A SIMPLE INTRODUCTION TO CRYSTALS $B^{2, s}$ FOR KIRILLOV-RESHETIKHIN MODULES OF TYPE $D^{(1)}_n$ (Combinatorial Aspect of Integrable Systems)

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A SIMPLE INTRODUCTION TO CRYSTALS $B^{2,s}$ FOR KIRILLOV-RESHETIKHIN MODULES OF TYPE $D^{(1)}_n$

ANNE SCHILLING AND PHILIP STERNBERG

ABSTRACT. The Kirillov–Reshetikhin modules $W^{r,s}$ are finite-dimensional representations of quantum affine algebras $U_q'(g)$, labeled by a Dynkin node $r$ of the affine Kac–Moody algebra $g$ and a positive integer $s$. In this paper we explain the combinatorial structure of the crystal basis $B^{2,s}$ corresponding to $W^{2,s}$ for the algebra of type $D^{(1)}_n$. Proofs of all claims, as well as more specific details of all constructions, may be found in [16].

1. INTRODUCTION

At the workshop on the Combinatorial Aspect of Integrable Systems held at RIMS Kyoto, one of the recurring themes was the $X = M$ conjecture of [1, 2]. Briefly, this conjecture states that the one-dimensional configuration sums $X$ of a certain class of lattice models can be expressed as fermionic formulas $M$, reflecting the corner transfer matrix method and the Bethe ansatz as methods for solving these lattice models. The combinatorial tools of these methods are Young tableaux/crystal bases and rigged configurations, respectively. The following table summarizes the three regimes of this conjecture.

<table>
<thead>
<tr>
<th>formulas</th>
<th>$X$: 1-D sum</th>
<th>$M$: fermionic formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>stat. mech. methods</td>
<td>CTM</td>
<td>Bethe ansatz</td>
</tr>
<tr>
<td>comb. objects</td>
<td>tableaux/crystals</td>
<td>rigged configurations</td>
</tr>
</tbody>
</table>

More specifically, the theory of crystal bases is used to label the highest weight vectors of irreducible representations (i.e., Bethe vectors) of a certain algebra by crystal basis elements. Since each Bethe vector corresponds to a solution of the Bethe equations and these solutions are indexed by rigged configurations, there should be a natural bijection between highest weight crystal elements and rigged configurations. Such bijections have been found by Kirillov and Reshetikhin [7] for type $A^{(1)}_n$ (see also [8]), and later for all nonexceptional types for the vector representation [10] and symmetric powers [15]. For type $D^{(1)}_n$ the bijection was given in [14] for the fundamental representations.

The $X = M$ conjecture depends upon the existence of the crystals $B^{r,s}$ for the Kirillov–Reshetikhin modules $W^{r,s}$. The Kirillov–Reshetikhin (KR) modules are finite-dimensional irreducible representations of quantum affine algebras $U_q'(g)$. In general, it is not known yet whether the $B^{r,s}$ exist and what their combinatorial structure is. It is the purpose of this note to give the combinatorial structure of $B^{2,s}$ of type $D^{(1)}_n$. The KR crystals of type $A^{(1)}_n$ have been explicitly described [4, 13], as well as $B^{r,1}$ and $B^{2,s}$ for most types [4, 6]. Furthermore, according to the theory of virtual crystals [11, 12], the following algebra

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embeddings have been explicitly extended to the crystals of their KR modules:

\[
\begin{align*}
C_n^{(1)} & \leftrightarrow A_{2n}^{(2)}, \\
A_{2n}^{(2)} & \leftrightarrow D_{n+1}^{(2)}, \\
A_{2n-1}^{(2)}, & \quad E_6^{(2)}, F_4^{(1)} \leftrightarrow E_6^{(1)}, \\
D_4^{(3)}, C_2^{(1)} & \leftrightarrow D_4^{(1)}.
\end{align*}
\]

The next case to explore is therefore \( B^{2,s} \) for type \( D_{n}^{(1)} \), which is the focus of this paper. Here, we present the combinatorial construction of \( B^{2,s} \) assuming existence as recently given in [16]. The combinatorial crystal is denoted by \( \tilde{B}^{2,s} \); we illustrate our main definition with examples. Proofs and further details can be found in [16]. The main result of [16] is:

**Theorem 1.1.** If \( B^{2,s} \) exists with the properties as in Conjecture 2.1, then \( \tilde{B}^{2,s} \cong B^{2,s} \).

