Relaxation in the Cauchy problem for Hamilton-Jacobi equations (Viscosity Solution Theory of Differential Equations and its Developments)

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Relaxation in the Cauchy problem for Hamilton-Jacobi equations

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1. Introduction. In this note we study a little further the relaxation of Hamilton-Jacobi equations developed recently in [4,5]. In [4] we initiated the study of the relaxation of Hamilton-Jacobi equations of eikonal type and in [5] we extended this study to a larger class of Hamilton-Jacobi equations.

Let us recall the relaxation in calculus of variations. In general a non-convex variational problem (P) does not have its minimizer. A natural way to attack such a variational problem is to introduce its relaxed (or convexified) variational problem (RP) which has a minimizer and to regard such a minimizer as a generalized solution of the original problem (P). The main result (or principle) in this direction states that \( \min(RP) = \inf(P) \). That is, any accumulation point of a minimizing sequence of (P) is a minimizer of (RP). This fact or principle is called the relaxation of non-convex variational problems. See [3] for a treatment of the relaxation of non-convex variational problems.

Relaxation of Hamilton-Jacobi equations is the principle which says that the pointwise supremum over a suitable collection of Lipschitz continuous subsolutions in the almost everywhere sense of a non-convex Hamilton-Jacobi equation yields a viscosity solution of the equation with convexified Hamiltonian. See [4,5].

Here we are concerned with the Cauchy problem for Hamilton-Jacobi equations and generalize some results obtained in [5].

2. Main result for the Cauchy Problem. We consider the Cauchy Problem

\[
\begin{align*}
(1) & \quad u_t(x,t) + H(x,D_xu(x,t)) = 0 \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,T), \\
(2) & \quad u|_{t=0} = g.
\end{align*}
\]

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where $H$ and $g$ are given continuous functions respectively on $\mathbb{R}^{2n}$ and $\mathbb{R}^n$, $T$ is a given positive number or $T = \infty$, $u = u(x, t)$ is the unknown continuous function on $\mathbb{R}^n \times [0, T)$, $u_t$ denotes the $t$-derivative of $u$, and $D_x u$ denotes the $x$-gradient of $u$.

Let $\hat{H}$ denote the convex envelope of the function $H$, that is,

$$\hat{H}(x, p) = \sup \{l(p) \mid l \text{ affine function}, \, l(q) \leq H(x, q) \text{ for } q \in \mathbb{R}^n\}.$$ 

We also consider the convexified Hamilton-Jacobi equation

$$u_t(x, t) + \hat{H}(x, D_x u(x, t)) = 0 \quad \text{for} \quad (x, t) \in \mathbb{R}^n \times (0, T). \quad (3)$$

We use the notation: for $a \in \mathbb{R}^n$ and $r \geq 0$, $B^n(a, r)$ denotes the $n$-dimensional closed ball of radius $r$ centered at $a$. For $\Omega \subset \mathbb{R}^n$, $\mathrm{BUC}(\Omega)$ and $\mathrm{UC}(\Omega)$ denote the spaces of bounded uniformly continuous functions on $\Omega$ and of uniformly continuous functions on $\Omega$, respectively. Furthermore, $\mathrm{Lip}(\Omega)$ denotes the space of Lipschitz continuous functions on $\Omega$. Notice that $f \in \mathrm{Lip}(\Omega)$ is not assumed to be a bounded function.

Throughout this note we assume:

(4) $H, \hat{H} \in \mathrm{BUC}(\mathbb{R}^n \times B^n(0, R))$ for all $R > 0$.

(5) $\lim_{R \to \infty} \inf \{\frac{H(x, p)}{|p|} \mid (x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus B^n(0, R))\} > 0$.

For $R > 0$ we define the function $H_R : \mathbb{R}^{2n} \to \mathbb{R} \cup \{\infty\}$ by

$$H_R(x, p) = \begin{cases} H(x, p) & \text{if } x \in B^n(0, R), \\ \infty & \text{if } x \not\in B^n(0, R), \end{cases}$$

and write $\hat{H}_R$ for $\hat{G}$, where $G = H_R$.

(6) For each $R > 0$ and $\epsilon > 0$ there is a constant $\rho \geq R$ such that

$$\hat{H}_\rho(x, p) \leq \hat{H}(x, p) + \epsilon \quad \text{for} \quad (x, p) \in \mathbb{R}^n \times B^n(0, R).$$

(7) $g \in \mathrm{UC}(\mathbb{R}^n)$.

