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Kyoto University
Relaxation in the Cauchy problem
for Hamilton-Jacobi equations

Hitoshi Ishii* (早稻田大学 教育・総合科学学術院) and Paola Loreti *

1. Introduction. In this note we study a little further the relaxation of Hamilton-Jacobi equations developed recently in [4,5]. In [4] we initiated the study of the relaxation of Hamilton-Jacobi equations of eikonal type and in [5] we extended this study to a larger class of Hamilton-Jacobi equations.

Let us recall the relaxation in calculus of variations. In general a non-convex variational problem (P) does not have its minimizer. A natural way to attack such a variational problem is to introduce its relaxed (or convexified) variational problem (RP) which has a minimizer and to regard such a minimizer as a generalized solution of the original problem (P). The main result (or principle) in this direction states that \( \min(RP) = \inf(P) \). That is, any accumulation point of a minimizing sequence of (P) is a minimizer of (RP). This fact or principle is called the relaxation of non-convex variational problems. See [3] for a treatment of the relaxation of non-convex variational problems.

Relaxation of Hamilton-Jacobi equations is the principle which says that the pointwise supremum over a suitable collection of Lipschitz continuous subsolutions in the almost everywhere sense of a non-convex Hamilton-Jacobi equation yields a viscosity solution of the equation with convexified Hamiltonian. See [4,5].

Here we are concerned with the Cauchy problem for Hamilton-Jacobi equations and generalize some results obtained in [5].

2. Main result for the Cauchy Problem. We consider the Cauchy Problem

(1) \( u_t(x, t) + H(x, D_x u(x, t)) = 0 \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T), \)

(2) \( u|_{t=0} = g, \)

* Department of Mathematics, School of Education, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan. Supported in part by Grant-in-Aid for Scientific Research, No. 15340051 and No. 14654032, JSPS. (ishii@edu.waseda.ac.jp)

* Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza", Via Scarpa n. 16, 00161 Roma, Italy (loreti@dm.uniroma1.it)
where \( H \) and \( g \) are given continuous functions respectively on \( \mathbb{R}^{2n} \) and \( \mathbb{R}^{n} \), \( T \) is a given positive number or \( T = \infty \), \( u = u(x,t) \) is the unknown continuous function on \( \mathbb{R}^{n} \times [0,T) \), \( u_{t} \) denotes the \( t \)-derivative of \( u \), and \( D_{x}u \) denotes the \( x \)-gradient of \( u \).

Let \( \hat{H} \) denote the convex envelope of the function \( H \), that is,

\[
\hat{H}(x,p) = \sup \{ l(p) \mid l \text{ affine function, } l(q) \leq H(x,q) \text{ for } q \in \mathbb{R}^{n} \}.
\]

We also consider the convexified Hamilton-Jacobi equation

\[
(3) \quad u_{t}(x,t) + \hat{H}(x, D_{x}u(x,t)) = 0 \quad \text{for } (x,t) \in \mathbb{R}^{n} \times (0,T).
\]

We use the notation: for \( a \in \mathbb{R}^{n} \) and \( r \geq 0 \), \( B^{n}(a, r) \) denotes the \( n \)-dimensional closed ball of radius \( r \) centered at \( a \). For \( \Omega \subset \mathbb{R}^{m} \), \( \text{BUC}(\Omega) \) and \( \text{UC}(\Omega) \) denote the spaces of bounded uniformly continuous functions on \( \Omega \) and of uniformly continuous functions on \( \Omega \), respectively. Furthermore, \( \text{Lip}(\Omega) \) denotes the space of Lipschitz continuous functions on \( \Omega \). Notice that \( f \in \text{Lip}(\Omega) \) is not assumed to be a bounded function.

Throughout this note we assume:

\[
(4) \quad H, \hat{H} \in \text{BUC}(\mathbb{R}^{n} \times B^{n}(0,R)) \quad \text{for all } R > 0.
\]

\[
(5) \lim_{R \to \infty} \inf \{ \frac{H(x,p)}{|p|} \mid (x,p) \in \mathbb{R}^{n} \times (\mathbb{R}^{n} \setminus B^{n}(0,R)) \} > 0.
\]

For \( R > 0 \) we define the function \( H_{R} : \mathbb{R}^{2n} \to \mathbb{R} \cup \{ \infty \} \) by

\[
H_{R}(x,p) = \begin{cases} 
H(x,p) & \text{if } x \in B^{n}(0,R), \\
\infty & \text{if } x \notin B^{n}(0,R),
\end{cases}
\]

and write \( \hat{H}_{R} \) for \( \hat{G} \), where \( G = H_{R} \).

