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On numerical range and norm of the generalized Aluthge transform

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ABSTRACT
In this report, we have attempted to reveal some relationships between a bounded linear operator $T$ acting on a Hilbert space and its generalized Aluthge transformation $T(s, t)$ in terms of their numerical ranges and norms.

1. INTRODUCTION

Let $B(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. By the polar decomposition of $T \in B(\mathcal{H})$, we mean the expression $T = U|T|$, where $U$ is a partial isometry and $|T|$ is the positive square root of $T^*T$ such that $\ker U = \ker |T|$. In [1], Aluthge introduced the class of $p$-hyponormal operators that generalizes the widely studied class of hyponormal operators. In order to reveal some important features of $p$-hyponormal operators, he exploited the operator $\overline{T}$ which is now popularly known as the Aluthge Transformation and which is defined as

$$\overline{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Motivated by this article [1], several authors explored and studied new classes of operators closely connected to $p$-hyponormal operators with the help of the Aluthge transformation and its generalization, known as the generalized Aluthge transformation. By the generalized Aluthge transformation of $T \in B(\mathcal{H})$, we mean the bounded operator $T(s, t)$ on $\mathcal{H}$ for which

$$T(s, t) = |T|^sU|T|^t,$$

where $s \geq 0$ and $t \geq 0$.

Especially, $T(1, 0) = |T|^1U|T|^0 = |T|UU^*U = |T|U$ and $T(0, 1) = |T|^0U|T|^1 = U^*UU|T|$.

In recent years, one can find number of articles in which various relations among $T$, $\overline{T}$ and $T(s, t)$ are obtained. It is obvious that $\|\overline{T}\| \leq \|T\|$. Okubo [8] gave a non-obvious extension of this inequality by deriving $\|f(\overline{T})\| \leq \|f(T)\|$ for any polynomial $f(t)$ by proving more general result. As corollary to this inequality, he showed that $W(f(\overline{T})) \subseteq W(f(T))$.
extending some results known to be true [6] or in case either \( f(t) = t \) \([7, 9, 11]\). Our main object of the present report is to compare the numerical range of \( T \) with that of \( T(s, t) \) for some restricted values of \( s \) and \( t \).

In section 2, some results are given that will be of use in the succeeding sections. Section 3 is devoted to establishing inclusion relations among the numerical ranges of rational functions of operators \( T(0, 1), T(1, 0) \) and \( T \). The inequality that says \( \|f(T)\| \leq \|f(T(s, t))\| \) with \( s + t = 1 \) for every rational function \( f \) for which \( f(T) \) exists. Finally, in section 5, we introduce a numerical range value function on \([0, 1] \] and obtain an improvement over a characterization of convexoid matrices due to Ando [2].

In what follows, we assume, unless it is stated otherwise, that \( f \) will be a rational function with poles off \( \sigma(T) \).

2. FUNDAMENTAL PROPERTIES

**Lemma 2.1.** Let \( T = U|T| \) be the polar decomposition of \( T \). Then \( \dim \ker T \leq \dim \ker T^* \) if and only if there exists an isometry \( V \) such that \( V|T| = U|T| \).

Although our first lemma is well known \([5, p. 75], [10, p. 4]\), we would like to present it with a proof.

**Proof.** Let \( \mathcal{H} = \overline{R(|T|)} \oplus R(|T|) = \overline{R(T)} \oplus R(T) \). Then \( U \) is an isometry from \( \overline{R(|T|)} \) to \( \overline{R(T)} \). On the other hand, there exists an isometry \( U_1: R(|T|) \to R(T) \) if and only if \( \dim(R(|T|)) \leq \dim(R(T)) \). By \( R(|T|) = \ker |T| = \ker T \) and \( R(T) = \ker T^* \), it is equivalent to \( \dim \ker T \leq \dim \ker T^* \). So the underlying kernel condition ensures the existence of an isometry \( U_1: R(|T|) \to R(T) \). Let

\[
V = UU^*U + U_1(I - U^*U) = U + U_1(I - U^*U).
\]

