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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2005), 1429: 12-24</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/47333">http://hdl.handle.net/2433/47333</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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Path Model for a Level-Zero Extremal Weight Module over a Quantum Affine Algebra

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0 Introduction.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over the field $\mathbb{Q}$ of rational numbers, and let $P$ be an integral weight lattice of $\mathfrak{g}$. In [L1] and [L2], Littelmann introduced the path model consisting of Lakshmibai-Seshadri paths (LS paths for short) for a representation of the symmetrizable Kac-Moody algebra $\mathfrak{g}$; for an integral weight $\lambda \in P$, an LS path of shape $A$ is, by definition, a path $\pi : [0, 1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ (i.e., piecewise linear, continuous maps such that $\pi(0) = 0$ and $\pi(1) \in P$) determined by a pair of a sequence of elements in $W\lambda$, where $W$ is the Weyl group of $\mathfrak{g}$, and a sequence of rational numbers satisfying a certain combinatorial condition (see §1.2 below). We denote by $\mathcal{B}(\lambda)$ the set of all LS paths of shape $\lambda$. Littelmann showed that the set $\mathcal{B}(\lambda)$ together with root operators (see §1.3 below) and the weight map $\text{wt}(\pi) := \pi(1)$, $\pi \in \mathcal{B}(\lambda)$, is a crystal with weight lattice $P$. Then he proved that if $\lambda \in P$ is a dominant integral weight, then the crystal graph of the crystal $\mathcal{B}(\lambda)$ is connected, and the formal sum $\sum_{\pi \in \mathcal{B}(\lambda)} e(\pi(1))$ is equal to the character $\text{ch} L(\lambda)$ of the integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ of highest weight $\lambda$. Moreover, it was proved independently by Kashiwara [Kas3] and Joseph [J] that the $\mathcal{B}(\lambda)$ for dominant $\lambda$ is, as a crystal, isomorphic to the crystal base of the highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of highest weight $\lambda$, where $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in $q$. Now, quite a natural question arises: Is there any $U_q(\mathfrak{g})$-module whose crystal base is isomorphic to the crystal $\mathcal{B}(\lambda)$ for general $\lambda \in P$? In a series of papers [NS1] ~ [NS3], we gave a kind of answer to this question in the case where $\mathfrak{g}$ is an affine Lie algebra.
For a more precise description, we need some notation. Let \( \mathfrak{g} \) be an affine Lie algebra over \( \mathbb{Q} \) with Cartan subalgebra \( \mathfrak{h} \), simple roots \( \{ \alpha_j \}_{j \in I} \subset \mathfrak{h}^* \), simple coroots \( \{ \varpi_j \}_{j \in I} \subset \mathfrak{h} \), and Weyl group \( W = \langle r_j \mid j \in I \rangle \subset \text{GL}(\mathfrak{h}^*) \), where \( r_j, j \in I \), are the simple reflections. We denote by \( \delta = \sum_{j \in I} a_j \alpha_j \in \mathfrak{h}^* \) the null root, and by \( c = \sum_{j \in I} a_j^\vee \varpi_j \in \mathfrak{h} \) the canonical central element. An integral weight \( \lambda \in P \) is said to be of positive (resp., negative) level if \( \lambda(c) > 0 \) (resp., \( \lambda(c) < 0 \)), and to be of level zero if \( \lambda(c) = 0 \):

\[
P = \{ \lambda \in P \mid \lambda(c) > 0 \} \cup \{ \lambda \in P \mid \lambda(c) = 0 \} \cup \{ \lambda \in P \mid \lambda(c) < 0 \}.
\]

If \( \lambda \in P \) is of positive (resp., negative) level, then there exists a unique dominant (resp., anti-dominant) integral weight in \( W \lambda \). Denote it by \( \mu \). Because \( \mathcal{B}(\lambda) = \mathcal{B}(w \lambda) \) for all \( w \in W \), we have that the set \( \mathcal{B}(\lambda) \) is the same as the set \( \mathcal{B}(\mu) \) of all LS paths of shape \( \mu \); accordingly, it follows from the result due to Kashiwara [Kas3] and Joseph [J] that \( \mathcal{B}(\lambda) \) is, as a crystal, isomorphic to the crystal base of the highest (resp., lowest) weight module \( V(\mu) \) of highest (resp., lowest) weight \( \mu \) over the quantum affine algebra \( U_q(\mathfrak{g}) \).