## 2. Review

For background on quantum groups, crystal bases, perfect crystals, and other well-understood concepts, please refer to [16] or any of the standard references on these topics.

The fermionic formulas suggest not only the existence of the crystals \( B^{r,s} \), but also several conjectures about the structure of these crystals as well [1]. In the case of \( B^{2,s} \), this specializes to

**Conjecture 2.1** ([1]). The crystal \( B^{2,s} \) of type \( D_{n}^{(1)} \) exists and has the following properties:

1. As a classical crystal \( B^{2,s} \) decomposes as \( B^{2,s} \cong \bigoplus_{k=0}^{s} B(k\Lambda_{2}) \).
2. \( B^{2,s} \) is perfect of level \( s \).
3. \( B^{2,s} \) is equipped with an energy function \( D_{B^{2,s}}(b) \) such that \( D_{B^{2,s}}(b) = k - s \) if \( b \) is in the component of \( B(k\Lambda_{2}) \) (in accordance with the energy \( D \) as in [16]).

To construct \( \tilde{B}^{2,s} \) so that it satisfies these properties, we first find a way to label the vertices of the crystal. Our approach is to define a set of rules for what a legal "affine tableau" is, and then show that this set is in bijection with the direct sum \( \bigoplus_{k=0}^{s} B(k\Lambda_{2}) \).

This bijection provides the action of the crystal operators \( \tilde{e}_i \) and \( \tilde{f}_i \) for \( 1 \leq i \leq n \), but we still need to know the action of \( \tilde{e}_0 \) and \( \tilde{f}_0 \). To define these crystal operators, we use an auxiliary construction called the branching component graph. It can be shown that the resulting affine crystal \( \tilde{B}^{2,s} \) is perfect of level \( s \). In fact it was proved in [16] that this is the unique perfect level \( s \) crystal for which the energy function is as stated in Conjecture 2.1.

## 3. Affine Tableaux

We briefly recall the labelling by tableaux of the vertices of classical highest weight crystals \( B(k\Lambda_2) \) of highest weight \( k\Lambda_2 \), following the construction by Kashiwara and Nakashima [5]. Each crystal element can be represented by a tableau of shape \( \lambda = (k, k) \) on the partially ordered alphabet

\[ 1 < 2 < \cdots < n - 1 < \frac{n}{2} < \frac{n-1}{2} < \cdots 2 < 1 \]

such that the following conditions hold [3, page 202]:

**Criterion 3.1.**

1. If \( ab \) is in the filling, then \( a \leq b \);
2. If \( \frac{a}{b} \) is in the filling, then \( b \not\leq a \).
(3) No configuration of the form $a\overline{a}$ or $\overline{a}a$ appears;
(4) No configuration of the form $n-1 \overline{n}$ or $\overline{n} n-1$ appears;
(5) No configuration of the form $\frac{1}{1}$ appears.

Note that for $k \geq 2$, condition 5 follows from conditions 1 and 3.

We define the set of affine tableau in $\tilde{B}^{2,s}$ by removing parts 3 and 5 from Criterion 3.1. The bijection between $\tilde{B}^{2,s}$ and $\bigoplus_{k=0}^{s} B(k\Lambda_{2})$ is as follows. Given an affine tableau $T$ which is not a classical tableau (i.e., a tableau that satisfies parts 1, 2, and 4 of 3.1, but violates part 3 or 5) there must be a configuration of the form $a \overline{a}$ or $\frac{1}{1}$. Remove columns of the form $a \overline{a}$ (possibly with $a = 1$) until the resulting tableau satisfies Criterion 3.1. It can be shown that this procedure gives a well-defined bijection between the two sets.

The following examples are taken from $\tilde{B}^{2,5}$ for $D_{4}^{(1)}$.

**Example 3.2.** The affine tableau

\[
\begin{array}{cccccc}
1 & 2 & 2 & 3 & 3 \\
4 & 2 & 2 & 2 & 1 \\
\end{array}
\]

... by removing the second and third columns.

