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Relaxation in the Cauchy problem for Hamilton-Jacobi equations

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1. Introduction. In this note we study a little further the relaxation of Hamilton-Jacobi equations developed recently in [4,5]. In [4] we initiated the study of the relaxation of Hamilton-Jacobi equations of eikonal type and in [5] we extended this study to a larger class of Hamilton-Jacobi equations.

Let us recall the relaxation in calculus of variations. In general a non-convex variational problem (P) does not have its minimizer. A natural way to attack such a variational problem is to introduce its relaxed (or convexified) variational problem (RP) which has a minimizer and to regard such a minimizer as a generalized solution of the original problem (P). The main result (or principle) in this direction states that min (RP) = inf (P). That is, any accumulation point of a minimizing sequence of (P) is a minimizer of (RP). This fact or principle is called the relaxation of non-convex variational problems. See [3] for a treatment of the relaxation of non-convex variational problems.

Relaxation of Hamilton-Jacobi equations is the principle which says that the point-wise supremum over a suitable collection of Lipschitz continuous subsolutions in the almost everywhere sense of a non-convex Hamilton-Jacobi equation yields a viscosity solution of the equation with convexified Hamiltonian. See [4,5].

Here we are concerned with the Cauchy problem for Hamilton-Jacobi equations and generalize some results obtained in [5].

2. Main result for the Cauchy Problem. We consider the Cauchy Problem

(1) \[ u_t(x, t) + H(x, D_x u(x, t)) = 0 \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T), \]

(2) \[ u|_{t=0} = g, \]

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where \( H \) and \( g \) are given continuous functions respectively on \( \mathbb{R}^{2n} \) and \( \mathbb{R}^n \), \( T \) is a given positive number or \( T = \infty \), \( u = u(x, t) \) is the unknown continuous function on \( \mathbb{R}^n \times [0, T) \), \( u_t \) denotes the \( t \)-derivative of \( u \), and \( D_x u \) denotes the \( x \)-gradient of \( u \).

Let \( \hat{H} \) denote the convex envelope of the function \( H \), that is,

\[
\hat{H}(x, p) = \sup \{ l(p) \mid l \text{ affine function}, l(q) \leq H(x, q) \text{ for } q \in \mathbb{R}^n \}.
\]

We also consider the convexified Hamilton-Jacobi equation

\[
(3) \quad u_t(x, t) + \hat{H}(x, D_x u(x, t)) = 0 \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T).
\]

We use the notation: for \( a \in \mathbb{R}^n \) and \( r \geq 0 \), \( B^n(a, r) \) denotes the \( n \)-dimensional closed ball of radius \( r \) centered at \( a \). For \( \Omega \subset \mathbb{R}^m \), \( \text{BUC}(\Omega) \) and \( \text{UC}(\Omega) \) denote the spaces of bounded uniformly continuous functions on \( \Omega \) and of uniformly continuous functions on \( \Omega \), respectively. Furthermore, \( \text{Lip}(\Omega) \) denotes the space of Lipschitz continuous functions on \( \Omega \). Notice that \( f \in \text{Lip}(\Omega) \) is not assumed to be a bounded function.

Throughout this note we assume:

1. \( H, \hat{H} \in \text{BUC}(\mathbb{R}^n \times B^n(0, R)) \) for all \( R > 0 \).
2. \( \lim_{R \to \infty} \inf \left\{ \frac{H(x, p)}{|p|} \mid (x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus B^n(0, R)) \right\} > 0 \).
3. For each \( R > 0 \) and \( \epsilon > 0 \) there is a constant \( \rho \geq R \) such that
   \[
   \hat{H}_\rho(x, p) \leq \hat{H}(x, p) + \epsilon \quad \text{for } (x, p) \in \mathbb{R}^n \times B^n(0, R).
   \]
4. \( g \in \text{UC}(\mathbb{R}^n) \).
5. Proposition 1. (i) If \( u \in \text{USC}(\mathbb{R}^n \times [0, T)) \) and \( v \in \text{LSC}(\mathbb{R}^n \times [0, T)) \) are a viscosity subsolution and a viscosity supersolution of (3) respectively. Assume that \( u(x, 0) \leq v(x, 0) \) for \( x \in \mathbb{R}^n \) and that there is a (concave) modulus \( \omega \) such that for all \( (x, t) \in \mathbb{R}^n \times [0, T) \) and \( y \in \mathbb{R}^n \),
   \[
   \begin{cases}
   u(x, t) \leq u(y, 0) + \omega(|x - y| + t), \\
   v(x, t) \geq v(y, 0) - \omega(|x - y| + t).
   \end{cases}
   \]
   Then \( u \leq v \) on \( \mathbb{R}^n \times [0, T) \). (ii) There is a (unique) viscosity solution \( u \in \text{UC}(\mathbb{R}^n \times [0, \infty)) \) of (3) which satisfies (2). If, in addition, \( g \in \text{Lip}(\mathbb{R}^n) \), then \( u \in \text{Lip}(\mathbb{R}^n \times [0, \infty)) \).
We remark that the same proposition as above is valid for (1). We omit giving the proof of the above proposition.

