$

for $A, B > 0$,

and their representing functions are $f_\nabla(t) = \frac{1}{2}(1 + t)$ and $f_\#(t) = t^\frac{1}{2}$, respectively. We remark that "$T \in$ class A $\implies T$ is paranormal" can be shown as follows:

$$T \in \text{class A-}\# \iff T \text{ is A-}\# \text{-paranormal} \quad \text{since } f_\#(\lambda^t) = \lambda^t f_\#(t)$$

$$\implies T \text{ is A-}\nabla \text{-paranormal} \quad \text{since } f_\nabla(t) \leq f_\nabla(t).$$

Moreover, we introduce further generalizations of class $A(s, t)$ and absolute-$(s, t)$-paranormality.
**Definition 2.2.** Let $f$ be a non-negative continuous function on $[0, \infty)$, and $s, t > 0$.

(i) $T \in \text{class } A(s, t)-f \iff f(|T^*|^t|T|^2s|T^*|^t) \geq |T^*|^{2t}$.

(ii) $T$ is $A(s, t)-f$-paranormal $\iff \lambda T \in \text{class } A(s, t)-f$ for all $\lambda > 0$.

When $f$ is a representing function of an operator connection $\sigma$, we also call class $A(s, t)-f$ and $A(s, t)-\sigma$-paranormal class $A(s, t)-\sigma$ and $A(s, t)-\sigma$-paranormal, respectively.

In fact, for each $s, t > 0$, class $A(s, t)$ and absolute-$(s, t)$-paranormality coincide with class $A(s, t)-\nabla_{\alpha}$ and $A(s, t)-\nabla_{\alpha}$-paranormality, where $\nabla_{\alpha}$ and $\sharp_{\alpha}$ are generalized arithmetic and geometric means for $\alpha \in [0, 1]$, that is,

$A \nabla_{\alpha} B = (1 - \alpha)A + \alpha B \quad \text{and} \quad A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}$ for $A, B > 0$,

and their representing functions are $f_{\nabla_{\alpha}}(t) = (1 - \alpha) + \alpha t$ and $f_{\sharp_{\alpha}}(t) = t^\alpha$, respectively.

3 Properties of class $A-f$ and $A-f$-paranormality

The following results have been shown on class $A(s, t)$ and absolute-$(s, t)$-paranormal operators.

**Theorem 3.A** ([8][15][17][25][26]).

(i) $T$ is $p$-hyponormal for some $p > 0$ or log-hyponormal $\implies T \in \text{class } A(s, t)$ for all $s, t > 0$.

(ii) For each $s, t > 0$, $T \in \text{class } A(s, t) \implies T$ is absolute-$(s, t)$-paranormal.

(iii) $T$ is absolute-$(s, t)$-paranormal for some $s, t > 0 \implies T$ is normaloid (i.e., $\|T\| = r(T)$), where $r(T)$ is the spectral radius of $T$.

(iv) For each $0 < s_1 \leq s_2$ and $0 < t_1 \leq t_2$,

$T \in \text{class } A(s_1, t_1) \implies T \in \text{class } A(s_2, t_2),$

$T$ is absolute-$(s_1, t_1)$-paranormal $\implies T$ is absolute-$(s_2, t_2)$-paranormal.

(v) $T$ is invertible and absolute-$(p, p)$-paranormal for all $p > 0 \implies T$ is log-hyponormal.

**Theorem 3.B** ([17][27]). Let $s, t \in (0, 1]$. Then

$T \in \text{class } A(s, t) \implies T^n \in \text{class } A\left(\frac{s}{n}, \frac{t}{n}\right)$ for every positive integer $n$.

These were obtained as applications of the following result.
Theorem F (Furuta inequality [10]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

We remark that Theorem F yields Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" when we put $r=0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [5],[19] and also an elementary one-page proof in [11]. It is shown in [22] that the domain of $p, q$ and $r$ drawn in Figure is the best possible for Theorem F.

First, we show monotonicity of class $A(s, t)-f_{s,t}$ as a generalization of (iv) in Theorem 3.A.

**Theorem 3.1.** Let $s_0, t_0 > 0$ and $\{f_{s,t} \mid s \geq s_0, t \geq t_0\}$ be a family of non-negative operator monotone functions on $[0, \infty)$ satisfying $f_{s,t}(x^t g(x)^s) = x^t$, where $g$ is a continuous increasing function. Then

$$T$$ is invertible and $T \in$ class $A(s_0, t_0)-f_{s_0,t_0}$

$$\implies T \in$ class $A(s,t)-f_{s,t}$ for all $s \geq s_0$ and $t \geq t_0$.