**Proposition 1.** (i) If $u \in \mathrm{USC}(\mathbb{R}^n \times [0, T))$ and $v \in \mathrm{LSC}(\mathbb{R}^n \times [0, T))$ are a viscosity subsolution and a viscosity supersolution of (3) respectively. Assume that $u(x, 0) \leq v(x, 0)$ for $x \in \mathbb{R}^n$ and that there is a (concave) modulus $\omega$ such that for all $(x, t) \in \mathbb{R}^n \times [0, T)$ and $y \in \mathbb{R}^n$,

$$\begin{cases} u(x, t) \leq u(y, 0) + \omega(|x - y| + t), \\
v(x, t) \geq v(y, 0) - \omega(|x - y| + t).\end{cases}$$

Then $u \leq v$ on $\mathbb{R}^n \times [0, T)$. (ii) There is a (unique) viscosity solution $u \in \mathrm{UC}(\mathbb{R}^n \times [0, \infty))$ of (3) which satisfies (2). If, in addition, $g \in \mathrm{Lip}(\mathbb{R}^n)$, then $u \in \mathrm{Lip}(\mathbb{R}^n \times [0, \infty)).$
We remark that the same proposition as above is valid for (1). We omit giving the proof of the above proposition.

Let $\mathcal{V}_T$ denote the set of functions $v \in \text{Lip} (\mathbb{R}^n \times [0, T])$ such that

$$v_t(x, t) + H(x, D_x v(x, t)) \leq 0 \quad \text{a.e. } (x, t) \in \mathbb{R}^n \times (0, T).$$

The following theorem is the main result in this note.

**Theorem 2.** Assume that (4)–(7) hold. Let $u \in \text{UC}(\mathbb{R}^n \times [0, T])$ be the unique viscosity solution of (3) satisfying (2). Then, for $(x, t) \in \mathbb{R}^n \times [0, T)$,

$$u(x, t) = \sup\{v(x, t) \mid v \in \mathcal{V}_T, v|_{t=0} \leq g\}.$$

**Remark.** In general the above formula does not give a subsolution of

$$u_t(x, t) + H(x, D_x u(x, t)) = 0 \quad \text{a.e. } (x, t) \in \mathbb{R}^n \times (0, \infty).$$

For instance, let $n = 2$ and define $H \in C(\mathbb{R}^2)$ and $g \in \text{UC}(\mathbb{R}^2)$ by $H(p, q) = (|p|^\frac{1}{2} + |q|^\frac{1}{2})^2$ and $g(x, y) = -|x| - |y|$, respectively. Note that $\hat{H}(p, q) = |p| + |q|$ for $(p, q) \in \mathbb{R}^2$. We set $\rho(x, y, t) = -2t - |x| - |y|$. Then, for instance, by computing $D^\pm \rho(x, y, t)$, we infer that $\rho$ is the viscosity solution of

$$\begin{cases}
  u_t(x, y, t) + |u_x(x, y, t)| + |u_y(x, y, t)| = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
  u(x, y, 0) = g(x, y) & \text{for } (x, y) \in \mathbb{R}^2.
\end{cases}$$

On the other hand, since at any point $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$, where $x, y \neq 0$, we have

$$H(\rho_x(x, y, t), \rho_y(x, y, t)) = 4, \quad \rho_t(x, y, t) = -2,$$

$\rho$ is not a subsolution of

$$u_t(x, y, t) + (|u_x(x, y, t)|^\frac{1}{2} + |u_y(x, y, t)|^\frac{1}{2})^2 = 0 \quad \text{a.e. } (x, y, t) \in \mathbb{R}^n \times (0, \infty).$$

**Theorem 2** is an easy consequence of the following theorem.

**Theorem 3.** Assume that (4)–(6) hold. Let $u \in \text{UC}(\mathbb{R}^n \times [0, T])$ be a viscosity subsolution of (3). Then, for all $(x, t) \in \mathbb{R}^n \times [0, T)$,

$$u(x, t) = \sup\{v(x, t) \mid v \in \mathcal{V}_T, v \leq u \text{ in } \mathbb{R}^n \times [0, T]\}.$$

Conceding Theorem 3 for the moment, we finish the proof of Theorem 2 as follows.
Proof of Theorem 2. We write $w(x, t)$ for the right hand side of (9). By Theorem 3 we find that $u \leq w$ on $\mathbb{R}^n \times [0, T)$. Let $v \in \mathcal{V}_T$ satisfy $v(\cdot, 0) \leq g$ on $\mathbb{R}^n$. Then, since $\hat{H} \leq H$, we have

$$v_t(x, t) + \hat{H}(x, D_x v(x, t)) \leq 0 \quad \text{a.e.} \quad (x, t) \in \mathbb{R}^n \times (0, T).$$

Since $\hat{H}(x, \cdot)$ is convex, $v$ is a viscosity subsolution of (3). By (i) of Proposition 1, we have $v \leq u$ on $\mathbb{R}^n \times (0, T)$, from which we get $w \leq u$ on $\mathbb{R}^n \times (0, T)$. Thus we have $u = w$ on $\mathbb{R}^n \times (0, T)$.