Let $\mathcal{V}_T$ denote the set of functions $v \in \text{Lip} (\mathbb{R}^n \times [0, T])$ such that

\begin{equation}
\tag{8}
\forall (x, t) \in \mathbb{R}^n \times (0, T), \quad v_t(x, t) + H(x, D_x v(x, t)) \leq 0 \quad \text{a.e.}
\end{equation}

The following theorem is the main result in this note.

**Theorem 2.** Assume that (4)–(7) hold. Let $u \in \text{UC}(\mathbb{R}^n \times [0, T])$ be the unique viscosity solution of (3) satisfying (2). Then, for $(x, t) \in \mathbb{R}^n \times [0, T)$,

\begin{equation}
\tag{9}
u(x, t) = \sup\{v(x, t) \mid v \in \mathcal{V}_T, \ v|_{t=0} \leq g\}.
\end{equation}

**Remark.** In general the above formula does not give a subsolution of

$$\forall (x, t) \in \mathbb{R}^n \times (0, \infty), \quad u_t(x, t) + H(x, D_x u(x, t)) = 0 \quad \text{a.e.}$$

For instance, let $n = 2$ and define $H \in C(\mathbb{R}^2)$ and $g \in \text{UC}(\mathbb{R}^2)$ by $H(p, q) = (|p|^{\frac{1}{2}} + |q|^{\frac{1}{2}})^2$ and $g(x, y) = -|x| - |y|$, respectively. Note that $\hat{H}(p, q) = |p| + |q|$ for $(p, q) \in \mathbb{R}^2$. We set $\rho(x, y, t) = -2t - |x| - |y|$. Then, for instance, by computing $D^\pm \rho(x, y, t)$, we infer that $\rho$ is the viscosity solution of

\begin{equation}
\begin{cases}
u(x, y, t) + |u_x(x, y, t)| + |u_y(x, y, t)| = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
u(x, y, 0) = g(x, y) & \text{for } (x, y) \in \mathbb{R}^2.
\end{cases}
\end{equation}

On the other hand, since at any point $(x, y, t) \in \mathbb{R}^2 \times (0, \infty)$, where $x, y \neq 0$, we have

$$H(\rho_x(x, y, t), \rho_y(x, y, t)) = 4, \quad \rho_t(x, y, t) = -2,$$

$\rho$ is not a subsolution of

$$\forall (x, y, t) \in \mathbb{R}^n \times (0, \infty), \quad u_t(x, y, t) + (|u_x(x, y, t)|^{\frac{1}{2}} + |u_y(x, y, t)|^{\frac{1}{2}})^2 = 0 \quad \text{a.e.}$$

Theorem 2 is an easy consequence of the following theorem.

**Theorem 3.** Assume that (4)–(6) hold. Let $u \in \text{UC}(\mathbb{R}^n \times [0, T])$ be a viscosity subsolution of (3). Then, for all $(x, t) \in \mathbb{R}^n \times [0, T)$,

\begin{equation}
\tag{10}
u(x, t) = \sup\{v(x, t) \mid v \in \mathcal{V}_T, \ v \leq u \text{ in } \mathbb{R}^n \times [0, T]\}.
\end{equation}

Conceding Theorem 3 for the moment, we finish the proof of Theorem 2 as follows.
Proof of Theorem 2. We write $w(x, t)$ for the right hand side of (9). By Theorem 3 we find that $u \leq w$ on $\mathbb{R}^n \times [0, T)$. Let $v \in \mathcal{V}_T$ satisfy $v(\cdot, 0) \leq g$ on $\mathbb{R}^n$. Then, since $\hat{H} \leq H$, we have

$$v_t(x, t) + \hat{H}(x, D_xv(x, t)) \leq 0 \quad \text{a.e.} \quad (x, t) \in \mathbb{R}^n \times (0, T).$$

Since $\hat{H}(x, \cdot)$ is convex, $v$ is a viscosity subsolution of (3). By (i) of Proposition 1, we have $v \leq u$ on $\mathbb{R}^n \times (0, T)$, from which we get $w \leq u$ on $\mathbb{R}^n \times (0, T)$. Thus we have $u = w$ on $\mathbb{R}^n \times (0, T)$. □

For our proof of Theorem 3, we need several lemmas. For a proof of the next three lemmas, we refer to [5].