The facts that \( I - U^*U \) is the projection onto \( \ker U = \ker T = \ker |T| = \overline{R(|T|)} \) \( = (\ker U_1) \), \( U_1^*U_1x = x \) on \( R(|T|) \) and \( R(U_1) \subseteq R(T) \) \( = \ker T^* = \ker U^* \) give

\[
V^*V = \{U + U_1(I - U^*U)\}^*\{U + U_1(I - U^*U)\}
\]

\[
= U^*U + (I - U^*U)U_1^*U + U^*U_1(I - U^*U) + (I - U^*U)U_1^*U_1(I - U^*U)
\]

\[
= U^*U + U^*U_1(I - U^*U) + U^*U_1(I - U^*U) + I - U^*U
\]

\[
= U^*U + I - U^*U
\]

\[
= I.
\]

Thus \( V \) is an isometry. Moreover,

\[
V|T| = \{U + U_1(I - U^*U)\}|T| = U|T|.
\]
Lemma 2.2. Let $A \in B(\mathcal{H})$. Then the following assertions hold:

(i) If $P$ is a projection with $PAP = AP$, then

$$f(AP) = Pf(A)P + f(0)(I - P).$$

(ii) If $V$ is an isometry, then

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

Proof. (i). Let $\mathcal{H} = (\ker P)^\perp \oplus \ker P$. Then by the assumption $PAP = AP$, $A$ can be expressed as follows:

$$A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \quad \text{on } \mathcal{H} = (\ker P)^\perp \oplus \ker P.$$

Hence

$$f(AP) = f(\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f(X) & 0 \\ 0 & f(0)I \end{pmatrix}.$$

On the other hand,

$$f(A) = f(\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}) = \begin{pmatrix} f(X) & Y' \\ 0 & f(Z) \end{pmatrix}.$$

Hence we have

$$Pf(A)P + f(0)(I - P) = \begin{pmatrix} f(X) & 0 \\ 0 & f(0)I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f(0)I \end{pmatrix} = f(AP).$$

(ii). For an isometry $V$, note that $\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix}$ is unitary. Then we have

$$\begin{pmatrix} f(VAV^*) & 0 \\ 0 & f(0)I \end{pmatrix} = f(\begin{pmatrix} VAV^* & 0 \\ 0 & 0 \end{pmatrix})$$

$$= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} f(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} V^* & 0 \\ 0 & V \end{pmatrix}$$

$$= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} f(A) & 0 \\ 0 & f(0)I \end{pmatrix} \begin{pmatrix} V^* & 0 \\ 0 & f(0)I \end{pmatrix}.$$

Hence

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

The following result is a modification of [6, Proposition 4.5].
Proposition 2.3. Let $A, B \in B(\mathcal{H})$. Then the following assertions are mutually equivalent:

(i) $\overline{W(f(A))} \subseteq \overline{W(f(B))}$ for all $f$.
(ii) $w(f(A)) \leq w(f(B))$ for all $f$.
(iii) $\|f(A)\| \leq \|f(B)\|$ for all $f$.

The proof is almost identical to the one given for Proposition 4.5 of [6].

3. Numerical Ranges of $T(0,1)$ and $T(1,0)$

The primary object of the present section is to establish the connection among the numerical ranges of $T$, $T(0,1)$ and $T(1,0)$.

Theorem 3.1. Let $T \in B(\mathcal{H})$. Then the following assertions hold:

(i) $W(f(T(0,1))) \subseteq W(f(T))$.
(ii) $W(f(T(1,0))) \subseteq W(f(T))$.

Proof. (i). Let $T = U|T|$ be the polar decomposition of $T$, and $\mathcal{H} = (\ker T)^{\perp} \oplus \ker T$. Then

$$T(0,1) = U^*UU|T| = U^*UT = U^*U^*U.$$

Since $U^*U$ is a projection, (i) of Lemma 2.2 yields

$$f(T(0,1)) = U^*Uf(T)U + f(0)(I - U^*U).$$

In case $\ker T = \{0\}$. In this case $U$ must be isometry. Then by (3.1), $f(T(0,1)) = f(T)$, and hence

$$W(f(T(0,1))) = W(f(T)).$$

In case $\ker T \neq \{0\}$. By (3.1), we obtain

$$W(f(T(0,1))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\}.$$

Here by $\ker T \neq \{0\}$, we have $f(0) \in W(f(T))$, and

$$W(f(T(0,1))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\} = W(f(T)).$$

(ii). Step 1. We shall show the following equality:

$$f(T(1,0)) = U^*f(T)U + f(0)(I - U^*U).$$

We shall establish this equality separately for each of the cases when $\dim \ker T \leq \dim \ker T^*$ and $\dim \ker T \geq \dim \ker T^*$.