For our proof of Theorem 3, we need several lemmas. For a proof of the next three lemmas, we refer to [5].

Lemma 4. Let $K$ be a non-empty convex subset of $\mathbb{R}^m$ and set

$$L(\xi) = \sup\{\xi \cdot p \mid p \in K\} \in \mathbb{R} \cup \{\infty\} \quad \text{for all } \xi \in \mathbb{R}^m.$$

Let $U$ be an open subset of $\mathbb{R}^m$ and let $v \in C(\overline{U})$ satisfy

$$D^+ v(x) \subset K \quad \text{for all } x \in U.$$

Let $x, y \in \overline{U}$, and assume that the open line segment $l_0(x, y) := \{tx + (1-t)y \mid t \in (0, 1)\} \subset U$. Then

$$u(x) \leq u(y) + L(x - y).$$

In the above lemma and in what follows, for $v \in C(U)$ and $x \in U$, $D^+ v(x)$ denotes the superdifferential of $v$ at $x$.

Lemma 5. Let $\Omega$ be an open subset of $\mathbb{R}^m$ and $f_1, \ldots, f_N \in \text{Lip}(\Omega)$, with $N \in \mathbb{N}$. Set

$$f(x) = \max\{f_1(x), \ldots, f_N(x)\} \quad \text{for } x \in \Omega.$$

Then $f \in \text{Lip}(\Omega)$ and $f_1, \ldots, f_N$ are almost everywhere differentiable. Moreover for almost every $x \in \Omega$,

$$Df(x) \in \{Df_1(x), \ldots, Df_N(x)\},$$

where $Df(x)$ denotes the gradient of $f$ at $x$.

Lemma 6. Let $Z$ be a non-empty closed subset of $\mathbb{R}^m$. Define $L : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ by

$$L(\xi) = \sup\{\xi \cdot p \mid p \in Z\}.$$

Let $\xi \in \mathbb{R}^m$ be a point where $L$ is differentiable. Then

$$DL(\xi) \in Z \cap \partial(\overline{\text{co} Z})$$
We introduce the notation: for \((x, r) \in \mathbb{R}^n \times \mathbb{R}\) let 
\[
Z(x, r) := \{(p, q) \in \mathbb{R}^{n+1} \mid q + H(x, p) \leq r\}
\]
and \(K(x, r) := \overline{\text{co}} Z(x, r)\), the closed convex hull of \(Z(x, r)\). We note that 
\[
K(x, r) = \{(p, q) \in \mathbb{R}^{n+1} \mid q + \hat{H}(x, p) \leq r\}.
\]
For \(\delta > 0\), let 
\[
\Delta(\delta) := \{(x, y) \in \mathbb{R}^{2n} \mid |x - y| \leq \delta\}.
\]

**Lemma 7.** Assume that (4) holds. For any \(R > 0\) and \(\varepsilon > 0\) there exists a constant \(\delta > 0\) such that for any \((x, y) \in \Delta(\delta)\) and \(r \in \mathbb{R}\),
\[
Z_R(x, r) + B^{n+1}(0, \delta) \subset Z_{R+1}(y, r + \varepsilon),
\]
where, for \(R > 0\), \(Z_R(x, r) = Z(x, r) \cap B^{n+1}(0, R)\).

**Proof.** Fix \(\varepsilon > 0\) and \(R > 0\). Let \(\omega\) denote the modulus of continuity of \(H\) on \(\mathbb{R}^n \times B^n(0, R+1)\).

Fix a constant \(\delta \in (0, 1)\) so that \(\delta + \omega(2\delta) \leq \varepsilon\). Fix \((\xi, \eta) \in B^{n+1}(0, \delta), (x, y) \in \Delta(\delta), (p, q) \in Z_R(x, 0),\) and \(r \in \mathbb{R}\).

Noting that \((p, q) + (\xi, \eta) \in B^{n+1}(0, R+1)\), we observe that 
\[
q + \eta + H(y, p + \xi) \leq q + H(x, p) + \eta + \omega(|x - y| + |\xi|) \leq r + \delta + \omega(2\delta) \leq r + \varepsilon.
\]
Thus we have 
\[
(p + \xi, q + \eta) \in Z_{R+1}(y, r + \varepsilon),
\]
which concludes the proof. \(\square\)

**Lemma 8.** Assume that (4)–(6) hold. For any \(R > 0\) and \(\varepsilon > 0\) there exists a constant \(M \geq R\) such that for any \(x \in \mathbb{R}^n\),
\[
K_R(x, 0) \subset \text{co} Z_M(x, \varepsilon),
\]
where \(K_R(x, r) = K(x, r) \cap B^{n+1}(0, R)\).