Lemma 4. Let $K$ be a non-empty convex subset of $\mathbb{R}^m$ and set

$$L(\xi) = \sup\{\xi \cdot p \mid p \in K\} \in \mathbb{R} \cup \{\infty\} \quad \text{for all } \xi \in \mathbb{R}^m.$$

Let $U$ be an open subset of $\mathbb{R}^m$ and let $v \in C(\overline{U})$ satisfy

$$D^+v(x) \subset K \quad \text{for all } x \in U.$$

Let $x, y \in \overline{U}$, and assume that the open line segment $l_0(x, y) := \{tx + (1-t)y \mid t \in (0, 1)\} \subset U$. Then

$$u(x) \leq u(y) + L(x - y).$$

In the above lemma and in what follows, for $v \in C(U)$ and $x \in U$, $D^+v(x)$ denotes the superdifferential of $v$ at $x$.

Lemma 5. Let $\Omega$ be an open subset of $\mathbb{R}^m$ and $f_1, \ldots, f_N \in \text{Lip}(\Omega)$, with $N \in \mathbb{N}$. Set

$$f(x) = \max\{f_1(x), \ldots, f_N(x)\} \quad \text{for } x \in \Omega.$$

Then $f \in \text{Lip}(\Omega)$ and $f_1, \ldots, f_N$ are almost everywhere differentiable. Moreover for almost every $x \in \Omega$,

$$Df(x) \in \{Df_1(x), \ldots, Df_N(x)\},$$

where $Df(x)$ denotes the gradient of $f$ at $x$.

Lemma 6. Let $Z$ be a non-empty closed subset of $\mathbb{R}^m$. Define $L : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ by

$$L(\xi) = \sup\{\xi \cdot p \mid p \in Z\}.$$

Let $\xi \in \mathbb{R}^m$ be a point where $L$ is differentiable. Then

$$DL(\xi) \in Z \cap \partial(\overline{\text{co} Z})$$
We introduce the notation: for \((x, r) \in \mathbb{R}^n \times \mathbb{R}\) let
\[
Z(x, r) := \{(p, q) \in \mathbb{R}^{n+1} \mid q + H(x, p) \leq r\}
\]
and \(K(x, r) := \overline{\text{co}} Z(x, r)\), the closed convex hull of \(Z(x, r)\). We note that
\[
K(x, r) = \{(p, q) \in \mathbb{R}^{n+1} \mid q + \hat{H}(x, p) \leq r\}.
\]

For \(\delta > 0\), let \(\Delta(\delta) := \{(x, y) \in \mathbb{R}^{2n} \mid |x - y| \leq \delta\}\).

**Lemma 7.** Assume that (4) holds. For any \(R > 0\) and \(\epsilon > 0\) there exists a constant \(\delta > 0\) such that for any \((x, y) \in \Delta(\delta)\) and \(r \in \mathbb{R}\),
\[
Z_R(x, r) + B^{n+1}(0, \delta) \subset Z_{R+1}(y, r + \epsilon),
\]
where, for \(R > 0\), \(Z_R(x, r) = Z(x, r) \cap B^{n+1}(0, R)\).

**Proof.** Fix \(\epsilon > 0\) and \(R > 0\). Let \(\omega\) denote the modulus of continuity of \(H\) on \(\mathbb{R}^n \times B^n(0, R+1)\).

Fix a constant \(\delta \in (0, 1)\) so that \(\delta + \omega(2\delta) \leq \epsilon\). Fix \((\xi, \eta) \in B^{n+1}(0, \delta)\), \((x, y) \in \Delta(\delta)\), \((p, q) \in Z_R(x, 0)\), and \(r \in \mathbb{R}\).

Noting that \((p, q) + (\xi, \eta) \in B^{n+1}(0, R + 1)\), we observe that
\[
q + \eta + H(y, p + \xi) \leq q + H(x, p) + \eta + \omega(|x - y| + |\xi|) \leq r + \delta + \omega(2\delta) \leq r + \epsilon.
\]
Thus we have
\[
(p + \xi, q + \eta) \in Z_{R+1}(y, r + \epsilon),
\]
which concludes the proof. \(\Box\)

**Lemma 8.** Assume that (4)–(6) hold. For any \(R > 0\) and \(\epsilon > 0\) there exists a constant \(M \geq R\) such that for any \(x \in \mathbb{R}^n\),
\[
K_R(x, 0) \subset \overline{\text{co}} Z_M(x, \epsilon),
\]
where \(K_R(x, r) = K(x, r) \cap B^{n+1}(0, R)\).