(a) The case $\dim \ker T \leq \dim \ker T^*$. By the Lemma 2.1, there is an isometry $V$ satisfying $U|T| = V|T|$. Note that in the proof of Lemma 2.1,

$$V = U + U_1(I - U^*U),$$

where $U_1$ is isometry with $R(U_1) \subseteq \ker T^*$. 

(3.3)
Then by (3.3), we have
\[ UU^*TUU^* = TUU^* = U|T|UU^* = V|T|UV^*. \]
Hence by (ii) Lemma 2.2, we obtain
\[ f(TUU^*) = f(V|T|UV^*) = Vf(|T|U)V^* + f(0)(I - VV^*). \]
Moreover since \( V \) is isometry, we have
\[ f(|T|U) = V^*f(TUU^*)V. \]
Therefore
\[
f(|T|U) = V^*f(TUU^*)V
= V^{*}\{UU^{*}f(T)UU^{*} + f(0)(I - UU^{*})\}V \quad \text{by (i) of Lemma 2.2}
= U^{*}f(T)U + f(0)(I - U^{*}U) \quad \text{by (3.3)}.
\]
On the other hand, by (3.3),
\[ U|T^*| = UU^*UU|T^*| = VU^*UTV^*. \]
Then by Lemma 2.2, we obtain
\[
f(U|T^*|) = f(VU^*UTV^*)
= Vf(U^*UT)V^* + f(0)(I - VV^*) \quad \text{by (ii) of Lemma 2.2}
= V\{U^*Uf(T)U^*U + f(0)(I - U^*U)\}V^* + f(0)(I - VV^*) \quad \text{by (i) of Lemma 2.2}
= Uf(T)U^* + f(0)(I - UU^*) \quad \text{by (3.3)}.
\]
(b) The case \( \dim \ker T \geq \dim \ker T^* \). Replacing \( T \) by \( T^* \) in (3.4), we have
\[
f(U^*|T|) = U^*f(T^*)U + f(0)(I - U^*U)
\iff f(|T|U) = U^*f(T)U + f(0)(I - U^*U).
\]
Step 2. In case \( \ker T = \{0\} \). In this case \( U \) must be isometry. Then by (3.2), \( f(T(1,0)) = f(|T|U) = U^*f(T)U \), and hence
\[ W(f(T(1,0))) \subseteq W(f(T)). \]
In case \( \ker T \neq \{0\} \). By (3.2), we obtain
\[ W(f(T(1,0))) = W(f(|T|U)) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\}. \]
Here by \( \ker T \neq \{0\} \), we have \( f(0) \in W(f(T)) \), and
\[ W(f(T(1,0))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\} = W(f(T)). \]
Hence the proof is complete. \( \square \)
Corollary 3.2. Let $T = U|T|$. Then

(i) $W(T(1,0)) = W(T)$ if $\ker T^* \subseteq \ker T$.
(ii) $W(T(0,1)) = W(T)$ if $\ker T \subseteq \ker T^*$.
(iii) $W(T(0,1)) \subseteq W(T(1,0))$ if $\ker T^* \subseteq \ker T$.
(iv) $W(T(1,0)) \subseteq W(T(0,1))$ if $\ker T \subseteq \ker T^*$.