Now we are left with the case where \( \lambda \in P \) is of level zero. We take (and fix) a special vertex \( 0 \in I \) such that \( a_0^\vee = 1 \), and set \( I_0 := I \setminus \{0\} \). Let \( \Lambda_i, i \in I \), be the fundamental weights for \( \mathfrak{g} \), and set \( \varpi_i := \Lambda_i - a_i^\vee \Lambda_0 \) for \( i \in I_0 \) (note that \( \varpi_i, i \in I \), is a level-zero integral weight). In the case where \( \lambda = m \varpi_i \) for some \( m \in \mathbb{Z}_{\geq 1} \) and \( i \in I_0 \), we proved in [NS1] and [NS2] that the LS path crystal is isomorphic to the crystal base of the extremal weight module over \( U_q(\mathfrak{g}) \) (Theorem 1). Here the extremal weight module \( V(\lambda) \) over \( U_q(\mathfrak{g}) \) with \( \lambda \) as an extremal weight is an integrable module over \( U_q(\mathfrak{g}) \) generated by a single element \( v_\lambda \) with the defining relations that the \( v_\lambda \) is an extremal weight vector of weight \( \lambda \) (see §1.4 below); we know from [Kas1, Proposition 8.2.2] that the extremal weight module \( V(\lambda) \) admits a crystal base, denoted by \( \mathcal{B}(\lambda) \).

**Theorem 1.** For \( m \in \mathbb{Z}_{\geq 1} \) and \( i \in I_0 \), the crystal \( \mathcal{B}(m \varpi_i) \) of all LS paths of shape \( m \varpi_i \) is, as a crystal, isomorphic to the crystal base \( \mathcal{B}(m \varpi_i) \) of the extremal weight module \( V(m \varpi_i) \) over \( U_q(\mathfrak{g}) \) with \( m \varpi_i \) as an extremal weight.

We know from [NS1, Remark 5.2] and [NS3, §3.1] that for a general integral weight \( \lambda \in P \) of level zero, there is no isomorphism of crystals between the set
\( \mathcal{B}(\lambda) \) of all LS paths of shape \( \lambda \) and the crystal base \( \mathcal{B}(\lambda) \) of the extremal weight \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) of extremal weight \( \lambda \). We do not know whether or not there exists a \( U_q(\mathfrak{g}) \)-module having a crystal base isomorphic to \( \mathcal{B}(\lambda) \), except for the case mentioned in Theorem 1.

Now we turn to a fundamental module of level zero (see §1.5 below). Let \( \text{cl} : \mathfrak{h}^* \to \mathfrak{h}^*/\mathbb{Q}\delta \) be the canonical projection. Denote by \( U'_q(\mathfrak{g}) \) the quantized universal enveloping algebra with \( P_{\text{cl}} := \text{cl}(P) \) the integral weight lattice. In [Kas4, §5.2], Kashiwara introduced a finite-dimensional irreducible \( U'_q(\mathfrak{g}) \)-module \( W(\varpi_i) \), called a fundamental module of level zero, and proved that it has a global basis with a simple crystal (see [Kas4, Theorem 5.17]). The fundamental module \( W(\varpi_i) \) of level zero seems to be isomorphic to the Kirillov-Reshetikhin module \( W_1^{(i)} \) in the notation of [HKOTT, §2.3] for \( i \in I_0 \) (see [HKOTT, Remark 2.3]). In [NS1] and [NS2], we gave a path model for \( W(\varpi_i) \cong W_1^{(i)} \) as follows. Let \( \lambda \in P \) be a level-zero integral weight. For an LS path \( \pi \in \mathcal{B}(\lambda) \) of shape \( \lambda \), we define a path \( \text{cl}(\pi) : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P_{\text{cl}} \) by: \( (\text{cl}(\pi))(t) = \text{cl}(\pi(t)) \) for \( t \in [0,1] \), and set \( \mathcal{B}(\lambda)_{\text{cl}} := \text{cl}(\mathcal{B}(\lambda)) \). Then the set \( \mathcal{B}(\lambda)_{\text{cl}} \) has a crystal structure with weight lattice \( P_{\text{cl}} \), which is naturally induced from that of \( \mathcal{B}(\lambda) \).

**Theorem 2.** The crystal \( \mathcal{B}(\varpi_i)_{\text{cl}} \) is isomorphic to the crystal base of the fundamental module \( W(\varpi_i) \) of level zero.

In [NS3], we studied the crystal structure of \( \mathcal{B}(\lambda)_{\text{cl}} = \text{cl}(\mathcal{B}(\lambda)) \) for a general integral weight \( \lambda \in P \) of level zero. Before stating our main result of [NS3], we make some comments. If \( \lambda' = \lambda + R\delta \) for some \( R \in \mathbb{Q} \), then it follows from the definition of LS paths that \( \mathcal{B}(\lambda') = \{ \pi + \pi_{R\delta} \mid \pi \in \mathcal{B}(\lambda) \} \), where \( (\pi + \pi_{R\delta})(t) := \pi(t) + tR\delta, \ t \in [0,1] \), and from the definition of the root operators that the crystal graph of \( \mathcal{B}(\lambda + R\delta) \) is the same shape as that of \( \mathcal{B}(\lambda) \), up to \( R\delta \)-shift of weight. In addition, we have that \( \mathcal{B}(\lambda) = \mathcal{B}(w\lambda) \) for all \( w \in W \). Therefore we may assume that the \( \lambda \in P \) is of the form \( \lambda = \sum_{i \in I_0} m_i \varpi_i \) with \( m_i \in \mathbb{Z}_{\geq 0} \) from the beginning.

