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Regularity and speed of the Hele-Shaw flow

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Abstract

This article summarizes the results of [CJK], where it is proven that if the Lipschitz constant of the initial free boundary is small, then for small positive time the solution is smooth and satisfies the Hele-Shaw equation in the classical sense. We will discuss the key ingredients of the proof and give a sketch of the main theorem at the end.

0 Introduction

Consider a compact set $K \subset \mathbb{R}^n$ with smooth boundary ∂K. Suppose that a bounded domain Ω contains K and let $\Omega_0 = \Omega - K$ and $\Gamma_0 = \partial\Omega$ (Figure 1). Note that $\partial\Omega_0 = \Gamma_0 \cup \partial K$.

Let u_0 be the harmonic function in Ω_0 with $u_0 = f > 0$ on K and zero on Γ_0. The classical Hele-Shaw problem models an incompressible viscous fluid which occupies part of the space between two parallel, narrowly placed plates. Suppose the fluid is being injected from ∂K into $\mathbb{R}^n - K$ with injection rate f. Assuming the effect of surface tension of the fluid to be

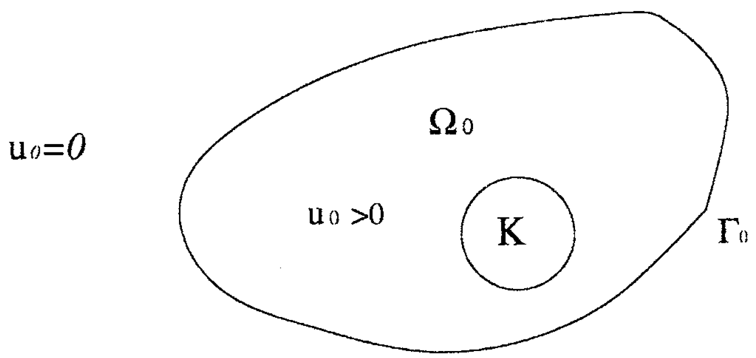

Figure 1.

zero, then u, the pressure of the fluid , solves the one phase Hele-Shaw problem

$$(HS) \qquad \begin{cases} -\Delta u = 0 & \text{in } \{u > 0\} \cap Q, \\ u_t - |Du|^2 = 0 & \text{on } \partial\{u > 0\} \cap Q, \\ u(x,0) = u_0(x); \quad u(x,t) = f \text{ for } x \in \partial K. \end{cases}$$

where $Q = (I\!R^n - K) \times (0, \infty)$. We refer to $\Omega_t(u) := \{u(\cdot, t) > 0\}$ as the *positive set* of u at time t and $\Gamma_t(u) := \partial\Omega_t(u) - \partial K$ as the *free boundary* of u at time t. Note that if u is smooth up to the free boundary, then the free boundary moves with normal velocity $V = u_t/|Du|$, and hence the second equation in (HS) implies that $V = |Du|$.

The short-time existence of classical solutions when Γ_0 is $C^{2+\alpha}$ was proved by Escher and Simonett [ES]. When $n = 2$, Elliot and Janovsky [EJ] showed the existence and uniqueness of weak solutions formulated by a parabolic variational inequality in $H^1(Q)$.

For our investigation we use a notion of viscosity solutions introduced in [K1], which we will explain in more details in section 2. We assume that Ω_0 is a Lipschitz domain in $I\!R^n$ with Lipschitz constant less than a dimensional constant a_n (In particular $a_2 = 1$.) For simplicity we only consider the case $f = 1$ and $K = B_r(0)$ for some $r > 0$. Consider a point $P \in B_1(0) \cap (I\!R^n - \bar{\Omega})$, and define

$$t(P) = \sup\{t > 0 : u(P, t) = 0\}.$$

In other words $t(P)$ is the time the free boundary reaches P. Our is an estimate on the size of $t(P)$ in terms of the initial data. Define $\delta = \delta(P) = dist(P, \bar{\Omega})$. Choose any point $z = z(P)$ in Ω such that $|P - z| = 2\delta$ and $dist(z, \partial\Omega) \geq \delta/2$.