**Proof.** For \(R > 0\) and \(\varepsilon > 0\) let \(\rho \equiv \rho(R, \varepsilon) \geq R\) be the constant from (6). That is, \(\rho = \rho(R, \varepsilon)\) is a constant for which
\[
\tilde{H}_\rho(x, p) \leq \tilde{H}(x, p) + \varepsilon \quad \text{for} \quad (x, p) \in \mathbb{R}^n \times B^n(0, R).
\]
In view of (4), for \(R > 0\) let \(M_R \geq 0\) be the constant defined by
\[
M_R = \sup\{|H(x, p)| \mid (x, p) \in \mathbb{R}^n \times B^n(0, R)\}.
\]
Fix $R > 0$, $\varepsilon > 0$, $x \in \mathbb{R}^n$, and $(p, q) \in K_R(x, 0)$. We have

$$\hat{H}(x, p) + q \leq 0,$$

and hence

$$\hat{H}_\rho(x, p) + q \leq \varepsilon.$$

Choose sequences $\{\lambda_i\}_{i=1}^m \subset (0, 1]$ and $\{p_i\}_{i=1}^m \subset B^n(0, \rho)$, with $m \in \mathbb{N}$, so that

$$\sum_{i=1}^m \lambda_i p_i = p, \quad \sum_{i=1}^m \lambda_i = 1,$$

$$\sum_{i=1}^m \lambda_i H(x, p_i) + q \leq 2\varepsilon.$$

(See the proof of Lemma 10 below.) Setting

$$h = q + \sum_{i=1}^m \lambda_i H(x, p_i), \quad q_i = h - H(x, p_i) \quad \text{for} \ i = 1, 2, \ldots, m,$$

we observe that

$$h \leq 2\varepsilon, \quad h \geq -|q| - M_\rho \geq -R - M_\rho,$$

$$|q_i| \leq |h| + M_\rho \leq 2\varepsilon + R + 2M_\rho \quad \text{for} \ i = 1, 2, \ldots, m,$$

and that

$$(p_i, q_i) \in Z(x, h) \subset Z(x, 2\varepsilon) \quad \text{for} \ i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^m \lambda_i q_i = h - \sum_{i=1}^m \lambda_i H(x, p_i) = q,$$

$$\sum_{i=1}^m \lambda_i (p_i, q_i) = (p, q).$$

These together show that $(p, q) \in \text{co} Z_M(x, 2\varepsilon)$, with $M = (\rho^2 + (2\varepsilon + R + 2M_\rho)^2)^{1/2}$. \hfill \square

**Proof of Theorem 3.** We write $Q = \mathbb{R}^n \times (0, T)$ and $Q_\delta = \mathbb{R}^n \times (-\delta, T + \delta)$ for $\delta > 0$.

Firstly, without loss of generality we may assume that $u$ is defined and Lipschitz continuous on $Q_\delta$ for some constant $\delta > 0$ and that

$$u_t(x, t) + \hat{H}(x, D_x u(x, t)) \leq 0 \quad \text{in} \ Q_\delta$$

(11)
in the viscosity sense. Indeed, we have

\[(12) \quad u(x, t) = \sup\{v(x, t) \mid v \in \text{Lip}(Q_\delta) \text{ for some } \delta > 0, v \text{ is a viscosity solution of (11), } v \leq u \text{ on } Q\}.\]

To see this, assuming \(T < \infty\), we solve the Cauchy problem

\[w_t(x, t) + \tilde{H}(x, D_x w(x, t)) \leq 0 \quad \text{in } \mathbb{R}^n \times (T, T + 1)\]

with the initial condition

\[(13) \quad w(x, T) = \lim_{t \searrow T} u(x, t) \quad \text{for } x \in \mathbb{R}^n.\]

In view of \((4)\) and \((5)\), there is a constant \(C > 0\) such that \(\tilde{H}(x, p) \geq -C\) for all \((x, p) \in \mathbb{R}^{2n}\), which shows that \(w\) is a viscosity solution of \(u_t \leq C\) in \(\mathbb{R}^n \times (0, T)\). This monotonicity of the function \(u(x, t)\) in \(t\) and the uniform continuity of \(u\) guarantee that the limit on the right hand side of \((13)\) defines a uniform continuous function on \(\mathbb{R}^n\).

By (ii) of Proposition 1, there is a unique viscosity solution \(w \in \text{UC}(\mathbb{R}^n \times [T, T + 1])\) for which \((13)\) holds. We extend the domain of definition of \(w\) to \(\mathbb{R}^n \times (0, T + 1)\) by setting

\[w(x, t) = u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T).\]

It is easy to see that \(w \in \text{UC}(\mathbb{R}^n \times (0, T + 1))\) that \(w\) is a viscosity subsolution of

\[w_t(x, t) + \tilde{H}(x, D_x w(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, T + 1).\]

Now, if \(T = \infty\), we define \(w \in \text{UC}(\mathbb{R}^n \times [0, \infty))\) by setting \(w = u\).