Now we are ready to state our main result in [NS3].

**Theorem 3.** Let \( \lambda = \sum_{i \in I_0} m_i \varpi_i \) with \( m_i \in \mathbb{Z}_{\geq 0} \). Then, there exists a unique isomorphism \( \mathcal{B}(\lambda)_{\text{cl}} \overset{\sim}{\to} \bigotimes_{i \in I_0} (\mathcal{B}(\varpi_i)_{\text{cl}})^{\otimes m_i} \) of crystals (with weight lattice \( P_{\text{cl}} \)) between the crystal \( \mathcal{B}(\lambda)_{\text{cl}} \) and the tensor product \( \bigotimes_{i \in I_0} (\mathcal{B}(\varpi_i)_{\text{cl}})^{\otimes m_i} \).
By combining Theorems 2 and 3, we can get the following corollary.

**Corollary.** Let \( \lambda = \sum_{i \in I_0} m_i \omega_i \) with \( m_i \in \mathbb{Z}_{\geq 0} \). Then, the crystal \( B(\lambda)_{cl} \) is, as a crystal with weight lattice \( P_{cl} \), isomorphic to the crystal base of the tensor product \( U'_q(\mathfrak{g}) \)-module \( \bigotimes_{i \in I_0} W(\omega_i)^{\otimes m_i} \).

### 1 Preliminaries.

#### 1.1 Affine Lie algebras and quantum affine algebras.

Let \( \mathfrak{g} \) be an affine Lie algebra over the field \( \mathbb{Q} \) of rational numbers with Cartan subalgebra \( \mathfrak{h} \). Denote by \( \Pi := \{ \alpha_j \}_{j \in I} \subset \mathfrak{h}^* := \text{Hom}_\mathbb{Q}(\mathfrak{h}, \mathbb{Q}) \) the set of simple roots, and by \( \Pi^\vee := \{ h_j \}_{j \in I} \subset \mathfrak{h} \) the set of simple coroots, where \( I = \{ 0, 1, 2, \ldots, \ell \} \) is an index set for the simple roots \( \Pi \). Throughout this article, we use the numbering of the simple roots as in [Kac, §4.8 and §6]. Let \( c \in \mathfrak{h}^* \) and

\[
c = \sum_{j \in I} a_j^\vee h_j \in \mathfrak{h}
\]

be the null root and the canonical central element of \( \mathfrak{g} \), respectively. Denote by \( W = \langle r_j \mid j \in I \rangle \subset \text{GL}(\mathfrak{h}^*) \) the Weyl group of the affine Lie algebra \( \mathfrak{g} \), where \( r_j \in \text{GL}(\mathfrak{h}^*) \) is the simple reflection in \( \alpha_j \) for \( j \in I \). We call an element of the set \( \Delta^\text{re} := W\Pi \) a real root, and denote by \( \Delta^\text{re}_+ \) the set of positive real roots. Let \( \Lambda_j, j \in I \), be the fundamental weights for the affine Lie algebra \( \mathfrak{g} \). We take (and fix) an integral weight lattice \( P \subset \mathfrak{h}^* \) that contains all the simple roots \( \alpha_j, j \in I \), and fundamental weights \( \Lambda_j, j \in I \). For each \( i \in I_0 := I \setminus \{ 0 \} \), we define a level-zero fundamental weight \( \omega_i \in P \) by

\[
\omega_i := \Lambda_i - a_i^\vee \Lambda_0.
\]

Note that \( \omega_i(c) = 0 \); an integral weight \( \lambda \in P \) is said to be level-zero if \( \lambda(c) = 0 \). An integral weight \( \lambda \in P \) of level zero is said to be dominant if \( \lambda(h_i) \geq 0 \) for all \( i \in I_0 \). Let

\[
\text{cl} : \mathfrak{h}^* \rightarrow \mathfrak{h}^*/\mathbb{Q}\delta
\]

be the canonical projection, and set \( P_{cl} := \text{cl}(P) \).

Let \( U_q(\mathfrak{g}) \) be the quantized universal enveloping algebra (with weight lattice \( P \)) of the affine Lie algebra \( \mathfrak{g} \) over the field \( \mathbb{Q}(q) \) of rational functions in \( q \). We
denote by $E_j, F_j, j \in I$, and $q^h, h \in P^\vee := \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$ the Chevalley generators of $U_q(\mathfrak{g})$, where $E_j$ (resp., $F_j$) corresponds to the simple root $\alpha_j$ (resp., $-\alpha_j$). Denote by $U'_q(\mathfrak{g})$ the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $E_j, F_j, j \in I$, and $q^h, h \in (P_\mathbb{Z})^\vee := \text{Hom}_\mathbb{Z}(P_\mathbb{Z}, \mathbb{Z})$, which is the quantized universal enveloping algebra of $\mathfrak{g}$ with weight lattice $P_\mathbb{Z}$.