**Proof.** For \(R > 0\) and \(\epsilon > 0\) let \(\rho \equiv \rho(R, \epsilon) \geq R\) be the constant from (6). That is, \(\rho = \rho(R, \epsilon)\) is a constant for which
\[
\hat{H}_\rho(x, p) \leq \hat{H}(x, p) + \epsilon \quad \text{for} \quad (x, p) \in \mathbb{R}^n \times B^n(0, R).
\]
In view of (4), for \(R > 0\) let \(M_R \geq 0\) be the constant defined by
\[
M_R = \sup\{|H(x, p)| \mid (x, p) \in \mathbb{R}^n \times B^n(0, R)\}.
\]
Fix $R > 0$, $\varepsilon > 0$, $x \in \mathbb{R}^n$, and $(p, q) \in K_R(x, 0)$. We have

$$\hat{H}(x, p) + q \leq 0,$$

and hence

$$\hat{H}_\rho(x, p) + q \leq \varepsilon.$$ 

Choose sequences $\{\lambda_i\}_{i=1}^m \subset (0, 1]$ and $\{p_i\}_{i=1}^m \subset B^n(0, \rho)$, with $m \in \mathbb{N}$, so that

$$\sum_{i=1}^m \lambda_i p_i = p, \quad \sum_{i=1}^m \lambda_i = 1,$$ 

and

$$\sum_{i=1}^m \lambda_i H(x, p_i) + q \leq 2\varepsilon.$$ 

(See the proof of Lemma 10 below.) Setting

$$h = q + \sum_{i=1}^m \lambda_i H(x, p_i), \quad q_i = h - H(x, p_i) \quad \text{for } i = 1, 2, \ldots, m,$$

we observe that

$$h \leq 2\varepsilon, \quad h \geq -|q| - M_\rho \geq -R - M_\rho,$$

$$|q_i| \leq |h| + M_\rho \leq 2\varepsilon + R + 2M_\rho \quad \text{for } i = 1, 2, \ldots, m,$$

and that

$$(p_i, q_i) \in Z(x, h) \subset Z(x, 2\varepsilon) \quad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^m \lambda_i q_i = h - \sum_{i=1}^m \lambda_i H(x, p_i) = q,$$

$$\sum_{i=1}^m \lambda_i (p_i, q_i) = (p, q).$$

These together show that $(p, q) \in \co Z_M(x, 2\varepsilon)$, with $M = (\rho^2 + (2\varepsilon + R + 2M_\rho)^2)^{1/2}$.

□

Proof of Theorem 3. We write $Q = \mathbb{R}^n \times (0, T)$ and $Q_\delta = \mathbb{R}^n \times (-\delta, T + \delta)$ for $\delta > 0$.

Firstly, without loss of generality we may assume that $u$ is defined and Lipschitz continuous on $Q_\delta$ for some constant $\delta > 0$ and that

$$(11) \quad u_t(x, t) + \hat{H}(x, D_x u(x, t)) \leq 0 \quad \text{in } Q_\delta$$
in the viscosity sense. Indeed, we have

\begin{equation}
(12) \quad u(x, t) = \sup \{v(x, t) \mid v \in \text{Lip}(Q_\delta) \text{ for some } \delta > 0, \quad v \text{ is a viscosity solution of (11), } v \leq u \text{ on } Q\}\.
\end{equation}

To see this, assuming $T < \infty$, we solve the Cauchy problem

\begin{equation}
(13) \quad w_t(x, t) + \tilde{H}(x, D_x w(x, t)) \leq 0 \quad \text{in } \mathbb{R}^n \times (T, T + 1)
\end{equation}

with the initial condition

\begin{equation}
(14) \quad w(x, T) = \lim_{\substack{t \nearrow T}} u(x, t) \quad \text{for } x \in \mathbb{R}^n.
\end{equation}

In view of (4) and (5), there is a constant $C > 0$ such that $\tilde{H}(x, p) \geq -C$ for all $(x, p) \in \mathbb{R}^{2n}$, which shows that $u$ is a viscosity solution of $u_t \leq C$ in $\mathbb{R}^n \times (0, T)$. This monotonicity of the function $u(x, t)$ in $t$ and the uniform continuity of $u$ guarantee that the limit on the right hand side of (13) defines a uniform continuous function on $\mathbb{R}^n$.

By (ii) of Proposition 1, there is a unique viscosity solution $w \in \text{UC}(\mathbb{R}^n \times [T, T + 1])$ for which (13) holds. We extend the domain of definition of $w$ to $\mathbb{R}^n \times (0, T + 1)$ by setting

\begin{equation}
(15) \quad w(x, t) = u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T).
\end{equation}

It is easy to see that $w \in \text{UC}(\mathbb{R}^n \times (0, T + 1))$ that $w$ is a viscosity subsolution of

\begin{equation}
(16) \quad w_t(x, t) + \tilde{H}(x, D_x w(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, T + 1).
\end{equation}

Now, if $T = \infty$, we define $w \in \text{UC}(\mathbb{R}^n \times [0, \infty))$ by setting $w = u$.