\[
(6) \quad \text{For each } R > 0 \text{ and } \epsilon > 0 \text{ there is a constant } \rho \geq R \text{ such that}
\]

\[
\hat{H}_{\rho}(x,p) \leq \hat{H}(x,p) + \epsilon \quad \text{for } (x,p) \in \mathbb{R}^{n} \times B^{n}(0,R).
\]

\[
(7) \quad g \in \text{UC}(\mathbb{R}^{n}).
\]

**Proposition 1.** (i) If \( u \in \text{USC}(\mathbb{R}^{n} \times [0,T)) \) and \( v \in \text{LSC}(\mathbb{R}^{n} \times [0,T)) \) are a viscosity subsolution and a viscosity supersolution of (3) respectively. Assume that \( u(x,0) \leq v(x,0) \) for \( x \in \mathbb{R}^{n} \) and that there is a (concave) modulus \( \omega \) such that for all \( (x,t) \in \mathbb{R}^{n} \times [0,T) \) and \( y \in \mathbb{R}^{n} \),

\[
\left\{ \begin{array}{l}
u(x,t) \leq u(y,0) + \omega(|x-y| + t), \\
u(x,t) \geq v(y,0) - \omega(|x-y| + t).
\end{array} \right.
\]

Then \( u \leq v \) on \( \mathbb{R}^{n} \times [0,T) \). (ii) There is a (unique) viscosity solution \( u \in \text{UC}(\mathbb{R}^{n} \times [0,\infty)) \) of (3) which satisfies (2). If, in addition, \( g \in \text{Lip}(\mathbb{R}^{n}) \), then \( u \in \text{Lip}(\mathbb{R}^{n} \times [0,\infty)) \).
We remark that the same proposition as above is valid for (1). We omit giving the proof of the above proposition.

Let $\mathcal{V}_T$ denote the set of functions $v \in \text{Lip}(\mathbb{R}^n \times [0,T))$ such that

$$u_t(x,t) + H(x,D_xu(x,t)) \leq 0 \quad \text{a.e. } (x,t) \in \mathbb{R}^n \times (0,T).$$

The following theorem is the main result in this note.

**Theorem 2.** Assume that (4)--(7) hold. Let $u \in \text{UC}(\mathbb{R}^n \times [0,T))$ be the unique viscosity solution of (3) satisfying (2). Then, for $(x,t) \in \mathbb{R}^n \times [0,T),

$$u(x,t) = \sup \{v(x,t) \mid v \in \mathcal{V}_T, v|_{t=0} \leq g\}.$$ \hfill (9)

**Remark.** In general the above formula does not give a subsolution of

$$u_t(x,t) + H(x,D_xu(x,t)) = 0 \quad \text{a.e. } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

For instance, let $n = 2$ and define $H \in C(\mathbb{R}^2)$ and $g \in \text{UC}(\mathbb{R}^2)$ by $H(p,q) = (|p|^{\frac{1}{2}} + |q|^{\frac{1}{2}})^2$ and $g(x,y) = -|x| - |y|$, respectively. Note that $\tilde{H}(p,q) = |p| + |q|$ for $(p,q) \in \mathbb{R}^2$. We set $\rho(x,y,t) = -2t - |x| - |y|$. Then, for instance, by computing $D^\pm \rho(x,y,t)$, we infer that $\rho$ is the viscosity solution of

$$\begin{cases}
  u_t(x,y,t) + |u_x(x,y,t)| + |u_y(x,y,t)| = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\
  u(x,y,0) = g(x,y) & \text{for } (x,y) \in \mathbb{R}^2.
\end{cases}$$

On the other hand, since at any point $(x,y,t) \in \mathbb{R}^2 \times (0,\infty)$, where $x, y \neq 0$, we have $H(\rho_x(x,y,t),\rho_y(x,y,t)) = 4$, $\rho_t(x,y,t) = -2$, $\rho$ is not a subsolution of

$$u_t(x,y,t) + (|u_x(x,y,t)|^{\frac{1}{2}} + |u_y(x,y,t)|^{\frac{1}{2}})^2 = 0 \quad \text{a.e. } (x,y,t) \in \mathbb{R}^n \times (0,\infty).$$

Theorem 2 is an easy consequence of the following theorem.

**Theorem 3.** Assume that (4)--(6) hold. Let $u \in \text{UC}(\mathbb{R}^n \times [0,T))$ be a viscosity subsolution of (3). Then, for all $(x,t) \in \mathbb{R}^n \times [0,T),

$$u(x,t) = \sup \{v(x,t) \mid v \in \mathcal{V}_T, v \leq u \text{ in } \mathbb{R}^n \times [0,T]\}.$$ \hfill (10)