1.2 Lakshmibai–Seshadri paths. A path (with weight in $P$) is, by definition, a piecewise linear, continuous map $\pi : [0, 1] \to \mathbb{Q} \otimes_\mathbb{Z} P$ from $[0, 1] := \{ t \in \mathbb{Q} \mid 0 \leq t \leq 1 \}$ to $\mathbb{Q} \otimes_\mathbb{Z} P$ such that $\pi(0) = 0$ and $\pi(1) \in P$. In this subsection, we recall the definition of a Lakshmibai–Seshadri path (an LS path for short) from [L2, §4] (see also [NS2, §1.4] and [NS3, §2.1]).

We first recall some auxiliary notations. Let $\lambda \in P$ be an integral weight. For $\mu, \nu \in W\lambda$, we write $\mu \geq \nu$ if there exist a sequence $\mu = \xi_0, \xi_1, \ldots, \xi_n = \nu$ of elements in $W\lambda$ and a sequence $\beta_1, \ldots, \beta_n \in \Delta^+_\mathbb{R}$ of positive real roots such that $\xi_k = r_{\beta_k}(\xi_{k-1})$ and $\xi_{k-1}(\beta_k^\vee) < 0$ for $k = 1, 2, \ldots, n$, where for a positive real root $\beta \in \Delta^+_\mathbb{R}$, $r_\beta$ denotes the reflection with respect to $\beta$, and $\beta^\vee$ denotes the dual real root of $\beta$. If $\mu \geq \nu$, then we define $\text{dist}(\mu, \nu)$ to be the maximal length $n$ of all possible such sequences $\xi_0, \xi_1, \ldots, \xi_n$ for the pair $(\mu, \nu)$. Then, for $\mu, \nu \in W\lambda$ with $\mu \geq \nu$ and a rational number $0 < a < 1$, an $a$-chain for $(\mu, \nu)$ is, by definition, a sequence $\mu = \xi_0 > \xi_1 > \cdots > \xi_n = \nu$ of elements in $W\lambda$ such that $\text{dist}(\xi_{k-1}, \xi_k) = 1$ and $a \xi_{k-1}(\beta_k^\vee) \in \mathbb{Z}_{<0}$ for all $k = 1, 2, \ldots, n$, where $\beta_k$ is the positive real root corresponding to $(\xi_{k-1}, \xi_k)$ with $\xi_{k-1} > \xi_k$.

Now we are ready for the definition of an LS path. Let $\lambda \in P$ be an integral weight. An LS path of shape $\lambda$ is a path $\pi : [0, 1] \to \mathbb{Q} \otimes_\mathbb{Z} P$ associated to a pair $(\nu; a)$ of a sequence $\nu : \nu_1, \nu_2, \ldots, \nu_s$ of elements in $W\lambda$ and a sequence $a : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers satisfying the condition that there exists an $a_k$-chain for $(\nu_k, \nu_{k+1})$ for all $k = 1, 2, \ldots, s - 1$; to such a pair $(\nu; a) = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s)$, we associate the following path $\pi : [0, 1] \to \mathbb{Q} \otimes_\mathbb{Z} P$:

$$\pi(t) = \sum_{l=1}^{k-1} (a_l - a_{l-1})\nu_l + (t - a_{k-1})\nu_k \text{ for } a_{k-1} \leq t \leq a_k, 1 \leq k \leq s.$$  

Note that $\pi(0)$ is obviously equal to $0 \in P$, and it follows from [L2, Lemma 4.5 a)]
that $\pi(1) \in P$; namely, the $\pi$ above is, in fact, a path for all such pairs $(\nu; a) = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s)$. Denote by $\mathbb{B}(\lambda)$ the set of LS paths of shape $\lambda$.

**Remark 1.2.1.** (1) The straight line $\pi_{\nu}(t) := t\nu$, $t \in [0, 1]$, is contained in $\mathbb{B}(\lambda)$ for all $\nu \in W\lambda$ (put $s = 1$ and $\nu_1 = \nu$).

(2) It follows from the definition that $\mathbb{B}(w\lambda) = \mathbb{B}(\lambda)$ for all $w \in W$.

### 1.3 Root operators

In this subsection, we give a description of root operators $e_j$ and $f_j$, $j \in I$, which was introduced in [L2, §1], on the set $\mathbb{B}(\lambda)$ of all LS paths of shape $\lambda \in P$ (see also [NS2, §1.2] and [NS4, §2.1]).

Let $\lambda \in P$ be an integral weight. For an LS path $\pi \in \mathbb{B}(\lambda)$ and $j \in I$, we define $e_j\pi$ as follows: First, we set

$$
H^\pi_j(t) := (\pi(t))(h_j) \quad \text{for } t \in [0, 1],
$$

$$
m^\pi_j := \min\{H^\pi_j(t) | t \in [0, 1]\}.
$$

If $m^\pi_j > -1$, then we define $e_j\pi := \theta$. Here, $\theta$ is an extra element, which corresponds to the 0 in the theory of crystals (by convention, we put $e_j\theta = f_j\theta := \theta$).