Fix any $\epsilon > 0$. Since $w \in \text{UC}(\mathbb{R}^n \times (0, T + 1))$, there is a constant $\delta \in (0, 1/2)$ such that

\begin{equation}
(17) \quad u(x, t) - 2\epsilon \leq w(x, t - \delta) - \epsilon \leq u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T).
\end{equation}

It is clear that the function $z(x, t) := w(x, t - \delta) - 2\epsilon$ is defined and uniformly continuous on $Q_\delta$ and is a viscosity solution of (11).

Now, we take the sup-convolution of $z$ in the $t$-variable. That is, for $\gamma > 0$, we consider the function

\begin{equation}
(18) \quad z^\gamma(x, t) = \sup \{z(x, s) - \frac{1}{2\gamma}(t - s)^2 \mid s \in (-\delta, T + \delta)\} \quad \text{for } (x, t) \in \mathbb{R}^{n+1}.
\end{equation}

If $\gamma > 0$ is small enough, then $z^\gamma$ is a viscosity solution of (11) in $Q_{\delta/2}$ and

\begin{equation}
(19) \quad z(x, t) \leq z^\gamma(x, t) \leq z(x, t) + \epsilon \quad \text{for } (x, t) \in Q_{\delta}.
\end{equation}
Note also that, for each $\gamma > 0$, the collection of functions $z^\gamma(x, \cdot)$, with $x \in \mathbb{R}^n$, is equi-Lipschitz continuous on $(-\delta/2, T + \delta/2)$. By virtue of (5), we may choose constants $c_0 > 0$ and $C_1 > 0$ such that
\[
\hat{H}(x,p) \geq c_0|p| - C_1 \quad \text{for} \quad (x,p) \in \mathbb{R}^{2n}.
\]
Since $z^\gamma$ is a viscosity solution of
\[
c_0|D_x z^\gamma(x,t)| \leq C_1 + L_{\gamma}
\]
in $Q_{\delta/2}$, where $L_{\gamma} > 0$ is a uniform Lipschitz bound of the functions $z^\gamma(x, t)$ on $(-\delta/2, T + \delta/2)$, we see that the functions $z^\gamma(\cdot, t)$ are Lipschitz continuous on $\mathbb{R}^n$, with a Lipschitz bound independent of $t \in (-\delta/2, T + \delta/2)$.

Now, using (14) and (15) and writing $U(x, t)$ for the right hand side of (12), we see that for sufficiently small $\gamma > 0$ and for all $(x,t) \in Q$,
\[
u(x,t) \geq z(x,t) + \varepsilon \geq z^\gamma(x,t),
\]
and hence,
\[
U(x,t) \geq z^\gamma(x,t) \geq z(x,t) \geq u(x,t) - 3\varepsilon,
\]
which proves (12).

Henceforth we assume that, for some constant $\delta > 0$, $u$ is a member of $\operatorname{Lip}(Q_\delta)$ and satisfies (11) in the viscosity sense.

Let $R > 0$ be a Lipschitz bound of the function $u$. Fix any $\varepsilon \in (0, 1)$. Due to Lemma 8, there is a constant $\rho \geq R$ such that for all $x \in \mathbb{R}^n$,
\[
K_R(x, 0) \subset \co Z_\rho(x, \varepsilon).
\]
In view of Lemma 7, there is a constant $\gamma \in (0,1)$ such that for any $(x, y) \in \Delta(\gamma)$,
\[
Z_\rho(x, \varepsilon) + B^{n+1}(0, \gamma) \subset Z_{\rho+1}(y, 2\varepsilon).
\]
Consequently, for $(x, y) \in \Delta(\gamma)$, we have
\[
K_R(x, 0) + B^{n+1}(0, \gamma) \subset \co Z_{\rho+1}(y, 2\varepsilon), \tag{16}
\]
\[
Z_{\rho+1}(y, 2\varepsilon) \subset Z_{\rho+2}(x, 3\varepsilon). \tag{17}
\]
We may assume that $\gamma < \delta$. Let $\mu \in (0, \gamma)$ be a constant to be fixed later. We choose a set $Y_\mu \subset Q_\delta$ so that
\[
\#(Y_\mu \cap B^{n+1}(0, r)) < \infty \quad \text{for all} \quad r > 0, \tag{18}
\]
\[
\bigcup_{(y,s) \in Y_\mu} B^{n+1}(y, s, \mu) \supset Q_\delta. \tag{19}
\]
We set
\[ L(\xi, \eta; y) = \sup \{ \xi \cdot p + \eta q \mid (p, q) \in Z_{\rho+1}(y, 2\epsilon) \} \quad \text{for } \xi, y \in \mathbb{R}^n, \eta \in \mathbb{R} \]
and
\[ v(x, t; y, s) = u(y, s) + L(x - y, t - s; y) \quad \text{for } (x, t) \in \mathbb{R}^{n+1}, (y, s) \in Q_{\delta}. \]