Conceding Theorem 3 for the moment, we finish the proof of Theorem 2 as follows.
Proof of Theorem 2. We write \( w(x, t) \) for the right hand side of (9). By Theorem 3 we find that \( u \leq w \) on \( \mathbb{R}^n \times [0, T) \). Let \( v \in \mathcal{V}_T \) satisfy \( v(\cdot, 0) \leq g \) on \( \mathbb{R}^n \). Then, since \( \hat{H} \leq H \), we have

\[
v_t(x, t) + \hat{H}(x, D_x v(x, t)) \leq 0 \quad \text{a.e. } (x, t) \in \mathbb{R}^n \times (0, T).
\]

Since \( \hat{H}(x, \cdot) \) is convex, \( v \) is a viscosity subsolution of (3). By (i) of Proposition 1, we have \( v \leq u \) on \( \mathbb{R}^n \times (0, T) \), from which we get \( w \leq u \) on \( \mathbb{R}^n \times (0, T) \). Thus we have \( u = w \) on \( \mathbb{R}^n \times (0, T) \). \( \square \)

For our proof of Theorem 3, we need several lemmas. For a proof of the next three lemmas, we refer to [5].

Lemma 4. Let \( K \) be a non-empty convex subset of \( \mathbb{R}^m \) and set

\[
L(\xi) = \sup\{\xi \cdot p \mid p \in K\} \in \mathbb{R} \cup \{\infty\} \quad \text{for all } \xi \in \mathbb{R}^m.
\]

Let \( U \) be an open subset of \( \mathbb{R}^m \) and let \( v \in C(\overline{U}) \) satisfy

\[
D^+ v(x) \subset K \quad \text{for all } x \in U.
\]

Let \( x, y \in \overline{U} \), and assume that the open line segment \( l_0(x, y) := \{tx + (1-t)y \mid t \in (0, 1)\} \subset U \). Then

\[
u(x) \leq u(y) + L(x - y).
\]

In the above lemma and in what follows, for \( v \in C(U) \) and \( x \in U \), \( D^+ v(x) \) denotes the superdifferential of \( v \) at \( x \).

Lemma 5. Let \( \Omega \) be an open subset of \( \mathbb{R}^m \) and \( f_1, \ldots, f_N \in \text{Lip}(\Omega) \), with \( N \in \mathbb{N} \). Set

\[
f(x) = \max\{f_1(x), \ldots, f_N(x)\} \quad \text{for } x \in \Omega.
\]

Then \( f \in \text{Lip}(\Omega) \) and \( f, f_1, \ldots, f_N \) are almost everywhere differentiable. Moreover for almost every \( x \in \Omega \),

\[
Df(x) \in \{Df_1(x), \ldots, Df_N(x)\},
\]

where \( Df(x) \) denotes the gradient of \( f \) at \( x \).

Lemma 6. Let \( Z \) be a non-empty closed subset of \( \mathbb{R}^m \). Define \( L : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\} \) by

\[
L(\xi) = \sup\{\xi \cdot p \mid p \in Z\}.
\]

Let \( \xi \in \mathbb{R}^m \) be a point where \( L \) is differentiable. Then

\[
DL(\xi) \in Z \cap \partial(\overline{\text{co} Z})
\]
We introduce the notation: for \((x, r) \in \mathbb{R}^n \times \mathbb{R}\) let
\[
Z(x, r) := \{(p, q) \in \mathbb{R}^{n+1} | q + H(x, p) \leq r\}
\]
and \(K(x, r) := \overline{\text{co}} Z(x, r)\), the closed convex hull of \(Z(x, r)\). We note that
\[
K(x, r) = \{(p, q) \in \mathbb{R}^{n+1} | q + \hat{H}(x, p) \leq r\}.
\]
For \(\delta > 0\), let \(\Delta(\delta) := \{(x, y) \in \mathbb{R}^{2n} | |x - y| \leq \delta\}\).

**Lemma 7.** Assume that (4) holds. For any \(R > 0\) and \(\epsilon > 0\) there exists a constant \(\delta > 0\) such that for any \((x, y) \in \Delta(\delta)\) and \(r \in \mathbb{R}\),
\[
Z_{R}(x, r) + B^{n+1}(0, \delta) \subset Z_{R+1}(y, r + \epsilon),
\]
where, for \(R > 0\), \(Z_{R}(x, r) = Z(x, r) \cap B^{n+1}(0, R)\).

**Proof.** Fix \(\epsilon > 0\) and \(R > 0\). Let \(\omega\) denote the modulus of continuity of \(H\) on \(\mathbb{R}^n \times B^n(0, R+1)\).

Fix a constant \(\delta \in (0, 1)\) so that \(\delta + \omega(2\delta) \leq \epsilon\). Fix \((\xi, \eta) \in B^{n+1}(0, \delta)\), \((x, y) \in \Delta(\delta)\), \((p, q) \in Z_{R}(x, 0)\), and \(r \in \mathbb{R}\).