Fix any \(\epsilon > 0\). Since \(w \in \text{UC}(\mathbb{R}^n \times (0, T + 1))\), there is a constant \(\delta \in (0, 1/2)\) such that

\[(14) \quad u(x, t) - 2\epsilon \leq w(x, t - \delta) - \epsilon \leq u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T).\]

It is clear that the function \(z(x, t) := w(x, t - \delta) - 2\epsilon\) is defined and uniformly continuous on \(Q_\delta\) and is a viscosity solution of \((11)\).

Now, we take the sup-convolution of \(z\) in the \(t\)-variable. That is, for \(\gamma > 0\), we consider the function

\[z^\gamma(x, t) = \sup\{z(x, s) - \frac{1}{2\gamma}(t - s)^2 \mid s \in (-\delta, T + \delta)\} \quad \text{for } (x, t) \in \mathbb{R}^{n+1}.\]

If \(\gamma > 0\) is small enough, then \(z^\gamma\) is a viscosity solution of \((11)\) in \(Q_{\delta/2}\) and

\[(15) \quad z(x, t) \leq z^\gamma(x, t) \leq z(x, t) + \epsilon \quad \text{for } (x, t) \in Q_\delta.\]
Note also that, for each $\gamma > 0$, the collection of functions $z^\gamma(x, \cdot)$, with $x \in \mathbb{R}^n$, is equi-Lipschitz continuous on $(-\delta/2, T + \delta/2)$. By virtue of (5), we may choose constants $c_0 > 0$ and $C_1 > 0$ such that

$$\hat{H}(x, p) \geq c_0|p| - C_1 \quad \text{for} \quad (x, p) \in \mathbb{R}^{2n}.$$ 

Since $z^\gamma$ is a viscosity solution of

$$c_0|D_x z^\gamma(x, t)| \leq C_1 + L_{\gamma} \quad \text{in} \quad Q_{\delta/2},$$

where $L_{\gamma} > 0$ is a uniform Lipschitz bound of the functions $z^\gamma(x, \cdot)$ on $(-\delta/2, T + \delta/2)$, we see that the functions $z^\gamma(\cdot, t)$ are Lipschitz continuous on $\mathbb{R}^n$, with a Lipschitz bound independent of $t \in (-\delta/2, T + \delta/2)$.

Now, using (14) and (15) and writing $U(x, t)$ for the right hand side of (12), we see that for sufficiently small $\gamma > 0$ and for all $(x, t) \in Q$, 

$$u(x, t) \leq z(x, t) + \varepsilon \leq z^\gamma(x, t),$$

and hence, 

$$U(x, t) \geq z^\gamma(x, t) \geq z(x, t) \geq u(x, t) - 3\varepsilon,$$

which proves (12).

Henceforth we assume that, for some constant $\delta > 0$, $u$ is a member of Lip($Q_\delta$) and satisfies (11) in the viscosity sense.

Let $R > 0$ be a Lipschitz bound of the function $u$. Fix any $\varepsilon \in (0, 1)$. Due to Lemma 8, there is a constant $\rho \geq R$ such that for all $x \in \mathbb{R}^n$,

$$K_R(x, 0) \subset \co Z_\rho(x, \varepsilon).$$

In view of Lemma 7, there is a constant $\gamma \in (0, 1)$ such that for any $(x, y) \in \Delta(\gamma)$,

$$Z_\rho(x, \varepsilon) + B^{n+1}(0, \gamma) \subset Z_{\rho+1}(y, 2\varepsilon).$$

$$Z_{\rho+1}(y, 2\varepsilon) \subset Z_{\rho+2}(x, 3\varepsilon).$$

Consequently, for $(x, y) \in \Delta(\gamma)$, we have

\begin{align}
(16) & \quad K_R(x, 0) + B^{n+1}(0, \gamma) \subset \co Z_{\rho+1}(y, 2\varepsilon), \\
(17) & \quad Z_{\rho+1}(y, 2\varepsilon) \subset Z_{\rho+2}(x, 3\varepsilon).
\end{align}