By Lemma 6, we get for \((x, y) \in \Delta(\gamma),\)
\[ D_{\xi,\eta}L(\xi, \eta; y) \in Z_{\rho+1}(y, 2\epsilon) \subset Z_{\rho+2}(x, 3\epsilon) \quad \text{a.e. } (\xi, \eta) \in \mathbb{R}^{n+1}. \]

Noting that \(D^+u(x, t) \subset K_R(x, 0)\) for \((x, t) \in Q_{\delta},\)
and setting \(\tilde{u}(x, t) := u(x, t) + \gamma |(x, t) - (y, s)|\) for \((x, t), (y, s) \in Q_{\delta},\) we find that for \((x, t), (y, s) \in Q_{\delta},\) if \(0 < |x - y| \leq \gamma,\) then
\[ D^+\tilde{u}(x, t) \subset D^+u(x, t) + B^{n+1}(0, \gamma) \subset \text{co} Z_{\rho+1}(y, 2\epsilon). \]

Hence, by Lemma 4, we get
\[ u(x, t) + \gamma |(x, t) - (y, s)| \leq v(x, t; y, s) \quad \text{for } (x, t), (y, s) \in Q_{\delta}, \text{ with } |x - y| \leq \delta. \]

Set \(\beta = \gamma/5\) and define the function \(w : Q_{2\beta} \to \mathbb{R}\) by
\[ w(x, t) = \min \{ v(x, t; y, s) \mid (y, s) \in Y_{\mu} \cap B^{n+1}((x, t), 3\beta) \}. \]

Now, we show that if \(\mu\) is sufficiently small, then for \((\overline{x}, \overline{t}) \in Q_{\beta}\) and \((x, t) \in B^{n+1}((\overline{x}, \overline{t}), \beta)\)
\[ w(x, t) = \min \{ v(x, t; y, s) \mid (y, s) \in Y_{\mu} \cap B^{n+1}((\overline{x}, \overline{t}), 2\beta) \}. \]

To do this, fix \((\overline{x}, \overline{t}) \in Q_{\beta}\) and \((x, t) \in Y_{\mu} \cap B^{n+1}((\overline{x}, \overline{t}), 2\beta).\) Noting that \(Y_{\mu} \cap B^{n+1}((x, t), \mu) \neq \emptyset\) and \(B^{n+1}((x, t), \mu) \subset B^{n+1}((x, t), 5\beta)\) and choosing a point \((y, s) \in Y_{\mu} \cap B^{n+1}((x, t), \mu),\) we see that
\[ w(x, t) \leq v(x, t; y, s) \leq u(y, s) + (\rho + 1)||x, t) - (y, s)|| \leq u(x, t) + (R + \rho + 1)||x, t) - (y, s)||. \]

Here we have used the fact that the functions \(L(\xi, \eta; y)\) of \((\xi, \eta)\) are Lipschitz continuous functions with \(\rho + 1\) as a Lipschitz bound. Fix now \(\mu \in (0, \gamma)\) by setting
\[ \mu = \frac{1}{2} \min \{ \gamma, \frac{\gamma \beta}{R + \rho + 1} \}. \]
and observe that

\[(23) \quad w(x, t) < u(x, t) + \gamma \beta.\]

Fix \((y, s) \in Q_{\delta} \setminus B^{n+1}(\overline{x}, 2\beta)\) and note that \(|(y, s) - (x, t)| \geq \beta\). Using (21), we have

\[v(x, t; y, s) \geq u(x, t) + \gamma \beta.\]

From this and (23), we conclude that (22) holds.

Next, we observe from (22) that the function \(w\) is Lipschitz continuous on 
\(B^{n+1}(\overline{x}, \beta)\) for all \((\overline{x}, t) \in Q_{\beta}\), with \(\rho + 1\) as a Lipschitz bound, which guarantees that \(w \in \text{Lip}(Q_{\beta})\). Applying Lemma 5 and using (20), we observe that \(w\) is almost everywhere differentiable on \(Q_{\beta}\) and, at any point \((x, t) \in Q_{\beta}\) where \(w\) is differentiable,

\[Dw(x, t) \in \bigcup \{D_{x,t}v(x, t; y, s) \mid (y, s) \in Y_{\mu} \cap B^{n+1}(\overline{x}, t, 2\beta)\} \subset Z_{\rho+2}(x, 3\epsilon),\]

which yields readily

\[w_{t}(x, t) + H(x, D_{x}w(x, t)) \leq 3\epsilon \quad \text{a.e.} \quad (x, t) \in Q_{\beta}.\]