Noting that \((p, q) + (\xi, \eta) \in B^{n+1}(0, R + 1)\), we observe that
\[
q + \eta + H(y, p + \xi) \leq q + H(x, p) + \eta + \omega(|x - y| + |\xi|) \leq r + \delta + \omega(2\delta) \leq r + \epsilon.
\]
Thus we have
\[
(p + \xi, q + \eta) \in Z_{R+1}(y, r + \epsilon),
\]
which concludes the proof. \(\Box\)

**Lemma 8.** Assume that (4)--(6) hold. For any \(R > 0\) and \(\epsilon > 0\) there exists a constant \(M \geq R\) such that for any \(x \in \mathbb{R}^n\),
\[
K_{R}(x, 0) \subset \text{co} Z_{M}(x, \epsilon),
\]
where \(K_{R}(x, r) = K(x, r) \cap B^{n+1}(0, R)\).

**Proof.** For \(R > 0\) and \(\epsilon > 0\) let \(\rho \equiv \rho(R, \epsilon) \geq R\) be the constant from (6). That is, \(\rho = \rho(R, \epsilon)\) is a constant for which
\[
\hat{H}_{\rho}(x, p) \leq \hat{H}(x, p) + \epsilon \quad \text{for} \ (x, p) \in \mathbb{R}^n \times B^n(0, R).
\]
In view of (4), for \(R > 0\) let \(M_{R} \geq 0\) be the constant defined by
\[
M_{R} = \sup\{|H(x, p)| \ | (x, p) \in \mathbb{R}^n \times B^n(0, R)\}.\]
Fix $R > 0$, $\epsilon > 0$, $x \in \mathbb{R}^n$, and $(p, q) \in K_R(x, 0)$. We have

$$\hat{H}(x, p) + q \leq 0,$$

and hence

$$\hat{H}_\rho(x, p) + q \leq \epsilon.$$

Choose sequences $\{\lambda_i\}_{i=1}^m \subset (0, 1]$ and $\{p_i\}_{i=1}^m \subset B^n(0, \rho)$, with $m \in \mathbb{N}$, so that

$$\sum_{i=1}^m \lambda_i p_i = p, \quad \sum_{i=1}^m \lambda_i = 1,$$

$$\sum_{i=1}^m \lambda_i H(x, p_i) + q \leq 2\epsilon.$$

(See the proof of Lemma 10 below.) Setting

$$h = q + \sum_{i=1}^m \lambda_i H(x, p_i), \quad q_i = h - H(x, p_i) \quad \text{for } i = 1, 2, \ldots, m,$$

we observe that

$$h \leq 2\epsilon, \quad h \geq -|q| - M_\rho \geq -R - M_\rho,$$

$$|q_i| \leq |h| + M_\rho \leq 2\epsilon + R + 2M_\rho \quad \text{for } i = 1, 2, \ldots, m,$$

and that

$$(p_i, q_i) \in Z(x, h) \subset Z(x, 2\epsilon) \quad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^m \lambda_i q_i = h - \sum_{i=1}^m \lambda_i H(x, p_i) = q,$$

$$\sum_{i=1}^m \lambda_i (p_i, q_i) = (p, q).$$

These together show that $(p, q) \in \text{co} Z_M(x, 2\epsilon)$, with $M = (\rho^2 + (2\epsilon + R + 2M_\rho)^2)^{1/2}$.

\[\square\]

**Proof of Theorem 3.** We write $Q = \mathbb{R}^n \times (0, T)$ and $Q_\delta = \mathbb{R}^n \times (-\delta, T + \delta)$ for $\delta > 0$.

Firstly, without loss of generality we may assume that $u$ is defined and Lipschitz continuous on $Q_\delta$ for some constant $\delta > 0$ and that

$$u_t(x, t) + \hat{H}(x, D_x u(x, t)) \leq 0 \quad \text{in } Q_\delta$$

(11)
in the viscosity sense. Indeed, we have

\[(12) \quad u(x, t) = \sup \{v(x, t) \mid v \in \text{Lip}(Q_\delta) \text{ for some } \delta > 0, \ v \text{ is a viscosity solution of } (11), \ v \leq u \text{ on } Q} \].

To see this, assuming \( T < \infty \), we solve the Cauchy problem

\[
w_t(x, t) + \hat{H}(x, D_x w(x, t)) \leq 0 \quad \text{in } \mathbb{R}^n \times (T, T + 1)
\]

with the initial condition

\[(13) \quad w(x, T) = \lim_{t \nearrow T} u(x, t) \quad \text{for } x \in \mathbb{R}^n.
\]

In view of (4) and (5), there is a constant \( C > 0 \) such that \( \hat{H}(x, p) \geq -C \) for all \( (x, p) \in \mathbb{R}^{2n} \), which shows that \( u \) is a viscosity solution of \( u_t \leq C \) in \( \mathbb{R}^n \times (0, T) \). This monotonicity of the function \( u(x, t) \) in \( t \) and the uniform continuity of \( u \) guarantee that the limit on the right hand side of (13) defines a uniform continuous function on \( \mathbb{R}^n \).