We may assume that $\gamma < \delta$. Let $\mu \in (0, \gamma)$ be a constant to be fixed later. We choose a set $Y_\mu \subset Q_\delta$ so that

\begin{align}
(18) & \quad \#(Y_\mu \cap B^{n+1}(0, r)) < \infty \quad \text{for all} \ r > 0, \\
(19) & \quad \bigcup_{(y, s) \in Y_\mu} B^{n+1}((y, s), \mu) \supset Q_\delta.
\end{align}
We set
\[ L(\xi; \eta; y) = \sup\{\xi \cdot p + \eta q \mid (p, q) \in Z_{\rho+1}(y, 2\epsilon)\} \quad \text{for } \xi, y \in \mathbb{R}^n, \eta \in \mathbb{R} \]
and
\[ v(x, t; y, s) = u(y, s) + L(x - y, t - s; y) \quad \text{for } (x, t) \in \mathbb{R}^{n+1}, (y, s) \in Q_{\delta}. \]
By Lemma 6, we get for \((x, y) \in \Delta(\gamma),\)
\[ D_{\xi, \eta} L(\xi; \eta; y) \in Z_{\rho+1}(y, 2\epsilon) \subset Z_{\rho+2}(X_{3\epsilon}) \quad \text{a.e. } (\xi, \eta) \in \mathbb{R}^{n+1}. \]
Noting that \(D^+ u(x, t) \subset K_R(x, 0)\) for \((x, t) \in Q_{\delta}\), and setting \(\tilde{u}(x, t) := u(x, t) + \gamma |(x, t) - (y, s)|\) for \((x, t), (y, s) \in Q_{\delta}\), we find that for \((x, t), (y, s) \in Q_{\delta}\), if \(0 < |x - y| \leq \gamma\), then
\[ D^+ \tilde{u}(x, t) \subset D^+ u(x, t) + B^{n+1}(0, \gamma) \subset \text{co } Z_{\rho+1}(y, 2\epsilon). \]
Hence, by Lemma 4, we get
\[ u(x, t) + \gamma |(x, t) - (y, s)| \leq v(x, t; y, s) \quad \text{for } (x, t), (y, s) \in Q_{\delta}, \text{ with } |x - y| \leq \delta. \]
Set \(\beta = \gamma/5\) and define the function \(w: Q_{2\beta} \to \mathbb{R}\) by
\[ w(x, t) = \min\{v(x, t; y, s) \mid (y, s) \in Y_{\mu} \cap B^{n+1}((x, t), 3\beta)\}. \]
Now, we show that if \(\mu\) is sufficiently small, then for \((\bar{x}, \bar{t}) \in Q_{\beta}\) and \((x, t) \in B^{n+1}((\bar{x}, \bar{t}), \beta)\)
\[ w(x, t) = \min\{v(x, t; y, s) \mid (y, s) \in Y_{\mu} \cap B^{n+1}((\bar{x}, \bar{t}), 2\beta)\}. \]
To do this, fix \((\bar{x}, \bar{t}) \in Q_{\beta}\) and \((x, t) \in Y_{\mu} \cap B^{n+1}((\bar{x}, \bar{t}), 2\beta)\). Noting that \(Y_{\mu} \cap B^{n+1}((x, t), \mu) \neq \emptyset\) and \(B^{n+1}((x, t), \mu) \subset B^{n+1}((x, t), 5\beta)\) and choosing a point \((y, s) \in Y_{\mu} \cap B^{n+1}((x, t), \mu), \) we see that
\[ w(x, t) \leq u(x, t; y, s) \leq u(y, s) + (\rho + 1)|(|x, t) - (y, s)| \leq u(x, t) + (R + \rho + 1)|(|x, t) - (y, s)|. \]
Here we have used the fact that the functions \(L(\xi; \eta; y)\) of \((\xi, \eta)\) are Lipschitz continuous functions with \(\rho + 1\) as a Lipschitz bound. Fix now \(\mu \in (0, \gamma)\) by setting
\[ \mu = \frac{1}{2} \min\{\gamma, \frac{\gamma \beta}{R + \rho + 1}\}. \]
and observe that

\[(23) \quad w(x, t) < u(x, t) + \gamma \beta.\]

Fix \((y, s) \in Q_\delta \setminus B^{n+1}(\bar{x}, \bar{t}, 2\beta)\) and note that \(|(y, s) - (x, t)| \geq \beta\). Using (21), we have

\[v(x, t; y, s) \geq u(x, t) + \gamma \beta.\]

From this and (23), we conclude that (22) holds.

Next, we observe from (22) that the function \(w\) is Lipschitz continuous on \(B^{n+1}(\bar{x}, \bar{t}, \beta)\) for all \((\bar{x}, \bar{t}) \in Q_\beta\), with \(\rho + 1\) as a Lipschitz bound, which guarantees that \(w \in \text{Lip}(Q_\beta)\). Applying Lemma 5 and using (20), we observe that \(w\) is almost everywhere differentiable on \(Q_\beta\) and, at any point \((x, t) \in Q_\beta\) where \(w\) is differentiable,

\[Dw(x, t) \in \bigcup\{D_{x,t}v(x, t; y, s) \mid (y, s) \in Y_\mu \cap B^{n+1}(\bar{x}, \bar{t}, 2\beta)\} \subset Z_{\rho+2}(x, 3\epsilon),\]

which yields readily

\[w_t(x, t) + H(x, Dw(x, t)) \leq 3\epsilon \quad \text{a.e. } (x, t) \in Q_\beta.\]