Our main result is then as follows:

Theorem 0.1 (main theorem) *For $0 \leq t \leq t_0$, where t_0 only depends on the Lipschitz constant and the number of 'coordinate patches' of Γ_0,*

(i) the free boundary $\Gamma(u)$ of u is smooth in space and time.

(ii) The normal velocity of the $\Gamma_t(u)$ at P at $t = t(P)$ is indeed comparable to the average velocity $|Du_0(z(P))|$.

Remark

1. The proof of Theorem 0 (i) and (ii) is closely related, as we will discuss in the next section.

2. Even though the theorem is a local statement, for the proof it is necessary to observe not only local but also global movement of the free boundary, since irregularity of the free boundary from far away can affect the local behavior of the solution.

3. Even with smooth initial data, we do not expect the free boundary of (HS) to be smooth for a long time because different free boundary parts may collide into each other. See for example [H] where a solution of (HS) with initially smooth free boundary develops a cusp type singularity at a positive time. On the other hand if the initial data $u(\cdot, 0)$ is *starshaped*, that is if there is a uniform constant $C > 0$ such that

$$u((1+\epsilon)x, 0) \leq (1 + C\epsilon)u(x, 0) \text{ for any } \epsilon > 0,$$

then one can show from the results of [CJK] and [K2] that $u(\cdot, t)$ is also starshaped - in particular there is no collision of the free bounary parts - and u and $\Gamma(u)$ is smooth for all time.

4. If the Lipschitz constant of $\Omega_0(u)$ is too large then there may be a waiting time for the free boundary to move, in which case Theorem 0 fails. For example King, Lacey and Vazquez studied two dimensional global solutions of (HS) with [KLV] the initial positive set u is a global 'wedge' $\{(r, \theta) : \theta < a\}$. Here it is proven that there is a waiting time for the free boundary at $t = 0$ if and only if $a < \pi/4$. We also refer to [CK] for more general discussion on the waiting time phenomena of the solutions of (HS) in $I\!R^n$.

In section 2 we will introduce the key ingredients of the proof and then in section 3 we will sketch the proof of the main theorem.

1 Main Ingredients

1.1 Comparison principle

Due to the collision of different free boundary parts, the solution u may not even be continuous at certain times and the topology of the free boundary may change at different times. Hence it is necessary to adopt a weak notion of global-time solutions, and we use viscosity solutions of (HS) introduced in [K1].

Roughly speaking, the definition of viscosity solution is based on maximum principle-type statements and barrier arguments by smooth functions at 'regular' points. For example we define a lowersemicontinuous function $v(x,t)$ to be a viscosity supersolution of (HS) if no classical 'local subsolution' of (HS) can cross u from below. More precisely if v is a supersolution and if φ is a smooth, strictly subharmonic function in a parabolic neighborhood of (x,t) which touches v from below at (x,t) on the free boundary, then φ has to satisfy $\varphi_t(x,t) \geq |D\varphi|^2(x,t)$. We refer to [K1] for precise definition of sub- and supersolution of (HS). For $u(x,t)$ defined in a cylindrical domain $D \times (a,b)$, We define u^* as

$$u^*(x,t) = \limsup_{(\xi,s)\in D\times(a,b)\to(x,t)} u(\xi,s).$$

Definition 1.1 *u is a viscosity solution of (HS) if u is a supersolution of (HS) and u^* is a subsolution of (HS)*

In particular classical solutions of (HS) are viscosity solutions of (HS). The following properties of viscosity solutions are used frequently throughout our analysis:

Definition 1.2 *We say that a pair of functions $u_0, v_0 : \bar{D} \to [0,\infty)$ are (strictly) separated (denoted by $u_0 \prec v_0$) in $D \subset I\!R^n$ if*