By (ii) of Proposition 1, there is a unique viscosity solution \( w \in UC(\mathbb{R}^n \times [T, T + 1]) \) for which (13) holds. We extend the domain of definition of \( w \) to \( \mathbb{R}^n \times (0, T + 1) \) by setting

\[
w(x, t) = u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T).
\]

It is easy to see that \( w \in UC(\mathbb{R}^n \times (0, T + 1)) \) that \( w \) is a viscosity subsolution of

\[
w_t(x, t) + \hat{H}(x, D_x w(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, T + 1).
\]

Now, if \( T = \infty \), we define \( w \in UC(\mathbb{R}^n \times [0, \infty)) \) by setting \( w = u \).

Fix any \( \epsilon > 0 \). Since \( w \in UC(\mathbb{R}^n \times (0, T + 1)) \), there is a constant \( \delta \in (0, 1/2) \) such that

\[(14) \quad u(x, t) - 2\epsilon \leq w(x, t - \delta) - \epsilon \leq u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T).
\]

It is clear that the function \( z(x, t) := w(x, t - \delta) - 2\epsilon \) is defined and uniformly continuous on \( Q_\delta \) and is a viscosity solution of (11).

Now, we take the sup-convolution of \( z \) in the \( t \)-variable. That is, for \( \gamma > 0 \), we consider the function

\[
z^\gamma(x, t) = \sup \{z(x, s) - \frac{1}{2\gamma}(t - s)^2 \mid s \in (-\delta, T + \delta)\} \quad \text{for } (x, t) \in \mathbb{R}^{n+1}.
\]

If \( \gamma > 0 \) is small enough, then \( z^\gamma \) is a viscosity solution of (11) in \( Q_{\delta/2} \) and

\[(15) \quad z(x, t) \leq z^\gamma(x, t) \leq z(x, t) + \epsilon \quad \text{for } (x, t) \in Q_\delta.
\]
Note also that, for each $\gamma > 0$, the collection of functions $z^\gamma(x, \cdot)$, with $x \in \mathbb{R}^n$, is equi-Lipschitz continuous on $(-\delta/2, T + \delta/2)$. By virtue of (5), we may choose constants $c_0 > 0$ and $C_1 > 0$ such that

$$\hat{H}(x, p) \geq c_0 |p| - C_1 \quad \text{for } (x, p) \in \mathbb{R}^{2n}. $$

Since $z^\gamma$ is a viscosity solution of

$$c_0 |D_x z^\gamma(x, t)| \leq C_1 + L_{\gamma}$$

in $Q_{\delta/2}$, where $L_{\gamma} > 0$ is a uniform Lipschitz bound of the functions $z^\gamma(x, \cdot)$ on $(-\delta/2, T + \delta/2)$, we see that the functions $z^\gamma(\cdot, t)$ are Lipschitz continuous on $\mathbb{R}^n$, with a Lipschitz bound independent of $t \in (-\delta/2, T + \delta/2)$.

Now, using (14) and (15) and writing $U(x, t)$ for the right hand side of (12), we see that for sufficiently small $\gamma > 0$ and for all $(x, t) \in Q$,

$$u(x, t) \geq z(x, t) + \epsilon \geq z^\gamma(x, t),$$

and hence,

$$U(x, t) \geq z^\gamma(x, t) \geq z(x, t) \geq u(x, t) - 3\epsilon,$$

which proves (12).

Henceforth we assume that, for some constant $\delta > 0$, $u$ is a member of Lip($Q_\delta$) and satisfies (11) in the viscosity sense.

Let $R > 0$ be a Lipschitz bound of the function $u$. Fix any $\epsilon \in (0, 1)$. Due to Lemma 8, there is a constant $\rho \geq R$ such that for all $x \in \mathbb{R}^n$,

$$K_R(x, 0) \subset \text{co } Z_\rho(x, \epsilon).$$

In view of Lemma 7, there is a constant $\gamma \in (0, 1)$ such that for any $(x, y) \in \Delta(\gamma)$,

$$Z_\rho(x, \epsilon) + B^{n+1}(0, \gamma) \subset Z_{\rho+1}(y, 2\epsilon).$$

Consequently, for $(x, y) \in \Delta(\gamma)$, we have

(16) $$K_R(x, 0) + B^{n+1}(0, \gamma) \subset \text{co } Z_{\rho+1}(y, 2\epsilon).$$

(17) $$Z_{\rho+1}(y, 2\epsilon) \subset Z_{\rho+2}(x, 3\epsilon).$$

We may assume that $\gamma < \delta$. Let $\mu \in (0, \gamma)$ be a constant to be fixed later. We choose a set $Y_\mu \subset Q_\delta$ so that