Proof. (i). In view of Theorem 3.1, only we have to prove $W(T) \subseteq W(T(1,0))$.
If $\ker T^* \subseteq \ker T$, then $T = U|T| = U|T|UU^* = UT(1,0)U^*$, and we have

$$W(T) \subseteq \text{conv}\{W(T(1,0)) \cup \{0\}\}.$$ If $\ker T \neq \{0\}$, then $0 \in W(T(1,0))$ and we have

$$W(T) \subseteq \text{conv}\{W(T(1,0)) \cup \{0\}\} = W(T(1,0)).$$

If $\ker T = \{0\}$, then $\{0\} = \ker T \varsubsetneq \ker T^*$, and $U^*$ must be an isometry. Hence we have $W(T) \subseteq W(T(1,0))$.

(ii). If $\ker T \subseteq \ker T^*$, then $U^*UU = U$ holds. Hence $T(0,1) = U^*UU|T| = U|T| = T$, and $W(T(0,1)) = W(T)$.

(iii). If $\ker T^* \subseteq \ker T$, then by (i) and Theorem 3.1, we have

$$W(T(0,1)) \subseteq W(T) = W(T(1,0)).$$

(iv). If $\ker T \subseteq \ker T^*$, then by (ii) and Theorem 3.1, we have

$$W(T(1,0)) \subseteq W(T) = W(T(0,1)).$$

$\square$

Remark 3.3. If we drop the kernel condition from the statements of Corollary 3.2, then we may not get the same conclusions as following indicate.

Example 3.4. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $U = T$. Clearly $W(T(1,0)) = \{0\} \neq W(T)$.

Example 3.5. For $\alpha > 0$, let

$$T = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ Then

$$T(0,1) = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(1,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Also $W(T(0, 1)) = \{ z \in \mathbb{C} : |z| \leq \frac{\alpha}{2} \}$ and $W(T(1, 0)) = \{ z \in \mathbb{C} : |z| \leq \frac{1}{2} \}$. Then

(i) for $\alpha \in (0, 1)$, $W(T(0, 1)) \subseteq W(T(1, 0))$,
(ii) for $\alpha > 1$, $W(T(1, 0)) \subseteq W(T(0, 1))$.

4. Norm inequality involving a rational function of $T$ and $T(s,t)$.

**Theorem 4.1.** Let $T \in B(\mathcal{H})$. Then

$$\|f(T(s,t))\| \leq \|f(T)\|$$

holds for $s, t \geq 0$ with $s + t = 1$.

**Proof.** Let $T = U|T|$ be the polar decomposition of $T$. Let $|T_\varepsilon| = |T| + \varepsilon I > 0$. Note that

$$\lim_{\varepsilon \to +0} |T_\varepsilon|^{-1}|T| = \lim_{\varepsilon \to +0} (|T| + \varepsilon I)^{-1}|T| = U^*U.$$

We prepare the important inequality due to [4]. For $X \in B(\mathcal{H})$ and positive operators $A$ and $B$,

$$\|A^sXB^s\| \leq \|AXB\|^s\|X\|^{1-s} \quad (4.1)$$

holds for $s \in [0,1]$. Then we have

$$\|f(T(s,t))\| = \|f(|T|U|T|^t)\|$$

$$= \|f(|T_\varepsilon|^s|T_\varepsilon|^{-s}|T|^sU|T|^t|T_\varepsilon|^s|T_\varepsilon|^{-s})\|$$

$$= \|[T_\varepsilon]^s f(|T_\varepsilon|^{-s}|T|^sU|T|^t|T_\varepsilon|^s|T_\varepsilon|^{-s})\|$$

$$\leq \|[T_\varepsilon]^s f(|T_\varepsilon|^{-s}|T|^sU|T|^t|T_\varepsilon|^s|T_\varepsilon|^{-s})\|^s \|f(|T_\varepsilon|^{-s}|T|^sU|T|^t|T_\varepsilon|^s|T_\varepsilon|^{-s})\|^t \text{ by (4.1)}$$