We use the following result in order to give a proof of Theorem 3.1.

**Theorem 3.C ([23]).** Let $A$ and $B$ be positive operators, and let $\{\psi_r \mid r \geq a\} (a > 0)$ be a family of non-negative operator monotone functions on $[0, \infty)$ satisfying $\psi_r(x^r g(x)^s) = x^r$, i.e., $x^{-r} \sigma \psi_r g(x) = 1$, where $g$ is a continuous increasing function. Then the following hold:

(i) If $A^a \sigma \psi_r B \geq I$, then $A^r \sigma \psi_r B$ is increasing for $r \geq a$.

(ii) If $A$ and $B$ are invertible and if $A^a \sigma \psi_r B \leq I$, then $A^r \sigma \psi_r B$ is decreasing for $r \geq a$.

We also use the following result which is an extension of a result in [17].

**Theorem 3.D ([16]).** Let $A$ and $B$ be positive operators, and let $f$ and $g$ be non-negative continuous functions on $[0, \infty)$ satisfying $f(x)g(x) = x$. Then the following hold:

(i) $f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B$ ensures $A - g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \geq A^{\frac{1}{2}}E_B A^{\frac{1}{2}} - g(0)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$.

(ii) $B \geq f(B^{\frac{1}{2}}AB^{\frac{1}{2}})$ ensures $g(A^{\frac{1}{2}}BA^{\frac{1}{2}}) - A \geq g(0)E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}} - A^{\frac{1}{2}}E_B A^{\frac{1}{2}}$.

Here $E_B$ and $E_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}$ are the orthoprojections to $N(B)$ and $N(A^{\frac{1}{2}}BA^{\frac{1}{2}})$, respectively.
Proof of Theorem 3.1. $T$ belongs to class $A(s_0, t_0)-f_{s_0, t_0}$ if and only if

$$f_{s_0, t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0}) \geq |T^*|^{2t_0}.$$

Since $f_{s_0, t}(x^s g(x)^{s_0}) = x^t$ and $g(x)^{s_0}$ is a continuous increasing function,

$$|T^*|^{-t} f_{s_0, t}(|T^*|^t|T|^{2s_0}|T^*|^t) |T^*|^{-t} \geq |T^*|^{-t_0} f_{s_0, t_0}(|T^*|^{t_0}|T|^{2s_0}|T^*|^{t_0}) |T^*|^{-t_0}$$

holds for $t \geq t_0$ by (i) of Theorem 3.C. Hence

$$f_{s_0, t}(|T^*|^t|T|^{2s_0}|T^*|^t) \geq |T^*|^{2t_0}.$$

By (i) of Theorem 3.D, this implies

$$|T|^{2s_0} \geq f^\perp_{s_0, t}(|T|^s|T^*|^{2t}|T|^s),$$

where $f^\perp_{s, t}(x) = \frac{x}{f_{s, t}(x)}$. Since

$$f^\perp_{s, t}(x^s g^{-1}(x)^{s_0}) = \frac{x^s g^{-1}(x)^t}{f_{s, t}(x^s g^{-1}(x)^t)} = x^s$$

and $g^{-1}(x)^t$ is a continuous increasing function,

$$|T|^{-s} f^\perp_{s_0, t}(|T|^{s_0}|T^*|^{2t}|T|^{s_0}) |T|^{-s} \leq |T|^{-s_0} f^\perp_{s_0, t_0}(|T|^{s_0}|T^*|^{2t}|T|^{s_0}) |T|^{-s_0}$$

holds for $s \geq s_0$ by (ii) of Theorem 3.C. Hence

$$|T|^{2s} \geq f_{s, t}(|T|^s|T^*|^{2t}|T|^s).$$

By (ii) of Theorem 3.D, this implies

$$f_{s, t}(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t},$$

that is, $T$ belongs to class $A(s, t)-f_{s, t}$.

Secondly, we show a sufficient condition for log-hyponormality in terms of class $A(s, t)-f$ as a generalization of (v) in Theorem 3.A.

**Theorem 3.2.** Let $f$ be a non-negative, continuously differentiable and concave (or convex) function on $[0, \infty)$ satisfying $f(1) \leq 1$ and $0 < f'(1) < 1$, and $p_0 > 0$. Then

$$T \text{ is invertible and } T \in \text{class } A(\theta'p, \theta p)-f \text{ for all } p \in (0, p_0) \implies T \text{ is log-hyponormal},$$

where $\theta = f'(1)$ and $\theta + \theta' = 1$. 