It is easy to see that for any affine tableau the removed columns must be adjacent, as they are in these examples.

**Example 3.3.** The affine tableau

\[
\begin{array}{cccc}
2 & 3 & 3 & 3 & 4 \\
4 & 3 & 3 & 2 & 1 \\
\end{array}
\]

... by removing either the second or the third column.

As the above example indicates, if there is a choice about which column to remove, it has no effect on the outcome.

**Example 3.4.** The classical tableau

\[
\begin{array}{cc}
1 & 2 \\
4 & 2 \\
\end{array}
\]

... corresponds to the affine tableau

\[
\begin{array}{cccccc}
1 & 2 & 2 & 2 & 2 \\
4 & 2 & 2 & 2 & 2 \\
\end{array}
\]

While we could choose to add columns of the form $\frac{2}{2}$ either to the middle or to the right side of the first tableau, either choice results in the same affine tableau.

**Example 3.5.** The classical tableau

\[
\begin{array}{cccc}
2 & 3 & 3 & 3 \\
4 & 2 & 1 \\
\end{array}
\]

... corresponds to the affine tableau

\[
\begin{array}{cccc}
2 & 2 & 3 & 3 \\
4 & 2 & 2 & 1 \\
\end{array}
\]

By part 1 of Criterion 3.1, the only place that a column of the form $\frac{a}{a}$ may be inserted is between the first and second columns of $t$. However, we may choose between using this to create a configuration of either of the forms $\frac{a}{a} \overline{a}$ or $\overline{a}a$. Once again, this “choice” does not affect the outcome.

4. **The Branching Component Graph**

Since the Dynkin diagram for type $D_{n}^{(1)}$ has a graph automorphism interchanging nodes 0 and 1, we know that interchanging the role of 1-arrows and 0-arrows in $\tilde{B}^{2,s}$ will produce an affine crystal isomorphic to $\tilde{B}^{2,s}$. We may use this fact to our advantage at a
larger scale by considering the $D_{n-1}$-crystals that result from removing the 1-arrows from $\bigoplus_{k=0}^{s} B(k\Lambda_2)$, since this direct sum is isomorphic to $\tilde{B}^{2,s}$ with the 0-arrows removed.

The branching component graph of $\tilde{B}^{2,s}$, denoted $BC(\tilde{B}^{2,s})$, is defined as follows. Its vertices correspond to the $D_{n-1}$-crystals that remain connected after removing all 0-arrows and 1-arrows from $\tilde{B}^{2,s}$; we label the vertices (non-uniquely) by the partition $\lambda$ indicating the classical highest weight of the corresponding $U_q(D_{n-1})$-crystal. The edges of $BC(\tilde{B}^{2,s})$ are defined by placing an edge from $v$ to $w$ if there is a tableau $b \in B(v)$ such that $\tilde{f}_1(b) \in B(w)$, where $B(v)$ denotes the set of tableaux contained in the $D_{n-1}$-crystal indexed by $v$.

It suffices to describe the effect of removing the 1-arrows from $B(k\Lambda_2)$ for arbitrary $k$. We denote this branching component graph by $BC(k\Lambda_2)$, and use $v_k$ to denote the "highest weight branching vertex", i.e., the branching vertex such that the highest weight tableaux $b_{k\Lambda_2} \in B(v_k)$.

An intuitive way to construct $BC(k\Lambda_2)$ is as follows. Begin with a $1 \times k$ rectangle, which labels $v_k$. For $1 \leq j \leq k$, the partitions labeling the vertices of rank $j$ are those which are contained in a $2 \times k$ rectangle and which are joined by an edge in Young's lattice to some partition labeling a vertex in rank $j - 1$. In each rank, the partitions appear with multiplicity one. For $k + 1 \leq j \leq 2k$, the partitions in rank $j$ are the same as those in rank $2k - j$, again with multiplicity one. Finally, there is an edge from a vertex $v$ of rank $j$ to a vertex $w$ of rank $j + 1$ precisely when the corresponding partitions are joined by an edge in Young's lattice.