If $m^\pi_j \leq -1$, then

$$(e_j\pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) + \alpha_j & \text{if } t_1 \leq t \leq 1,
\end{cases}
$$

where we set

$$
t_1 := \min\{t \in [0, 1] | H^\pi_j(t) = m^\pi_j \},
$$

$$
t_0 := \max\{t' \in [0, t_1] | H^\pi_j(t) \geq m^\pi_j + 1 \text{ for all } t \in [0, t']\}.
$$

Similarly, $f_j\pi$ is given as follows: If $H^\pi_j(1) - m^\pi_j < 1$, then we set $f_j\pi := \theta$. If $H^\pi_j(1) - m^\pi_j \geq 1$, then

$$(f_j\pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) - \alpha_j & \text{if } t_1 \leq t \leq 1,
\end{cases}
$$

where we set

$$
t_0 := \max\{t \in [0, 1] | H^\pi_j(t) = m^\pi_j \},
$$

$$
t_1 := \min\{t' \in [t_0, 1] | H^\pi_j(t) \geq m^\pi_j + 1 \text{ for all } t \in [t', 1]\}.$$
Theorem 1.3.1 ([L2]). For every integral weight \( \lambda \in P \), the set \( \mathcal{B}(\lambda) \cup \{\theta\} \) is stable under the action of the root operators \( e_j \) and \( f_j \) for \( j \in I \). We define
\[
\begin{align*}
\text{wt}(\pi) & := \pi(1) \quad \text{for } \pi \in \mathcal{B}(\lambda), \\
\varepsilon_j(\pi) & := \max\{n \geq 0 \mid e_j^n \pi \neq \theta\} \quad \text{for } \pi \in \mathcal{B}(\lambda) \text{ and } j \in I, \\
\varphi_j(\pi) & := \max\{n \geq 0 \mid f_j^n \pi \neq \theta\} \quad \text{for } \pi \in \mathcal{B}(\lambda) \text{ and } j \in I.
\end{align*}
\]
Then, the set \( \mathcal{B}(\lambda) \) together with the root operators and the maps above is a crystal with weight lattice \( P \).

1.4 Extremal weight modules.

Definition 1.4.1 (cf. [Kas1, §8] and [Kas4, §3.1]). Let \( M \) be an integrable \( U_q(\mathfrak{g}) \)-module. A vector \( v \in M \) of weight \( \lambda \in P \) is said to be extremal, if there exists a family \( \{v_w\}_{w \in W} \) of weight vectors of \( M \) satisfying the following conditions:

for \( w \in W \) and \( j \in I \),

\[ a)\, v_w = v \text{ if } w = 1; \]

\[ b)\, \text{if } n := (w(\lambda))(h_j) \geq 0, \text{ then } E_j v_w = 0 \text{ and } E_j^{(n)} v_w = v_{r_j w}; \]

\[ c)\, \text{if } n := (w(\lambda))(h_j) \leq 0, \text{ then } F_j v_w = 0 \text{ and } F_j^{(-n)} v_w = v_{r_j w}. \]

Here, \( E_j^{(n)} \) and \( F_j^{(n)} \) are the \( n \)-th \( q \)-divided powers of the Chevalley generators \( E_j \) and \( F_j \) of \( U_q(\mathfrak{g}) \), respectively.

Definition 1.4.2 (cf. [Kas1, §8] and [Kas4, §3.1]). Let \( \lambda \in P \) be an integral weight. The extremal weight module \( V(\lambda) \) over \( U_q(\mathfrak{g}) \) with \( \lambda \) as an extremal weight is, by definition, the integrable \( U_q(\mathfrak{g}) \)-module generated by a single element \( v_\lambda \) with the defining relations that \( v_\lambda \) is an extremal vector of weight \( \lambda \).

We know the following theorem from [Kas1, Proposition 8.2.2].

Theorem 1.4.3. For every \( \lambda \in P \), the extremal weight module \( V(\lambda) \) has a crystal base, which we denote by \( \mathcal{B}(\lambda) \).

Remark 1.4.4. The extremal weight module is a natural generalization of an integrable highest and lowest weight module; in fact, we know from [Kas1, §8] that if \( \lambda \in P \) is dominant (resp. anti-dominant), then the extremal weight module \( V(\lambda) \) is isomorphic to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight \( \lambda \), and the crystal base \( \mathcal{B}(\lambda) \) of \( V(\lambda) \) is isomorphic to the crystal base of the integrable highest (resp., lowest) weight module as a crystal.
1.5 Fundamental module of level zero. We define a positive integer $d_i \in \mathbb{Z}_{\geq 1}$ by
\[
\{ n \in \mathbb{Z} | \varpi_i + n\delta \in W\varpi_i \} = \mathbb{Z}d_i.
\] (1.5.1)