Proof. There exists a continuous function \( g \) on \([0, \infty)\) such that \( f'(g(x)) = \frac{f(x)-f(1)}{x-1} \) for \( x \neq 1 \) by the mean value theorem and concavity (or convexity) of \( f \). Then we have

\[
\frac{|T^{*}|^{2\theta p} - I}{p} \leq \frac{f(|T^{*}|^{\theta p}|T|^{2\theta'p}|T^{*}|^{\theta p}) - f(1)}{p}
\]

\[
= f'(g(|T^{*}|^{\theta p}|T|^{2\theta'p}|T^{*}|^{\theta p})) \frac{|T^{*}|^{\theta p}|T|^{2\theta'p}|T^{*}|^{\theta p} - I}{p}
\]

for \( 0 < p < p_0 \). By tending \( p \to +0 \), we have

\[
\log |T^{*}|^{2\theta} \leq \theta \left( \log |T|^{2\theta'} + \log |T^{*}|^{2\theta} \right)
\]

hence \( T \) is log-hyponormal.

Thirdly, we show a result on powers of class \( A(s, t) - f \) operators as a generalization of Theorem 3.B.

**Theorem 3.3.** Let \( f \) be a non-negative operator monotone function on \([0, \infty)\) satisfying \( f(0) = 0 \), and \( s, t \in (0, 1] \). Then

\[
T \in \text{class } A(s, t) - f \text{ and } T \in \text{class } A \quad \implies T^n \in \text{class } A \left( \frac{s}{n}, \frac{t}{n} \right) - f \text{ for every positive integer } n.
\]

We use the following result in order to give a proof of Theorem 3.3.

**Theorem 3.E** ([17][27]). If \( T \) belongs to class \( A \), then

\[
|T^n|^\frac{2}{n} \geq |T|^2 \quad \text{and} \quad |T^{*}|^2 \geq |T^{*}|^\frac{2}{n}
\]

for every positive integer \( n \).

We also use the following which is an extension of results in [18][27].

**Lemma 3.4.** Let \( A, B \) and \( C \) be positive operators, and \( f \) be an operator monotone function on \([0, \infty)\) satisfying \( f(0) \leq 0 \). Then

\[
f(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \geq B \text{ and } B \geq C \implies f(C^{\frac{1}{2}}AC^{\frac{1}{2}}) \geq C.
\]

**Proof.** There exists an operator \( X \) such that

\[
B^{\frac{1}{2}}X = X^*B^{\frac{1}{2}} = C^{\frac{1}{2}} \quad \text{and} \quad \|X\| \leq 1
\]

by Douglas' theorem [4]. Then we have

\[
f(C^{\frac{1}{2}}AC^{\frac{1}{2}}) = f(X^*B^{\frac{1}{2}}AB^{\frac{1}{2}}X) \geq X^*f(B^{\frac{1}{2}}AB^{\frac{1}{2}})X = X^*BX = C
\]

by Hansen's inequality [13].
Proof of Theorem 3.3. By Löwner-Heinz theorem and Theorem 3.E,

\[ |T^n|^{\frac{2s}{n}} \geq |T|^{2s} \quad \text{and} \quad |T^*|^{2t} \geq |T^{*n}|^{\frac{2t}{n}}. \]

Hence \( f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t} \) implies

\[ f(|T^{*n}|^{\frac{t}{n}}|T|^{2s}|T^{*n}|^{\frac{t}{n}}) \geq |T^{*n}|^{\frac{2t}{n}} \]

by Lemma 3.4, and

\[ f(|T^{*n}|^{\frac{t}{n}}|T^n|^{\frac{2s}{n}}|T^{*n}|^{\frac{t}{n}}) \geq |T^{*n}|^{\frac{2t}{n}} \]

since \( f \) is operator monotone.

The following can be obtained similarly by using Lemma 3.4, which is an extension of a result on class \( A(s, t) \) operators in [21].

**Proposition 3.5.** Let \( f \) be a non-negative operator monotone function on \([0, \infty)\) satisfying \( f(0) = 0 \), and \( s, t \in (0, 1] \). Then

\[ T \in \text{class } A(s, t)-f \implies T|_{\mathcal{M}} \in \text{class } A(s, t)-f, \]

where \( \mathcal{M} \) is an invariant subspace of \( T \) and \( T|_{\mathcal{M}} \) is the restriction of \( T \) onto \( \mathcal{M} \).