$$= \|f(|T|U^*U)|T|\|^t \|f(U^*U|T|)\|^s \text{ as } \varepsilon \to +0$$

$$= \|f(T(1,0))\|^s \|f(T(0,1))\|^t$$

$$\leq \|f(T)\| \quad \text{by Theorem 3.1 and Proposition 2.3.}$$

Hence the proof is complete. \qed

**Remark 4.2.** Above theorem is not true if $s + t \neq 1$ as can be illustrated with the following Example 4.3.

**Example 4.3.** Let $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ on $\mathcal{H} = \mathbb{C}^2$. Then $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $|T| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Also $T(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It easy to find that $\overline{W(T(2,1))} = [0, 2]$. Moreover $\overline{W(T)}$ is a closed elliptic disc with foci at 0 and 1, and the major axis $\sqrt{2}$ and the minor axis 1. This fact shows that $\overline{W(T)}$ excludes 2 and therefore $\overline{W(T(2,1))}$ is not a
subset of \( \overline{W(T)} \). If the theorem were true for \( s + t \neq 1 \), then we would have in particular, 
\[ \|T(s, t) - zI\| \leq \|T - zI\| \text{ for all } z. \] Then \( \overline{W(T(s, t))} \subseteq \overline{W(T)} \), which is not correct.

As a simple consequence of Proposition 2.3 and Theorem 4.1, we obtain the following corollary.

**Corollary 4.4.** Let \( T \in B(\mathcal{H}) \). Then
\[
\overline{W(f(T(s, t)))} \subseteq \overline{W(f(T))}
\]
holds for \( s, t \geq 0 \) with \( s + t = 1 \).

**Theorem 5.1.** For an operator \( T \) and \( x \in [0, 1] \), let
\[
F(x) = \overline{W(f(T(x, 1-x)))}.
\]

Then
\[
F(\alpha x + (1-\alpha)y) \subseteq \alpha F(x) + (1-\alpha)F(y)
\]
holds for all \( x, y \in [0, 1] \) and \( \alpha \in [0, 1] \).

As a consequence of Theorem 5.1, the function \( \Phi(x) = w(T(x, 1-x)) \) turns out to be a convex function on \([0, 1]\).

**Proof.** Let \( T = U|T| \) be the polar decomposition. Firstly, we shall prove
\[
F\left(\frac{x+y}{2}\right) \subseteq \frac{1}{2}\{F(x) + F(y)\}. \tag{5.2}
\]

Note that for a positive invertible operator \( S \) and \( A \in B(\mathcal{H}) \),
\[
\|A\| \leq \frac{1}{2}\|SAS^{-1} + S^{-1}AS\|.
\]
in [3]. Let \( \epsilon > 0 \) and \( |T_\epsilon| = |T| + \epsilon I \geq 0 \). By the above inequality, we obtain
\[
\|f(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right))\| = \|f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}})|T_\epsilon|^{\frac{x+y}{2}} + |T_\epsilon|^{\frac{x+y}{2}}f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}})|T_\epsilon|^{\frac{x+y}{2}}
\]
\[
\leq \frac{1}{2}\|f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}})|T_\epsilon|^{\frac{x+y}{2}} + |T_\epsilon|^{\frac{x+y}{2}}f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}})|T_\epsilon|^{\frac{x+y}{2}}
\]
\[
= \frac{1}{2}\|f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}}) + f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}})|T_\epsilon|^{\frac{x+y}{2}}\|
\]
\[
\rightarrow \frac{1}{2}\|f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}}) + f(|T|^\frac{x+y}{2}U|T|^{1-\frac{x+y}{2}})|| \text{ as } \epsilon \rightarrow 0
\]
\[
= \frac{1}{2}\|f(T(x, 1-x)) + f(T(y, 1-y))\|.
\]
Hence for any complex number $\lambda$,
\[
||f(T\left(\frac{x+y}{2}, 1-\frac{x+y}{2}\right)) - \lambda I|| \leq ||\frac{f(T(x,1-x)) + f(T(y,1-y))}{2} - \lambda I||.
\]
Since
\[
\overline{W(T)} = \bigcap_{\lambda \in \mathbb{C}} \{z \in \mathbb{C} : |z - \lambda| \leq ||T - \lambda I||\},
\]
we have
\[
F\left(\frac{x+y}{2}\right) \subseteq \frac{1}{2}\{\overline{W(f(T(x,1-x)))} + \overline{W(f(T(y,1-y)))}\} = \frac{1}{2}\{F(x) + F(y)\}.
\]

Next, we will extend (5.2) to (5.1). From (5.2), one can easily derive
\[
F\left(\frac{x_1 + x_2 + \cdots + x_{2^n}}{2^n}\right) \subseteq \frac{1}{2^n}\{F(x_1) + F(x_2) + \cdots + F(x_{2^n})\}
\]
for all $x_i \in [0,1]$ ($i = 1, 2, \cdots$). Hence for any rational number $\alpha \in [0,1]$, we have (5.1). Since $F$ is continuous, we have (5.1) for any real number $\alpha \in [0,1]$.