Because $V(\varpi_i) \cong V(w\varpi_i)$ as $U_q(\mathfrak{g})$-modules for all $w \in W$ (see [Kas1, Proposition 8.2.2 iv]), we see that there exists a $U_q(\mathfrak{g})$-module isomorphism $V(\varpi_i + d_i\delta) \cong V(\varpi_i)$. In addition, there exists a $U'_q(\mathfrak{g})$-module isomorphism $V(\varpi_i) \cong V(\varpi_i + d_i\delta)$, which maps the $\varpi_i$-weight space $V(\varpi_i)_{\varpi_i}$ of $V(\varpi_i)$ to the $(\varpi_i + d_i\delta)$-weight space $V(\varpi_i + d_i\delta)_{\varpi_i + d_i\delta}$ of $V(\varpi_i + d_i\delta)$ (by [Kas4, Proposition 5.16], these weight spaces are 1-dimensional). Thus we get a $U'_q(\mathfrak{g})$-module automorphism $z_i: V(\varpi_i) \cong V(\varpi_i)$ of weight $d_i\delta$ (see [Kas4, §5.2]) as the composition of these maps. We now define a $U'_q(\mathfrak{g})$-module $W(\varpi_i)$ by
\[
W(\varpi_i) := V(\varpi_i)/(z_i - 1)V(\varpi_i),
\] (1.5.2)
which is called a fundamental module of level zero. We know from [Kas4, Theorem 5.17] that $W(\varpi_i)$ is a finite-dimensional irreducible $U'_q(\mathfrak{g})$-module, and has a simple crystal base, which is denoted by $B(\varpi_i)_c$.

2 Our results.

2.1 Isomorphism theorems. Our main result in [NS1] and [NS2] is the following theorem (see [NS1, Theorem 5.1] and [NS2, Corollaries 2.2.1 and 3.3.8]).

Theorem 2.1.1. For $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, the crystal $B(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal with weight lattice $P$, isomorphic to the crystal base $B(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ with $m\varpi_i$ as an extremal weight.

Here, let us give a sketch of our proof of Theorem 2.1.1. First we show the theorem for the case where $m = 1$. In [NS2, Theorem 2.1.1], we proved the following.

Theorem 2.1.2. For every $i \in I_0$, the crystal graph of the crystal $B(\varpi_i)$ is connected.
We know from [Kas4, Proposition 5.4(ii)] that the crystal graph of the crystal base $\mathcal{B}(\varpi_i)$ is also connected, and from [Kas4, Proposition 5.16(ii)] that the cardinality of the subset $\mathcal{B}(\varpi_i)_{w \varpi_i}$ is equal to 1 for all $w \in W$, where $\mathcal{B}(\varpi_i)_\mu$ is the subset of $\mathcal{B}(\varpi_i)$ consisting of all elements of weight $\mu \in P$. In addition, we see from [BN, Theorem 4.16(i)] that there exists a canonical embedding $\mathcal{B}_0(N\varpi_i) \hookrightarrow \mathcal{B}(\varpi_i)^{\otimes N}$ of crystals that sends $u_{N\varpi_i}$ to $u_{\varpi_i}^{\otimes N}$, where for each $A \in P$, $u_{\lambda}$ denotes the element of the crystal base $\mathcal{B}(\lambda)$ corresponding to the generator $v_{\lambda}$ of the extremal weight module $V(\lambda)$, and $\mathcal{B}_0(\lambda)$ denotes the connected component of $\mathcal{B}(\lambda)$ containing the element $u_{\lambda}$. Further we showed the following proposition.

**Proposition 2.1.3 ([NS1, Theorem 3.1])**. For every $N \in \mathbb{Z}_{>0}$ and $i \in I_0$, there exists an injective map $S_N : \mathcal{B}(\varpi_i) \hookrightarrow \mathcal{B}_0(N\varpi_i)$, which we call an $N$-multiple map, satisfying the following condition:

1. $S_N(u_{\varpi_i}) = u_{N \varpi_i}$,
2. $\text{wt}(S_N(b)) = N \text{wt}(b)$ for each $b \in \mathcal{B}(\varpi_i)$,
3. $S_N(e_i b) = e_i^N S_N(b)$, $S_N(f_i b) = f_i^N S_N(b)$ for $b \in \mathcal{B}(\varpi_i)$ and $i \in I$.

By using these facts, we can show that $\mathcal{B}(\varpi_i) \cong \mathcal{B}(\varpi_i)$ as crystals in exactly the same way as [Kas2, Theorem 4.1] (see [NS1, Theorem 5.1]).

As a consequence of Theorem 2.1.1 for the case where $m = 1$, we obtained the following corollary (cf. [NS1, Corollary 5.3]).

**Corollary 2.1.4.** For every $m \geq 1$ and $i \in I_0$, we have

$$\mathcal{B}_0(m\varpi_i) \cong \mathcal{B}_0(m\varpi_i)$$

as crystals, where $\mathcal{B}_0(m\varpi_i)$ is the connected component of the crystal $\mathcal{B}(m\varpi_i)$ containing the straight line $\pi_{m\varpi_i}(t) = t(m\varpi_i)$, $t \in [0,1]$.