Setting

\[z(x, t) = w(x, t) - \gamma \beta - 3\epsilon t \quad \text{for} \quad (x, t) \in Q_{\beta},\]

we have

\[z_{t}(x, t) + H(x, D_{x}z(x, t)) \leq 0 \quad \text{a.e.} \quad (x, t) \in Q_{\beta}.\]

By (23), we have \(z(x, t) \leq u(x, t) - 3\epsilon t\) for \((x, t) \in Q_{\beta}\) and, by (21), we have \(z(x, t) \geq u(x, t) - \gamma \beta - 3\epsilon t\) for \((x, t) \in Q_{\beta}\). In the above two inequalities, we may take \(\gamma > 0\) as small as we wish. Thus we get

\[u(x, t) = \sup \{z(x, t) \mid z \in \mathcal{V}_{T}, \ z \leq u \text{ on } Q\} \quad \text{for} \quad (x, t) \in Q,\]

which completes the proof. \(\square\)

3. Examples. In this section we consider some examples of Hamiltonians \(H\) and examine if \(H\) satisfies conditions (4)–(6) or not.

Let \(H \in C(\mathbb{R}^{2n})\) be a function of the form

\[H(x, p) = G(x, p)^{m} + f(x),\]

where \(G \in C(\mathbb{R}^{2n})\) satisfies

\[(24) \quad G \in \text{BUC}(\mathbb{R}^{n} \times B^{n}(0, R)) \quad \text{for} \quad R > 0,\]

\[(25) \quad G(x, \lambda p) = \lambda G(x, p) \quad \text{for} \quad \lambda \geq 0, (x, p) \in \mathbb{R}^{2n},\]

\[(26) \quad \delta_{G} := \inf_{\mathbb{R}^{n} \times \partial B^{n}(0,1)} G > 0.\]
$m$ is a constant satisfying $m \geq 1$, and $f \in \text{BUC}(\mathbb{R}^n)$.

**Proposition 9.** The function $H$ given above satisfies (4)–(6).

We need the following Lemma.

**Lemma 10.** For all $(x, p) \in \mathbb{R}^{2n}$, we have

$$(27) \quad \hat{G}(x, p) = \min\{r \in \mathbb{R} \mid p = \sum_{i=1}^{k} \lambda_i p_i, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, G(x, p_i) = r\}.$$ 

**Proof.** We fix $x \in \mathbb{R}^n$ and write $G(p)$ for $G(x, p)$ for notational simplicity. By using the separation theorem and Carathéodory's theorem in convex analysis, we see easily that

$$(28) \quad \hat{G}(p) = \inf\{\sum_{i=1}^{n+1} \lambda_i G(p_i) \mid \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i p_i = p\} \quad \text{for } p \in \mathbb{R}^n.$$ 

It is clear from the above representation formula that

$$\hat{G}(\lambda p) = \lambda \hat{G}(p) \quad \text{for } (\lambda, p) \in [0, \infty) \times \mathbb{R}^n,$$

$$G(p) \geq \hat{G}(p) \geq \delta_G |p| \quad \text{for } p \in \mathbb{R}^n.$$ 

Fix $p \in \mathbb{R}^n$. If $p = 0$, then it is clear that (27) holds. We may thus assume that $p \neq 0$. For any $r > \hat{G}(p)$, by the above formula, there are $\{\lambda_i\}_{i=1}^{n+1} \subset [0, 1]$ and $\{p_i\}_{i=1}^{n+1} \subset \mathbb{R}^n$ such that

$$r > \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \sum_{i=1}^{n+1} \lambda_i = 1, \quad \sum_{i=1}^{n+1} \lambda_i p_i = p.$$ 

Set

$$s = \sum_{i=1}^{n+1} \lambda_i G(p_i), \quad \mu_i = s^{-1} G(p_i).$$ 

Notice that $s \geq \hat{G}(p) > 0$ by (28). By rearranging the order in $i$ if necessary, we may assume that

$$\lambda_i \mu_i > 0 \quad \text{for } i \leq k, \quad \lambda_i \mu_i = 0 \quad \text{for } i > k$$ 

for some $k \in \{1, \ldots, n+1\}$. Note that if $i > k$ and $\lambda_i > 0$, then $p_i = 0$. We now have

$$\sum_{i=1}^{k} \lambda_i \mu_i = s^{-1} \sum_{i=1}^{n+1} \lambda_i G(p_i) = 1,$$

$$\sum_{i=1}^{k} \lambda_i \mu_i (\mu_i^{-1} p_i) = \sum_{i=1}^{k} \lambda_i p_i = \sum_{i=1}^{n+1} \lambda_i p_i = p,$$

$$G(\mu_i^{-1} p_i) = s G(p_i)^{-1} G(p_i) = s \quad \text{for } i = 1, \ldots, k.$$
Hence we get

$$\hat{G}(p) \geq \inf\{s \in \mathbb{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^{k} \lambda_i p_i = p, \ k \leq n + 1\}.$$ 