(18) $$\#(Y_\mu \cap B^{n+1}(0, r)) < \infty \quad \text{for all } r > 0,$$

(19) $$\bigcup_{(y, s) \in Y_\mu} B^{n+1}((y, s), \mu) \supset Q_\delta.$$
We set
\[ L(\xi, \eta; y) = \sup\{\xi \cdot p + \eta q | (p, q) \in Z_{\rho+1}(y, 2\epsilon)\} \quad \text{for } \xi, y \in \mathbb{R}^n, \eta \in \mathbb{R} \]
and
\[ v(x, t; y, s) = u(y, s) + L(x - y, t - s; y) \quad \text{for } (x, t) \in \mathbb{R}^{n+1}, (y, s) \in Q_\delta. \]

By Lemma 6, we get for \((x, y) \in \Delta(\gamma)\),
\[ D_{\xi, \eta}^*L(\xi, \eta; y) \in Z_{\rho+1}(y, 2\epsilon) \subset Z_{\rho+2}(x, 3\epsilon) \quad \text{a.e. } (\xi, \eta) \in \mathbb{R}^{n+1}. \]

Noting that \(D^+u(x, t) \subset K_R(x, 0)\) for \((x, t) \in Q_\delta\), and setting \(\tilde{u}(x, t) := u(x, t) + \gamma|\langle x, t \rangle - (y, s)|\) for \((x, t), (y, s) \in Q_\delta\), we find that for \((x, t), (y, s) \in Q_\delta\), if \(0 < |x - y| \leq \gamma\), then
\[ D^+\tilde{u}(x, t) \subset D^+u(x, t) + B^{n+1}(0, \gamma) \subset \text{co} Z_{\rho+1}(y, 2\epsilon). \]

Hence, by Lemma 4, we get
\[ u(x, t) + \gamma|\langle x, t \rangle - (y, s)| \leq v(x, t; y, s) \quad \text{for } (x, t), (y, s) \in Q_\delta, \text{ with } |x - y| \leq \delta. \]

Set \(\beta = \gamma/5\) and define the function \(w : Q_{2\beta} \rightarrow \mathbb{R}\) by
\[ w(x, t) = \min\{v(x, t; y, s) | (y, s) \in Y_\mu \cap B^{n+1}((\overline{x}, t\gamma, 2\beta))\}. \]

Now, we show that if \(\mu\) is sufficiently small, then for \((\overline{x}, \overline{t}) \in Q_\beta\) and \((x, t) \in B^{n+1}((\overline{x}, \overline{t}), \beta)\)
\[ w(x, t) = \min\{v(x, t; y, s) | (y, s) \in Y_\mu \cap B^{n+1}((\overline{x}, \overline{t}, 2\beta))\}. \]

To do this, fix \((\overline{x}, \overline{t}) \in Q_\beta\) and \((x, t) \in Y_\mu \cap B^{n+1}((\overline{x}, \overline{t}), 2\beta)\). Noting that \(Y_\mu \cap B^{n+1}((x, t), \mu) \neq \emptyset\) and \(B^{n+1}((x, t), \mu) \subset B^{n+1}((x, t), 5\beta)\) and choosing a point \((y, s) \in Y_\mu \cap B^{n+1}((x, t), \mu)\), we see that
\[ w(x, t) \leq v(x, t; y, s) \leq u(y, s) + (\rho + 1)|\langle x, t \rangle - (y, s)| \leq u(x, t) + (R + \rho + 1)|\langle x, t \rangle - (y, s)|. \]

Here we have used the fact that the functions \(L(\xi, \eta; y)\) of \((\xi, \eta)\) are Lipschitz continuous functions with \(\rho + 1\) as a Lipschitz bound. Fix now \(\mu \in (0, \gamma)\) by setting
\[ \mu = \frac{1}{2} \min\{\gamma, \frac{\gamma\beta}{R + \rho + 1}\}. \]
and observe that

(23) \[ w(x, t) < u(x, t) + \gamma \beta. \]

Fix \((y, s) \in Q_i \setminus B^{n+1}(\bar{x}, \bar{t}, 2\beta)\) and note that \(|(y, s) - (x, t)| \geq \beta\). Using (21), we have

\[ v(x, t; y, s) \geq u(x, t) + \gamma \beta. \]

From this and (23), we conclude that (22) holds.