Next we prove Theorem 2.1.1 for the case where $m \geq 2$ (as seen below, the crystal graph of $\mathcal{B}(m\varpi_i)$ is not connected when $m \geq 2$). Let $\text{Par}_{<m}$ be the set of partitions of length (i.e., the number of parts) strictly less than $m$. For each $\sigma = (k_1 \geq k_2 \geq \cdots \geq k_{m-1}) \in \text{Par}_{<m}$, we denote by $|\sigma|$ the weight of $\sigma$, i.e.,
Let \( |\sigma| := k_1 + k_2 + \cdots + k_{m-1} \). We can define a crystal structure on \( \text{Par}_{<m} \) as follows:

\[
\begin{align*}
\epsilon_j \sigma &= f_j \sigma = 0 & \text{for all } \sigma \in \text{Par}_{<m} \text{ and } j \in I, \\
\epsilon_j(\sigma) &= \varphi_j(\sigma) = 0 & \text{for all } \sigma \in \text{Par}_{<m} \text{ and } j \in I, \\
\text{wt}(\sigma) := -|\sigma|d_0 & \text{ for } \sigma \in \text{Par}_{<m}.
\end{align*}
\]

In [NS2, §§3.2 ~ 3.6], we showed the following.

**Lemma 2.1.5.** (1) For every \( \sigma = (k_1 \geq k_2 \geq \cdots \geq k_{m-1}) \in \text{Par}_{<m} \),

\[
\pi_\sigma := (m(\varpi_i - k_1d_0), \ldots, m(\varpi_i - k_{m-1}d_0), m\varpi_i; 0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1).
\]

is contained in \( \mathcal{B}(m\varpi_i) \).

(2) For each \( \pi \in \mathcal{B}(m\varpi_i) \), there exists a unique \( \sigma \in \text{Par}_{<m} \) such that the \( \pi \) is connected to \( \pi_\sigma \) in the crystal graph of \( \mathcal{B}(m\varpi_i) \).

For \( \sigma \in \text{Par}_{<m} \), we denote by \( \mathcal{B}_\sigma(m\varpi_i) \) the connected component of \( \mathcal{B}(m\varpi_i) \) containing the path \( \pi_\sigma \). Then it follows from the lemma above that

\[
\mathcal{B}(m\varpi_i) = \bigsqcup_{\sigma \in \text{Par}_{<m}} \mathcal{B}_\sigma(m\varpi_i).
\]

Here recall from §1.3 that the root operators \( e_j, f_j \) are defined in terms of the function given by the pairing of a path and the simple coroot \( h_j \). Because the path \( \pi_\sigma(t) \) is the same as the straight line \( \pi_{m\varpi_i}(t) = t(m\varpi_i) \), up to some \( \delta \)-shift, and because \( \delta(h_j) = 0 \) for all \( j \in I \), we deduce that the crystal graph of \( \mathcal{B}_0(m\varpi_i) \) is isomorphic to the crystal graph of \( \mathcal{B}_\sigma(m\varpi_i) \), up to some \( \delta \)-shift of weight. More precisely, we have

\[
\mathcal{B}_\sigma(m\varpi_i) \cong \{ \sigma \} \otimes \mathcal{B}_0(m\varpi_i) \hookrightarrow \text{Par}_{<m} \otimes \mathcal{B}_0(\varpi_i) \text{ as crystals,}
\]

which sends \( \pi_\sigma \) to \( \sigma \otimes \pi_{m\varpi_i} \). Thus we obtain

**Theorem 2.1.6.** For \( m \in \mathbb{Z}_{\geq 1} \) and \( i \in I_0 \), we have

\[
\mathcal{B}(m\varpi_i) \cong \text{Par}_{<m} \otimes \mathcal{B}_0(m\varpi_i) \text{ as crystals.}
\]

On the other hand, we know the following theorem from [BN, Theorem 4.16 (i)].
Theorem 2.1.7. For each \( m \in \mathbb{Z}_{\geq 1} \) and \( i \in I_0 \), we have

\[
\mathcal{B}(m\varpi_i) \cong \text{Par}_m \otimes \mathcal{B}_0(m\varpi_i) \quad \text{as crystals.}
\]

By combining Theorems 2.1.6 and 2.1.7 with Corollary 2.1.4, we can get our isomorphism theorem (Theorem 2.1.1).

Now, for an integral weight \( \lambda \in P \), we set

\[
\mathcal{B}(\lambda)_{\text{cl}} := \{ \text{cl}(\pi) \mid \pi \in \mathcal{B}(\lambda) \},
\]

where for a path \( \pi \), we define \( \text{cl}(\pi) : [0,1] \rightarrow \mathbb{Q} \otimes \mathbb{Z} P_{\text{cl}} \cong \mathfrak{h}^* / \mathbb{Q} \delta \) by:

\[
(\text{cl}(\pi))(t) := \text{cl}(\pi(t)) \quad \text{for } t \in [0,1].
\]