**Example 4.1.** Figure 1 depicts $BC(3\Lambda_2)$.

There is a unique inclusion of $BC(k\Lambda_2)$ in $BC((k+1)\Lambda_2)$ that agrees with the labelling of the vertices. We may define a rank function on all of $BC(\tilde{B}^{2,s})$ by setting the rank of
a vertex to the rank of its image in $BC(s\Lambda_2)$ under the appropriate composition of these inclusions. For example, every vertex labelled by $\emptyset$ always has rank $s$ in $BC(\tilde{B}^{2,s})$.

**Example 4.2.** Figure 2 depicts $BC(\tilde{B}^{2,s})$, which is the union of $BC(0)$, $BC(\Lambda_2)$, and $BC(\Lambda_2)$.

5. **Affine Kashiwara Operators**

In this section we describe how to “overlay” a set of arrows, called $F_0$ arrows, on $BC(\tilde{B}^{2,s})$ in a way that specifies $\tilde{e}_0$ and $\tilde{f}_0$. Let $v \in BC(\tilde{B}^{2,s})$ be a vertex of global rank $j$ in $BC(k\Lambda_2)$ associated with the partition $(\lambda_1, \lambda_2)$. Place an $F_0$ arrow from $v$ to the following vertices, *if they exist*:

- the vertex of global rank $j - 1$ in $BC((k - 1)\Lambda_2)$ with shape $(\lambda_1 - 1, \lambda_2)$;
- the vertex of global rank $j - 1$ in $BC(k\Lambda_2)$ with shape $(\lambda_1, \lambda_2 - 1)$;
- the vertex of global rank $j - 1$ in $BC((k + 1)\Lambda_2)$ with shape $(\lambda_1 + 1, \lambda_2)$;
- the vertex of global rank $j - 1$ in $BC(k\Lambda_2)$ with shape $(\lambda_1, \lambda_2 + 1)$.

The directed graph that consists of the vertices of $BC(\tilde{B}^{2,s})$ and the $F_0$ arrows is isomorphic to $BC(\tilde{B}^{2,s})$. Via this graph isomorphism, which we denote $\sigma$, we may define $\tilde{f}_0$ for $\tilde{B}^{2,s}$. Let $b \in B(v)$ be a tableau in $\tilde{B}^{2,s}$. Note that $B(v)$ is isomorphic to $B(\sigma(v))$ as a $D_{n-1}$-crystal; let $b' \in B(\sigma(v))$ denote the tableau corresponding to $b$ under this isomorphism. We may have $\tilde{f}_1(b') = c' \in B(w)$ for some branching vertex $w$, or we may have $\tilde{f}_1(b') = 0$. In the former case, we say that $\tilde{f}_0(b) = c$, where $c$ corresponds to $c'$ under the isomorphism between $B(w)$ and $B(\sigma(w))$; in the latter case, $\tilde{f}_0(b) = 0$. By the definition of crystals, this also determines $e_0$.

**Example 5.1.** In Figure 3 we have $BC(\tilde{B}^{2,2})$ with the original arrows removed and the $F_0$ arrows superimposed.

Of course, we could have chosen to define the graph isomorphism in terms of the branching vertices, and let the definition of the $F_0$ arrows follow. In fact, we did exactly that in [16], where $\sigma$ is used to denote the automorphism of the vertices of $\tilde{B}^{2,s}$ corresponding to interchanging nodes 0 and 1 of the Dynkin diagram.

We now present some examples taken from $\tilde{B}^{2,2}$. 
Example 5.2. Let $b = \frac{1}{2} \frac{2}{1}$, so $b \in B(v)$ where $v$ is the branching vertex of shape $(1, 0)$ with global rank 3 in $BC(\Lambda_2)$. We see from Figures 2 and 3 that $\sigma(v)$ is the vertex with the same shape with rank 1 in $BC(2\Lambda_2)$. The corresponding tableau in $\sigma(v)$ is $b' = \frac{1}{2} \frac{2}{2}$, and $c' = \tilde{f}_1(b') = \frac{1}{2} \frac{1}{1}$. The branching vertex containing $c'$ is the vertex of shape $(1, 1)$ with rank 2 in $BC(2\Lambda_2)$, which is fixed under $\sigma$, so $c = c'$. Therefore, $\tilde{f}_0(b) = \frac{1}{2} \frac{2}{1}$.