Next, we observe from (22) that the function \(w\) is Lipschitz continuous on \(B^{n+1}(\bar{x}, \bar{t}, \beta)\) for all \((\bar{x}, \bar{t}) \in Q_\beta\), with \(\rho + 1\) as a Lipschitz bound, which guarantees that \(w \in \text{Lip} (Q_\beta)\). Applying Lemma 5 and using (20), we observe that \(w\) is almost everywhere differentiable on \(Q_\beta\) and, at any point \((x, t) \in Q_\beta\) where \(w\) is differentiable,

\[ Dw(x, t) \in \bigcup \{D_{x, t}v(x, t; y, s) \mid (y, s) \in Y_{\mu} \cap B^{n+1}(\bar{x}, \bar{t}, 2\beta)\} \subset Z_{\rho+2}(x, 3\epsilon), \]

which yields readily

\[ w_t(x, t) + H(x, D_xw(x, t)) \leq 3\epsilon \quad \text{a.e.} \ (x, t) \in Q_\beta. \]

Setting

\[ z(x, t) = w(x, t) - \gamma \beta - 3\epsilon t \quad \text{for} \ (x, t) \in Q_\beta, \]

we have

\[ z_t(x, t) + H(x, D_xz(x, t)) \leq 0 \quad \text{a.e.} \ (x, t) \in Q_\beta. \]

By (23), we have \(z(x, t) \leq u(x, t) - 3\epsilon t\) for \((x, t) \in Q_\beta\) and, by (21), we have \(z(x, t) \geq u(x, t) - \gamma \beta - 3\epsilon t\) for \((x, t) \in Q_\beta\). In the above two inequalities, we may take \(\gamma > 0\) as small as we wish. Thus we get

\[ u(x, t) = \sup \{z(x, t) \mid z \in \mathcal{V}_T, \ z \leq u \text{ on } Q\} \quad \text{for} \ (x, t) \in Q, \]

which completes the proof. \(\square\)

3. Examples. In this section we consider some examples of Hamiltonians \(H\) and examine if \(H\) satisfies conditions (4)–(6) or not.

Let \(H \in C(\mathbb{R}^{2n})\) be a function of the form

\[ H(x, p) = G(x, p)^m + f(x), \]

where \(G \in C(\mathbb{R}^{2n})\) satisfies

(24) \[ G \in \text{BUC}(\mathbb{R}^n \times B^n(0, R)) \quad \text{for } R > 0, \]

(25) \[ G(x, \lambda p) = \lambda G(x, p) \quad \text{for } \lambda \geq 0, (x, p) \in \mathbb{R}^{2n}, \]

(26) \[ \delta_G := \inf_{\mathbb{R}^n \times \partial B^n(0,1)} G > 0. \]
Proposition 9. The function $H$ given above satisfies (4)–(6).

We need the following Lemma.

Lemma 10. For all $(x, p) \in \mathbb{R}^{2n}$, we have

$$
(27) \quad \hat{G}(x, p) = \min\{r \in \mathbb{R} \mid p = \sum_{i=1}^{k} \lambda_i p_i, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, G(x, p_i) = r\}.
$$

Proof. We fix $x \in \mathbb{R}^n$ and write $G(p)$ for $G(x, p)$ for notational simplicity. By using the separation theorem and Carathéodory's theorem in convex analysis, we see easily that

$$
(28) \quad \hat{G}(p) = \inf\{\sum_{i=1}^{n+1} \lambda_i G(p_i) \mid \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i p_i = p\} \quad \text{for } p \in \mathbb{R}^n.
$$

It is clear from the above representation formula that

$$
\hat{G}(\lambda p) = \lambda \hat{G}(p) \quad \text{for } (\lambda, p) \in [0, \infty) \times \mathbb{R}^n,
$$

$$
G(p) \geq \hat{G}(p) \geq \delta_G|p| \quad \text{for } p \in \mathbb{R}^n.
$$

Fix $p \in \mathbb{R}^n$. If $p = 0$, then it is clear that (27) holds. We may thus assume that $p \neq 0$. For any $r > \hat{G}(p)$, by the above formula, there are $\{\lambda_i\}_{i=1}^{n+1} \subset [0, 1]$ and $\{p_i\}_{i=1}^{n+1} \subset \mathbb{R}^n$ such that

$$
r > \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \sum_{i=1}^{n+1} \lambda_i = 1, \quad \sum_{i=1}^{n+1} \lambda_i p_i = p.
$$

Set

$$
s = \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \mu_i = s^{-1}G(p_i).
$$

Notice that $s \geq \hat{G}(p) > 0$ by (28). By rearranging the order in $i$ if necessary, we may assume that

$$
\lambda_i \mu_i > 0 \quad \text{for } i \leq k,
$$

$$
\lambda_i \mu_i = 0 \quad \text{for } i > k
$$

for some $k \in \{1, \ldots, n+1\}$. Note that if $i > k$ and $\lambda_i > 0$, then $p_i = 0$. We now have

$$
\sum_{i=1}^{k} \lambda_i \mu_i = s^{-1} \sum_{i=1}^{n+1} \lambda_i G(p_i) = 1,
$$

$$
\sum_{i=1}^{k} \lambda_i \mu_i (\mu_i^{-1} p_i) = \sum_{i=1}^{k} \lambda_i p_i = \sum_{i=1}^{n+1} \lambda_i p_i = p,
$$

$$
G(\mu_i^{-1} p_i) = sG(p_i)^{-1}G(p_i) = s \quad \text{for } i = 1, \ldots, k.
$$
Hence we get

\[ \hat{G}(p) \geq \inf\{ s \in \mathbb{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^{k} \lambda_i p_i = p, k \leq n + 1 \}. \]