**Proof.** Let \( P \) be the orthogonal projection onto \( \mathcal{M} \), and \( T_0 = TP \). Then

\[ |T_0|^{2s} = (P|T|^{2}P)^s \geq P|T|^{2s}P \]

by Hansen's inequality [13], so that \( |T_0^*|^t|T_0|^{2s}|T_0^*|^t \geq |T^*|^t|T|^{2s}|T^*|^t \). And also,

\[ |T_0^*|^{2t} = (TP^{*}T)^t \leq (TT^{*})^t = |T^{*}|^{2t} \]

by Löwner-Heinz theorem. Hence \( f(|T^*|^t|T|^{2s}|T^*|^t) \geq |T^*|^{2t} \) implies

\[ f(|T_0^*|^t|T_0|^{2s}|T_0^*|^t) \geq |T_0^*|^{2t} \]

by Lemma 3.4, and

\[ f(|T_0^*|^t|T_0|^{2s}|T_0^*|^t) \geq |T_0^*|^{2t} \]

since \( f \) is operator monotone.

Lastly, we show a result on Aluthge transformation of class \( A(s, t)-f \) operators. We remark that for each \( s, t > 0 \), if \( T \) belongs to class \( A(s, t) \), then \( \tilde{T}_{s,t} \) is \( \frac{\min\{s,t\}}{s+t} \)-hyponormal ([15],[17]). An operator \( T \) is said to be \( f \)-hyponormal if \( f(T^{*}T) \geq f(TT^{*}) \) for a continuous function \( f \) ([6]).

**Theorem 3.6.** Let \( f \) and \( g \) be non-negative continuous increasing functions on \([0, \infty)\) satisfying \( f(x)g(x) = x \) and \( g(0) = 0 \), and \( s, t > 0 \). If \( T \in \text{class } A(s, t)-f \), then the following hold, where \( T = U|T| \) is the polar decomposition and \( \tilde{T}_{s,t} = |T|^tU|T|^t \):
(i) $\tilde{T}_{s,t}$ is $f$-hyponormal if $f \circ g^{-1}$ is operator monotone and $x^f \geq (f \circ g^{-1})(x^s)$.

(ii) $\tilde{T}_{s,t}$ is $g$-hyponormal if $g \circ f^{-1}$ is operator monotone and $(g \circ f^{-1})(x^f) \geq x^s$.

We use the following result in order to give a proof of Theorem 3.6.

Lemma 3.F ([16]). Let $A$ be a positive operator and $U$ be a partial isometry such that $N(U) \subseteq N(A)$, and let $f$ be a continuous function on $[0, \infty)$. Then

$$Uf(A)U^* = f(UAU^*) - f(0)(I -UU^*).$$

Proof of Theorem 3.6. Noting that $N(U^*) = N(|T^*|) \subseteq N(|T^*|^t|T|^{2s}|T^*|^{t})$,

$$f \left( (\tilde{T}_{s,t})^* \tilde{T}_{s,t} \right) = f(|T|^tU^*|T|^{2s}U|T|^t)$$

$$= f(U^*|T|^t|T|^{2s}|T^*|^{t}U)$$

$$= f(U^*|T|^t|T|^{2s}|T^*|^{t})U + f(0)(I -UU^*) \text{ by Lemma 3.F}$$

$$\geq U^*|T|^t|T|^{2s}U$$

$$= |T|^{2t}.$$  

On the other hand, $f(|T|^t|T|^{2s}|T^*|^{t}) \geq |T|^{2t}$ implies

$$|T|^{2t} \geq g(|T|^t|T|^{2s}|T^*|^{t}) = g(|T|^tU|T|^{2s}U^*|T|^t) = g \left( \tilde{T}_{s,t}(\tilde{T}_{s,t})^* \right)$$

by (i) of Theorem 3.D. If $f \circ g^{-1}$ is operator monotone and $x^f \geq (f \circ g^{-1})(x^s)$,

$$f \left( (\tilde{T}_{s,t})^* \tilde{T}_{s,t} \right) \geq f \left( \tilde{T}_{s,t}(\tilde{T}_{s,t})^* \right),$$

hence $\tilde{T}_{s,t}$ is $f$-hyponormal. If $g \circ f^{-1}$ is operator monotone and $(g \circ f^{-1})(x^f) \geq x^s$,

$$g \left( (\tilde{T}_{s,t})^* \tilde{T}_{s,t} \right) \geq (g \circ f^{-1})(|T|^{2t}) \geq |T|^{2t} \geq g \left( \tilde{T}_{s,t}(\tilde{T}_{s,t})^* \right),$$

hence $\tilde{T}_{s,t}$ is $g$-hyponormal.

References


[10] T. Furuta, A ≥ B ≥ 0 assures \((B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}\) for \(r \geq 0, p \geq 0, q \geq 1\) with \((1+2r)q \geq p+2r\), Proc. Amer. Math. Soc. 101 (1987), 85–88.