(i) the support of u_0, $\mathrm{supp}(u_0) = \overline{\{u_0 > 0\}}$ restricted in $\bar{D}$ is compact and

(ii) in $\mathrm{supp}(u_0) \cap \bar{D}$ the functions are strictly ordered:

$$u_0(x) < v_0(x).$$

Theorem 1.3 (comparison principle) *Let u, v be respectively viscosity sub- and supersolutions in $D \times (0,T) \subset Q$ with initial data $u_0 \prec v_0$ in D. If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \bar{\Omega}(u)$ for $0 \leq t < T$, then $u(\cdot,t) \prec v(\cdot,t)$ in D for $t \in [0,T)$.*

Theorem 1.4 *For Ω_0 with small Lipschitz constant M, there is a unique viscosity solution u in Q with boundary data 1 and initial data u_0. Moreover u is harmonic in $\Omega(u)$.*

1.2 A Carlson-type Estimate

Our motivation for the analysis comes from a Carlson-type estimate for solutions of (HS). In the analysis of the free boundary behavior with initial wedge or cusp - type singularities in $\mathbb{R}^2$, Jerison and Kim [JeKi] observed that $t(P)$ satisfies the following estimate in terms of the initial data:

$$t(P) \simeq \delta(P)^2/u_0(z(P)) \tag{1.1}$$

if there is no initial waiting time of the free boundary. (Here $a \simeq b$ means that a/b is bounded above and below by positive constants.) This result also easily extends to the case of radially symmetric cones or cusps in higher dimensions. In particular (0.1) implies that the average normal velocity $\bar{V}_P$ of the free boundary moving from $P + z(P)/2$ to P between $t = 0$ and $t = t(P)$ is comparable to

$$\bar{V}_P \simeq u_0(z(P))/\delta(P) \simeq |Du_0(z(P))|. \tag{1.2}$$

In our investigation we were able to extend the estimates (1.1)-(1.2) to Lipschitz initial domains. The main step of the proof is showing that the global effect on the value of $u(P)$, caused by the movement of the free boundary far away from P, is under control. For simplicity of the statement, let us suppose that $u(x,0)$ is monotone decreasing in the direction $e_n = (0, ..., 1) \in \mathbb{R}^n$.

Lemma 1.5 *Let $P_0, Q_0 \in \Gamma_0$ and fix a time $t > 0$. suppose that we have $P_0 + re_n \in \Gamma_t$. Then there is $a < 1$ such that if $|P_0 - Q_0| \simeq 2^k r$ then $Q_0 + a^k 2^k re_n$ is outside $\Omega_t(u)$.*

(1.2) yields an estimate on the speed precisely up to order of magnitude of the speed of the free boundary in terms of initial condition. In fact Theorem 0.1 (ii) says indeed the normal velocity V of the free boundary at P is comparable to the average normal velocity $\bar{V}_P$.

1.3 Iteration argument

To prove the regularity of the free boundary, in several stages of the proof we adopt an iteration argument introduced by Caffarelli [C1],[ACS]. The iteration argument, even though quite involved, is based on the simple idea that the nice properties of the solution in the positive set 'propagates' to the free boundary in time.

To illustrate this idea,let us set $0 \in \Gamma_0(u)$ and suppose that for $t \in [0,1]$ $u(\cdot,t)$ is *ϵ-monotone* for a cone $\{x \in I\!R^n : (x,\nu) \geq |x|\cos\theta\}$ in $B_1(0)$ i.e.,

$$u(x) \geq \sup_{y \in B_{\epsilon \sin\theta}(x)} u(y - \epsilon e) \text{ for } x \in B_1(0).$$

(This holds for example if $\Gamma_t(u) \cap B_R(0)$, with large R, is bounded between two Lipschitz graphs $x_n = f_1(x'), x_n = f_2(x')$ by distance ϵ, where $x = (x', x_n)$.)