Since the set \( \{q \in \mathbb{R}^n \mid G(q) \leq \hat{G}(p) + 1\} \) is a compact set, it is not hard to see that the infimum on the right hand side of the above inequality is actually attained. That is, we have

$$\hat{G}(p) \geq \min\{s \in \mathbb{R} \mid \lambda_i > 0, G(p_i) = s, \sum_{i=1}^{k} \lambda_i p_i = p, \ k \leq n + 1\}.$$ 

The opposite inequality is obvious. The proof is now complete. \( \square \)

**Proof of Proposition 9.** First we observe that

\begin{equation}
\hat{H}(x, p) = \hat{G}(x, p)^m + f(x) \quad \text{for} \ (x, p) \in \mathbb{R}^{2n}.
\end{equation}

Indeed, since the function:

\[ p \mapsto \hat{G}(x, p)^m + f(x) \]

is convex on \( \mathbb{R}^n \) for every \( x \in \mathbb{R}^n \) and

$$\hat{G}(x, p)^m + f(x) \leq H(x, p) \quad \text{for} \ (x, p) \in \mathbb{R}^{2n},$$

we see that

$$\hat{G}(x, p)^m + f(x) \leq \hat{H}(x, p) \quad \text{for} \ (x, p) \in \mathbb{R}^{2n}.$$ 

On the other hand, by Lemma 10, for \((x, p) \in \mathbb{R}^{2n}\) we have

$$\hat{G}(x, p)^m = \min\{r^m \in \mathbb{R} \mid k \leq n + 1, \lambda_i > 0, G(x, p_i) = r, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p\}$$

$$\geq \inf\{\sum_{i=1}^{k} \lambda_i G(x, p_i)^m \mid k \in \mathbb{N}, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p\}.$$ 

Hence, by the formula

$$\hat{H}(x, p) = \inf\{\sum_{i=1}^{k} \lambda_i H(x, p_i) \mid k \in \mathbb{N}, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p\},$$

we have

$$\hat{G}(x, p)^m + f(x) \geq \hat{H}(x, p).$$
Thus we have shown (29).

To show that $H$ satisfies (4), we just need to prove that
\[ \hat{G} \in BUC(\mathbb{R}^n \times B^n (0, R)) \quad \text{for } R > 0. \]

Fix $R > 0$, set
\[ \rho_1 = \sup_{\mathbb{R}^n \times B^n (0, R)} G, \]
and, in view of (26), choose $\rho_2 > 0$ so that
\[ \inf_{\mathbb{R}^n \times (\mathbb{R}^n \times B^n (0, \rho_2))} G > \rho_1. \]

Then, by Lemma 10, we have
\[
\hat{G}(x, p) = \min \left\{ \sum_{i=1}^{k} \lambda_i G(x, p_i) \mid \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, \ G(x, p_i) \leq \rho_1, \sum_{i=1}^{k} \lambda_i p_i = p \right\}
\]
for $(x, p) \in \mathbb{R}^n \times B^n (0, R)$.

This shows that the collection of functions:
\[ x \mapsto \hat{G}(x, p), \]
with $p \in B^n (0, R)$, is equi-continuous on $\mathbb{R}^n$. On the other hand,
\[ \{ \hat{G}(x, \cdot) \mid x \in \mathbb{R}^n \} \]
is a uniformly bounded collection of convex functions on $B^n (0, R)$. Consequently, this collection is equi-Lipschitz continuous on $B^n (0, R)$. Thus we see that $\hat{G} \in BUC(\mathbb{R}^n \times B^n (0, R))$ for all $R > 0$.

By assumptions (25) and (26), $H$ clearly satisfies (5).

To show (6), fix $R > 0$ and choose $\rho_2 > 0$ as above. Then, by Lemma 10, we get
\[
\hat{G}(x, p)^m = \min \left\{ \sum_{i=1}^{k} \lambda_i G(x, p_i)^m \mid k \in \mathbb{N}, \ \lambda_i \geq 0, \ G(x, p_i) = \hat{G}(x, p), \ \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \right\}
\]
\[
= \min \left\{ \sum_{i=1}^{k} \lambda_i G(x, p_i)^m \mid k \in \mathbb{N}, \ \lambda_i \geq 0, \ p_i \in B^n (0, \rho_2), \ \sum_{i=1}^{k} \lambda_i = 1, \sum_{i=1}^{k} \lambda_i p_i = p \right\}.
\]
Hence we have

\[ \hat{H}(x, p) = \hat{H}_{\rho_{2}}(x, p) \quad \text{for} \ (x, p) \in \mathbb{R}^{n} \times B^{n}(0, R). \]

Thus \( H \) satisfies \((4)\)–\((6)\). \( \square \)

Bibliography