Setting

\[z(x, t) = w(x, t) - \gamma \beta - 3\epsilon t \quad \text{for } (x, t) \in Q_\beta,
\]

we have

\[z_t(x, t) + H(x, Dxz(x, t)) \leq 0 \quad \text{a.e. } (x, t) \in Q_\beta.
\]

By (23), we have \(z(x, t) \leq u(x, t) - 3\epsilon t\) for \((x, t) \in Q_\beta\) and, by (21), we have \(z(x, t) \geq u(x, t) - \gamma \beta - 3\epsilon t\) for \((x, t) \in Q_\beta\). In the above two inequalities, we may take \(\gamma > 0\) as small as we wish. Thus we get

\[u(x, t) = \sup\{z(x, t) \mid z \in \mathcal{V}_T, \ z \leq u \text{ on } Q\} \quad \text{for } (x, t) \in Q,
\]

which completes the proof. \(\square\)

3. Examples. In this section we consider some examples of Hamiltonians \(H\) and examine if \(H\) satisfies conditions (4)–(6) or not.

Let \(H \in C(\mathbb{R}^{2n})\) be a function of the form

\[H(x, p) = G(x, p)^m + f(x),\]

where \(G \in C(\mathbb{R}^{2n})\) satisfies

\[(24) \quad G \in \text{BUC}(\mathbb{R}^n \times B^n(0, R)) \quad \text{for } R > 0,
\]

\[(25) \quad G(x, \lambda p) = \lambda G(x, p) \quad \text{for } \lambda \geq 0, (x, p) \in \mathbb{R}^{2n},
\]

\[(26) \quad \delta_G := \inf_{\mathbb{R}^n \times \partial B^n(0,1)} G > 0.
\]
$m$ is a constant satisfying $m \geq 1$, and $f \in \text{BUC}(\mathbb{R}^n)$.

**Proposition 9.** The function $H$ given above satisfies (4)–(6).

We need the following Lemma.

**Lemma 10.** For all $(x, p) \in \mathbb{R}^{2n}$, we have

\[
\hat{G}(x, p) = \min\{r \in \mathbb{R} \mid p = \sum_{i=1}^{k} \lambda_i p_i, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, G(x, p_i) = r\}.
\]

**Proof.** We fix $x \in \mathbb{R}^n$ and write $G(p)$ for $G(x, p)$ for notational simplicity. By using the separation theorem and Carathéodory's theorem in convex analysis, we see easily that

\[
\hat{G}(p) = \inf\left\{ \sum_{i=1}^{n+1} \lambda_i G(p_i) \mid \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i p_i = p \right\} \quad \text{for } p \in \mathbb{R}^n.
\]

It is clear from the above representation formula that

\[
\hat{G}(\lambda p) = \lambda \hat{G}(p) \quad \text{for } (\lambda, p) \in [0, \infty) \times \mathbb{R}^n,
\]

\[
G(p) \geq \hat{G}(p) \geq \delta_G |p| \quad \text{for } p \in \mathbb{R}^n.
\]

Fix $p \in \mathbb{R}^n$. If $p = 0$, then it is clear that (27) holds. We may thus assume that $p \neq 0$. For any $r > \hat{G}(p)$, by the above formula, there are \(\{\lambda_i\}_{i=1}^{n+1} \subset [0, 1]\) and \(\{p_i\}_{i=1}^{n+1} \subset \mathbb{R}^n\) such that

\[
r > \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \sum_{i=1}^{n+1} \lambda_i = 1, \quad \sum_{i=1}^{n+1} \lambda_i p_i = p.
\]

Set

\[
s = \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \mu_i = s^{-1} G(p_i).
\]

Notice that $s \geq \hat{G}(p) > 0$ by (28). By rearranging the order in $i$ if necessary, we may assume that

\[
\lambda_i \mu_i > 0 \quad \text{for } i \leq k, \quad \lambda_i \mu_i = 0 \quad \text{for } i > k
\]

for some $k \in \{1, \ldots, n+1\}$. Note that if $i > k$ and $\lambda_i > 0$, then $p_i = 0$. We now have

\[
\sum_{i=1}^{k} \lambda_i \mu_i = s^{-1} \sum_{i=1}^{n+1} \lambda_i G(p_i) = 1,
\]

\[
\sum_{i=1}^{k} \lambda_i \mu_i (\mu_i^{-1} p_i) = \sum_{i=1}^{k} \lambda_i p_i = \sum_{i=1}^{n+1} \lambda_i p_i = p,
\]

\[
G(\mu_i^{-1} p_i) = sG(p_i)^{-1} G(p_i) = s \quad \text{for } i = 1, \ldots, k.
\]
Hence we get

\[ \hat{G}(p) \geq \inf \{ s \in \mathbb{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^{k} \lambda_i p_i = p, k \leq n + 1 \}. \]