Then Corollary 1 of [C2] says that in the positive set, $C\epsilon$-away from $\Gamma_t(u)$, the level sets of $u(\cdot,t)$ are Lipschitz graphs. Using this property of the positive level sets we can apply an iteration argument to prove that for $1/2 \leq t \leq 1$, $\Gamma_t(u)$ is a Lipschitz graph.

An important obstacle in applying the iteration argument to our case is that, since u is a harmonic function at each time, the regularity of u in time is not necessarily better in the positive set than on the free boundary. We use the strong spatial regularity of u and the fact that the space and time affects each other by free boundary motion to overcome this difficulty. Another difficulty arises because we do not assume anything for the solution at positive times, making it necessary for us to establish a starting point for the iteration argument, such as proving the nondegeneracy of u on the free boundary.

1.4 The estimate on u_t

As mentioned above there is no apriori estimate on the time derivative of the solution in the positive set. Nevertheless due to the free boundary motion law $u_t = |Du|^2$, one can at least formally write down u_t as a harmonic function $\Omega_t(u)$ with boundary data $|Du|^2$ on $\Gamma(u)$, that is,

$$u_t(x,t) = \int_{\Gamma_t(u)} P_{\Omega_t(u)}(x,y)|Du(y,t)|^2 d\sigma(y).$$

where $P_{\Omega_t(u)}(x,y)$ is the Poisson kernel of $\Omega_t(u)$ with pole $x \in \Omega_t(u)$ evaluated at $y \in \Gamma_t(u)$. Note that $|Du(y,t)|$ is comparable to $P_{\Omega_t}(x_0,y)$ where x_0 is a unit distance away from $\Gamma_t(u)$. Hence if the domain Ω_t has small Lipschitz constant, then it follows from the above equality and the reverse Hölder inequality (see [JK1] for example) that

$$u_t \leq C|Du|^2$$

which provides an upper bound of u_t in terms of $|Du|$ in the positive set, thus connecting the space and the time derivative. Similar but more refined argument, using the regularity properties of the Poisson kernel in Lipschitz domains shown in [JK1] and [Jk2], will be applied in the last stage of our analysis when we prove that the solution is smooth in time.

2 Sketch of the Proof

Finally we sketch the proof of the main theorem. First we start our investigation by proving (1.1)-(1.2) for Lipschitz initial domains.

Next we prove that u remains Lipschitz in space for small time if the Lipschitz constant is sufficiently small. The main idea is to first prove the ϵ-monotonicity of u in space for small time, and then to follow the iteration argument for improving the monotonicity to Lipschitz continuity in space. For this argument we show the nondegeneracy of u on the free boundary in a corresponding scale to the monotonicity of u. (For $n = 2$ a relatively simple reflection argument can be used to derive the Lipschitz continuity of u in space for small time. In this case we only require the Lipschitz constant to be smaller than one, which is the case where there is no waiting time.)

The rest of the proof is concerned with proving the upper bound of the free boundary. The lack of upper bound for the time derivative of the solution makes this step challenging. Due to the free boundary motion law $V = |Du|$ where u is the solution associated with the free boundary, it is equivalent to study the upper bound of $|Du|$ on the free boundary. We prove that $u_t \leq C|Du|^2$, which yields an upper bound for the time derivative in the positive region of u, away from the free boundary. We run several versions of iteration argument, paying careful attention to the lack of the upper bound of the time derivative on the free boundary, to show that this upper bound obtained in the positive region propagates to the free boundary over time and the free boundary becomes smooth for positive small times and the solution satisfies (HS) in the classical sense.

We hope to extend our result to the case of the one-phase Stefan problem:

$$(St) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ u_t = |Du|^2 & \text{on } \partial\{u = 0\}, \\ u(x,0) = u_0(x) \geq 0. \end{cases}$$

We expect one of the main difficulties to be the lack of scaling under which the problem is invariant.

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