This completes the proof. $\square$

**Remark 5.2.** The conclusion of Theorem 5.1 cannot be strengthened further to
\[
F(\alpha x + (1-\alpha)y) = \alpha F(x) + (1-\alpha)F(y)
\]
as Example 5.3 will show. However, whether the range of $F$ is convex remains as an open problem.

**Example 5.3.** For $\alpha > 0$, let
\[
T = \begin{pmatrix}
0 & 16 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then
\[
T(s, t) = \begin{pmatrix}
0 & 16^t & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then $F(x) = W(T(x, 1-x)) = \{z : |z| \leq \frac{16^{1-x}}{2}\}$. Let $x = \frac{1}{4}$, $y = \frac{3}{4}$ and $\alpha = \frac{1}{2}$. Then

(i) $F(\frac{1}{2}) = F(\alpha x + (1-\alpha)y) = \{z : |z| \leq 2\}$. 

(ii) \( F(x) = \{ z : |z| \leq 4 \} \).

(iii) \( F(y) = \{ z : |z| \leq 1 \} \).

Hence
\[
F \left( \frac{1}{2} \right) = F (\alpha x + (1 - \alpha) y) = \{ z : |z| \leq 2 \}
\subseteq \{ z : |z| \leq \frac{5}{2} \} = \alpha F(x) + (1 - \alpha) F(y).
\]

**Corollary 5.4.** Let \( T \) be an operator. Then
\[
\overline{W(f(\tilde{T}))} = F \left( \frac{1}{2} \right) \subseteq \frac{1}{2} \{ F(s) + F(1-s) \} \subseteq \overline{W(f(T))}
\]
holds for all \( \frac{1}{2} \leq s \leq t \leq 1 \).

**Proof.** Since \( \frac{1}{2} = \frac{s + 1 - s}{2} \) and \( F(x) = \overline{W(f(T(x,1-x)))} \subseteq \overline{W(f(T))} \) for \( x \in [0,1] \), we have
\[
\overline{W(f(\tilde{T}))} = F \left( \frac{1}{2} \right) \subseteq \frac{1}{2} \{ F(s) + F(1-s) \} \text{ by Theorem 5.1}
\subseteq \overline{W(f(T))} \text{ by Corollary 4.4.}
\]

Next, let \( \frac{1}{2} \leq s \leq t \leq 1 \). Then we have \( [1-s,s] \subseteq [1-t,t] \). Then there exist \( \alpha_1, \alpha_2 \in [0,1] \) such that
\[
s = \alpha_1 t + (1 - \alpha_1)(1 - t) \quad \text{and} \quad 1 - s = \alpha_2 t + (1 - \alpha_2)(1 - t).
\]

By an easy calculation, we have \( \alpha_1 + \alpha_2 = 1 \), and by Theorem 5.1, we have
\[
\frac{1}{2} \{ F(s) + F(1-s) \} \subseteq \frac{1}{2} \{ \alpha_1 F(t) + (1 - \alpha_1)F(1-t) + \alpha_2 F(t) + (1 - \alpha_2)F(1-t) \}
\]
\[
= \frac{1}{2} \{ F(t) + F(1-t) \}.
\]

As a simple consequence of Corollary 4.4, one can see that if \( T \) is convexoid then so is \( T(s,t) \) with \( \overline{W(T(s,t))} = \overline{W(T)} \). The converse is obvious. However, if we do not assume \( \overline{W(T(s,t))} = \overline{W(T)} \), then mere convexoidity of \( T(s,t) \) does not guarantee that \( T \) is convexoid even if \( \mathcal{H} \) is finite dimensional. To see this, we refer to Example 4.3.