Since the set \( \{ q \in \mathbb{R}^n \mid G(q) \leq \hat{G}(p) + 1 \} \) is a compact set, it is not hard to see that the infimum on the right hand side of the above inequality is actually attained. That is, we have

\[ \hat{G}(p) \geq \min\{ s \in \mathbb{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^{k} \lambda_i p_i = p, k \leq n + 1 \}. \]

The opposite inequality is obvious. The proof is now complete. \( \square \)

**Proof of Proposition 9.** First we observe that

(29) \[ \hat{H}(x, p) = \hat{G}(x, p)^m + f(x) \text{ for } (x, p) \in \mathbb{R}^{2n}. \]

Indeed, since the function: \[ p \mapsto \hat{G}(x, p)^m + f(x) \]

is convex on \( \mathbb{R}^n \) for every \( x \in \mathbb{R}^n \) and

\[ \hat{G}(x, p)^m + f(x) \leq H(x, p) \text{ for } (x, p) \in \mathbb{R}^{2n}, \]

we see that

\[ \hat{G}(x, p)^m + f(x) \leq \hat{H}(x, p) \text{ for } (x, p) \in \mathbb{R}^{2n}. \]

On the other hand, by Lemma 10, for \( (x, p) \in \mathbb{R}^{2n} \) we have

\[ \hat{G}(x, p)^m = \min\{ r^m \in \mathbb{R} \mid k \leq n + 1, \lambda_i > 0, G(x, p_i) = r, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \}. \]

Hence, by the formula

\[ \hat{H}(x, p) = \inf\{ \sum_{i=1}^{k} \lambda_i H(x, p_i) \mid k \in \mathbb{N}, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \}, \]

we have

\[ \hat{G}(x, p)^m + f(x) \geq \hat{H}(x, p). \]
Thus we have shown (29).

To show that $H$ satisfies (4), we just need to prove that

$$
\hat{G} \in \text{BUC}(\mathbb{R}^n \times B^n(0, R)) \quad \text{for } R > 0.
$$

Fix $R > 0$, set

$$
\rho_1 = \sup_{\mathbb{R}^n \times B^n(0, R)} G,
$$

and, in view of (26), choose $\rho_2 > 0$ so that

$$
\inf_{\mathbb{R}^n \times \left(\mathbb{R}^n \setminus B^n(0, \rho_2)\right)} G > \rho_1.
$$

Then, by Lemma 10, we have

$$
\hat{G}(x, p) = \min \left\{ \sum_{i=1}^{k} \lambda_i G(x, p_i) \mid \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, G(x, p_i) \leq \rho_1, \sum_{i=1}^{k} \lambda_i p_i = p \right\}
$$

for $(x, p) \in \mathbb{R}^n \times B^n(0, R)$.

This shows that the collection of functions:

$$
x \mapsto \hat{G}(x, p),
$$

with $p \in B^n(0, R)$, is equi-continuous on $\mathbb{R}^n$. On the other hand,

$$
\{ \hat{G}(x, \cdot) \mid x \in \mathbb{R}^n \}
$$

is a uniformly bounded collection of convex functions on $B^n(0, R)$. Consequently, this collection is equi-Lipschitz continuous on $B^n(0, R)$. Thus we see that $\hat{G} \in \text{BUC}(\mathbb{R}^n \times B^n(0, R))$ for all $R > 0$.

By assumptions (25) and (26), $H$ clearly satisfies (5).

To show (6), fix $R > 0$ and choose $\rho_2 > 0$ as above. Then, by Lemma 10, we get

$$
\hat{G}(x, p)^m = \min \left\{ \sum_{i=1}^{k} \lambda_i G(x, p_i)^m \mid k \in \mathbb{N}, \lambda_i \geq 0, G(x, p_i) = \hat{G}(x, p), \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \right\}
$$

$$
= \min \left\{ \sum_{i=1}^{k} \lambda_i G(x, p_i)^m \mid k \in \mathbb{N}, \lambda_i \geq 0, p_i \in B^n(0, \rho_2), \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \right\}.
$$
Hence we have
\[ \hat{H}(x,p) = \hat{H}_{\rho_2}(x,p) \quad \text{for } (x,p) \in \mathbb{R}^n \times B^n(0, R). \]
Thus $H$ satisfies (4)–(6). □

Bibliography