Since the set \( \{ q \in \mathbb{R}^n \mid G(q) \leq \hat{G}(p) + 1 \} \) is a compact set, it is not hard to see that the infimum on the right hand side of the above inequality is actually attained. That is, we have

\[ \hat{G}(p) \geq \min \{ s \in \mathbb{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^{k} \lambda_i p_i = p, k \leq n + 1 \}. \]

The opposite inequality is obvious. The proof is now complete. \( \square \)

**Proof of Proposition 9.** First we observe that

\[ (29) \quad \hat{H}(x,p) = \hat{G}(x,p)^m + f(x) \quad \text{for} \quad (x,p) \in \mathbb{R}^{2n}. \]

Indeed, since the function:

\[ p \mapsto \hat{G}(x,p)^m + f(x) \]

is convex on \( \mathbb{R}^n \) for every \( x \in \mathbb{R}^n \) and

\[ \hat{G}(x,p)^m + f(x) \leq H(x, p) \quad \text{for} \quad (x, p) \in \mathbb{R}^{2n}, \]

we see that

\[ \hat{G}(x,p)^m + f(x) \leq \hat{H}(x,p) \quad \text{for} \quad (x, p) \in \mathbb{R}^{2n}. \]

On the other hand, by Lemma 10, for \( (x,p) \in \mathbb{R}^{2n} \) we have

\[ \hat{G}(x,p)^m = \min \{ r^m \in \mathbb{R} \mid k \leq n + 1, \lambda_i > 0, G(x,p_i) = r, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \} \]

\[ \geq \inf \{ \sum_{i=1}^{k} \lambda_i G(x,p_i)^m \mid k \in \mathbb{N}, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \}. \]

Hence, by the formula

\[ \hat{H}(x,p) = \inf \{ \sum_{i=1}^{k} \lambda_i H(x,p_i) \mid k \in \mathbb{N}, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \}, \]

we have

\[ \hat{G}(x,p)^m + f(x) \geq \hat{H}(x,p). \]
Thus we have shown (29).

To show that $H$ satisfies (4), we just need to prove that

$$\hat{G} \in \text{BUC}(\mathbb{R}^n \times B^n(0,R))$$

for $R > 0$.

Fix $R > 0$, set

$$\rho_1 = \sup_{\mathbb{R}^n \times B^n(0,R)} G,$$

and, in view of (26), choose $\rho_2 > 0$ so that

$$\inf_{\mathbb{R}^n \times (\mathbb{R}^n \setminus B^n(0,\rho_2))} G > \rho_1.$$

Then, by Lemma 10, we have

$$\hat{G}(x,p) = \min \{ \sum_{i=1}^{k} \lambda_i G(x,p_i) \mid \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, G(x,p_i) \leq \rho_1, \sum_{i=1}^{k} \lambda_i p_i = p \},$$

for $(x,p) \in \mathbb{R}^n \times B^n(0,R)$.

This shows that the collection of functions:

$$x \mapsto \hat{G}(x,p),$$

with $p \in B^n(0,R)$, is equi-continuous on $\mathbb{R}^n$. On the other hand,

$$\{ \hat{G}(x,\cdot) \mid x \in \mathbb{R}^n \}$$

is a uniformly bounded collection of convex functions on $B^n(0,R)$. Consequently, this collection is equi-Lipschitz continuous on $B^n(0,R)$. Thus we see that $\hat{G} \in \text{BUC}(\mathbb{R}^n \times B^n(0,R))$ for all $R > 0$.

By assumptions (25) and (26), $H$ clearly satisfies (5).

To show (6), fix $R > 0$ and choose $\rho_2 > 0$ as above. Then, by Lemma 10, we get

$$\hat{G}(x,p)^m = \min \{ \sum_{i=1}^{k} \lambda_i G(x,p_i)^m \mid k \in \mathbb{N}, \lambda_i \geq 0, G(x,p_i) = \hat{G}(x,p), \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \},$$

$$= \min \{ \sum_{i=1}^{k} \lambda_i G(x,p_i)^m \mid k \in \mathbb{N}, \lambda_i \geq 0, p_i \in B^n(0,\rho_2), \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \}. $$
Hence we have

$$\hat{H}(x,p) = \hat{H}_{\rho_2}(x,p) \quad \text{for} \ (x,p) \in \mathbb{R}^n \times B^n(0,R).$$

Thus $H$ satisfies (4)–(6). □

Bibliography