That convexoidity of \( \tilde{T} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) is clear from the fact that it is selfadjoint. On the other hand as \( \text{conv} \ \sigma(T) = [0,1] \neq \overline{W(T)} \), \( T \) is not convexoid. However, if \( \mathcal{H} \) is finite-dimensional, then our next result will show that the condition \( W(T(s,t)) = \overline{W(T)} \) is just equivalent to the convexoidity of \( T \). In case \( \mathcal{H} \) is infinite dimensional, we do not know the validity of this result.
Corollary 5.5. For a $n \times n$ matrix $T$, the following assertions are mutually equivalent:

(i) $T$ is convexoid.
(ii) $W(\tilde{T}) = W(T)$.
(iii) $W(T(s_0, 1-s_0)) = W(T)$ for a fixed $s_0 \in (0, 1)$.
(iv) $W(T(s, 1-s)) = W(T)$ for all $s \in [0, 1]$.

In order to prove Corollary 5.5, we shall need the following theorem, a remarkable result due to Ando [2].

Theorem A ([2]). Let $T$ be a $n \times n$ matrix. Then $T$ is convexoid if and only if $W(\tilde{T}) = W(T)$.

Proof. (i) $\iff$ (ii) has been shown in Theorem A. (iv) $\implies$ (ii), (iii) are obvious. So only we have to show (ii) $\implies$ (iv) and (iii) $\implies$ (ii).

Proof of (ii) $\implies$ (iv). Since $W(T(s, 1-s)) \subseteq W(T)$ and $W(T(1-s, s)) \subseteq W(T)$ for all $s \in [0, 1]$ hold and Corollary 5.4, we have

$$W(T) = W(\tilde{T}) \subseteq \frac{1}{2} \{W(T(s, 1-s)) + W(T(1-s, s))\} \subseteq \frac{1}{2} \{W(T(s, 1-s)) + W(T)\} \subseteq W(T).$$

Then we have

$$\frac{1}{2} \{W(T(s, 1-s)) + W(T)\} = W(T).$$

For any $\theta \in [0, 2\pi)$, let $\lambda$ be an extreme point of $\text{Re} \ e^{i\theta} W(T)$. Then by (5.3), there exist $\lambda_1 \in \text{Re} \ e^{i\theta} W(T)$ and $\mu_1 \in \text{Re} \ e^{i\theta} W(T(s, 1-s))$ such that

$$\lambda = \frac{\lambda_1 + \mu_1}{2}.$$ 

Since $\text{Re} \ e^{i\theta} W(T)$ is a line segment, and $\lambda$ is a extreme point of $\text{Re} \ e^{i\theta} W(T)$, it must be $\lambda = \lambda_1 = \mu_1 \in \text{Re} \ e^{i\theta} W(T(s, 1-s))$, i.e., $\text{Re} \ e^{i\theta} W(T) \subseteq \text{Re} \ e^{i\theta} W(T(s, 1-s))$ for any $\theta \in [0, 2\pi)$. Since $W(T)$ is convex, and $W(T(s, 1-s)) \subseteq W(T)$ always holds, we have $W(T) = W(T(s, 1-s))$ for all $s \in [0, 1]$.

Proof of (iii) $\implies$ (ii). We may assume $s_0 > \frac{1}{2}$. For each $s_0 \in (\frac{1}{2}, 1)$, there exists $\alpha \in (0, 1)$ such that

$$s_0 = \alpha \frac{1}{2} + (1-\alpha) \cdot 1.$$ 

Then by Theorem 5.1,

$$W(T) = W(T(s_0, 1-s_0)) \subseteq \alpha W(\tilde{T}) + (1-\alpha) W(T(1,0)) \subseteq W(T).$$

By the same argument of the above one, we have $W(\tilde{T}) = W(T)$. \qed
Remark 5.6. In (iii) of Corollary 5.5, $s_0$ must not be 0 or 1, because if $T$ is invertible, then $U$ is unitary and $W(T) = W(T(0, 1)) = W(T(1, 0))$. But in general, $W(T) \neq W(\bar{T})$.

REFERENCES