We can endow \( \mathcal{B}(\lambda)_{\text{cl}} \) with a structure of crystal with weight lattice \( P_{\text{cl}} \) in such a way that

\[
\begin{align*}
e_j(\text{cl}(\pi)) &:= e_j(\text{cl}(\pi)), & f_j(\text{cl}(\pi)) &:= f_j(\text{cl}(\pi)), \\
\epsilon_j(\text{cl}(\pi)) &:= \epsilon_j(\text{cl}(\pi)), & \varphi_j(\text{cl}(\pi)) &:= \varphi_j(\text{cl}(\pi)), \\
\text{wt}(\text{cl}(\pi)) &:= \text{wt}(\text{cl}(\pi)).
\end{align*}
\]

for \( \pi \in \mathcal{B}(\lambda) \) and \( j \in I \) (see [NS2, §3.3] and [NS3, §§1.3 and 1.4]). The following is a consequence of Theorem 2.1.1 (see [NS1, Proposition 5.8] and [NS2, Proposition 3.2]).

Theorem 2.1.8. For each \( i \in I_0 \), the crystal \( \mathcal{B}(\varpi_i)_{\text{cl}} \) is isomorphic to the crystal base \( \mathcal{B}(\varpi_i)_{\text{cl}} \) of the fundamental module \( W(\varpi_i) \) of level zero as a crystal with weight lattice \( P_{\text{cl}} \).

2.2 Tensor product decomposition theorem. In [NS3], we studied the crystal structure of \( \mathcal{B}(\lambda)_{\text{cl}} = \text{cl}(\mathcal{B}(\lambda)) \) for a general integral weight \( \lambda \in P \) of level zero. Before stating our main result in [NS3], we make some comments. Let \( \lambda \in P \) be an integral weight of level zero. We can write the \( \lambda \in P \) in the form

\[
\lambda = \sum_{i \in I_0} m'_i \varpi_i + R\delta
\]

for some \( m'_i \in \mathbb{Z} \), \( i \in I_0 \), and \( R \in \mathbb{Q} \) (cf. [Kac, Chap. 6]). Then it follows from the definition of LS paths that

\[
\mathcal{B}(\lambda) = \{ \pi + \pi_{R\delta} \mid \pi \in \mathcal{B}(\sum_{i \in I_0} m'_i \varpi_i) \},
\]

where we set \((\pi + \pi_{R\delta})(t) := \pi(t) + tR\delta\), \( t \in [0,1] \), and from the definition of the root operators that the crystal graph of \( \mathcal{B}(\lambda) \) is the same shape as that
of $\mathbb{B}(\sum_{i \in I_0} m'_i \varpi_i)$, up to $R\delta$-shift of weight. Therefore we have that $\mathbb{B}(\lambda)_{c1} = \mathbb{B}(\sum_{i \in I_0} m'_i \varpi_i)_{c1}$. In addition, the integral weight $\sum_{i \in I_0} m'_i \varpi_i \in P$ is equivalent to the one that is dominant with respect to the simple coroots $\{h_j\}_{j \in I_0}$ under the Weyl group $\hat{W} := \langle r_j \mid j \in I_0 \rangle \subset W$ (of finite type). Hence there exist nonnegative integers $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I_0$, such that $\mathbb{B}(\sum_{i \in I_0} m'_i \varpi_i)_{c1} = \mathbb{B}(\sum_{i \in I_0} m_i \varpi_i)_{c1}$ by Remark 1.2.1(2). To sum up, for an integral weight $\lambda \in P$ of level zero, we have that $\mathbb{B}(\lambda)_{c1} = \mathbb{B}(\sum_{i \in I_0} m_i \varpi_i)_{c1}$.

Now we are ready to state our main result in [NS3].

**Theorem 2.2.1 ([NS3, Theorem 2.2.1]).** Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. Then, there exists an isomorphism $\mathbb{B}(\lambda)_{c1} \sim \otimes_{i \in I_0} (\mathbb{B}(\varpi_i)_{c1})^{\otimes m_i}$ of crystals (with weight lattice $P_{c1}$) between $\mathbb{B}(\lambda)_{c1}$ and the tensor product $\otimes_{i \in I_0} (\mathbb{B}(\varpi_i)_{c1})^{\otimes m_i}$ of the crystals $\mathbb{B}(\varpi_i)_{c1}$, $i \in I_0$.

By combining Theorems 2.1.8 and 2.2.1, we obtain the next corollary.

**Corollary 2.2.2.** Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. The crystal $\mathbb{B}(\lambda)_{c1}$ is, as a crystal (with weight lattice $P_{c1}$), isomorphic to the crystal base $\otimes_{i \in I_0} (\mathbb{B}(\varpi_i)_{c1})^{\otimes m_i}$ of the tensor product $\otimes_{i \in I_0} W(\varpi_i)^{\otimes m_i}$ of fundamental $U'_q(\mathfrak{g})$-modules $W(\varpi_i)$, $i \in I_0$, of level zero.

**Acknowledgements.**

We would like to thank Professor Masato Okado and Professor Atsuo Kuniba, the organizers of the workshop, very much for giving us a chance to talk about our results in the nice workshop.

**References**