Example 5.3. Let $b = \frac{3}{1} \frac{3}{1}$, so $b \in B(v)$ where $v$ is the branching vertex of shape $(2, 0)$ with rank 4 in $BC(2\Lambda_2)$. We see from Figures 2 and 3 that $\sigma(v)$ is the vertex of the same shape with rank 0 in $BC(2\Lambda_2)$. The corresponding tableau in $\sigma(v)$ is $b' = \frac{3}{3} \frac{1}{1}$, and $c' = \tilde{f}_1(b') = \frac{1}{2} \frac{3}{3}$. The branching vertex containing $c'$ is the vertex of shape $(2, 1)$ with rank 1 in $BC(2\Lambda_2)$. Its image under $\sigma$ is the vertex of the same shape with rank 3 in $BC(2\Lambda_2)$, so $\tilde{f}_0(b) = c = \frac{3}{1}$.

Example 5.4. Let $b_{k\Lambda_2}$ denote the classical highest weight tableau of $B(k\Lambda_2) \subset \tilde{B}^{2,s}$. Then $\tilde{f}_0(b_{k\Lambda_2}) = b_{(k+1)\Lambda_2}$ for $0 \leq k \leq s - 1$.

6. Perfectness

Several conditions must be satisfied for a crystal $B$ to be a perfect crystal of level $\ell$, but the most significant challenge is in the condition that the maps $\epsilon$ and $\varphi$ from $B_{\min}$ to $(P_{\mathrm{c}1}^+)_{\ell}$ are bijective. We briefly recall the definition of these sets and maps below; for more detail see [16] or [4].

For a crystal basis element $b \in B$, define the weights

$$\epsilon(b) = \sum_{i \in I} \epsilon_i(b) \Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i,$$

where

$$\epsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq \emptyset\},$$

$$\varphi_i(b) = \max\{n \geq 0 \mid \overline{f}_i^n(b) \neq \emptyset\}.$$

The level of a weight $\Lambda$ is $\langle c, \Lambda \rangle$, where $c = h_0 + h_1 + h_{n-1} + h_n + \sum_{n=2}^{n-2} 2h_i$ is the canonical central element of the algebra of type $D_n^{(1)}$. The set of minimal vertices, denoted
$B_{\text{min}},$ is the set of crystal elements $b$ for which $\langle c, \epsilon(b) \rangle$ is minimal. Finally, define $(P_{\ell}^{+})_{\ell}$ to be the set of level $\ell$ weights $\Lambda$ with no $\delta$ component for which $\langle h_{i}, \Lambda \rangle \geq 0$ for all $i \in I.$

We now outline the construction of a $2 \times s$ tableau $T$ such that given any level $s$ weight $\Lambda$, we have $\epsilon(T) = \varphi(T) = \Lambda.$ It was shown in [16] that these are precisely the tableaux in $B_{\text{min}}.$

For $i = 0, \ldots, n,$ let $k_{i} = (h_{i}, \lambda).$ We first construct a tableau $T_{\lambda'}$ corresponding to the weight $\lambda' = \sum_{i=2}^{n} k_{i} \Lambda_{i}.$ We begin with the middle $k_{n-1} + k_{n}$ columns of $T_{\lambda'}.$ If $k_{n-1} + k_{n}$ is even and $k_{n} \geq k_{n-1},$ these columns of $T_{\lambda'}$ are

$$
\begin{array}{cccccccc}
& n-2 & n-1 & n & n-1 & n-1 & n-2 & n-2 \\
n-1 & n & n-1 & n & n-1 & n-2 & n-2 & k_{n-1} \\
\end{array}
$$

If $k_{n-1} + k_{n}$ is odd and $k_{n} \geq k_{n-1},$ we have

$$
\begin{array}{cccccccc}
& n-2 & n-1 & n-1 & n & n & n-1 & n-1 \\
n-1 & n & n & n & n-1 & n & n-1 & k_{n-1} \\
\end{array}
$$

In either case, if $k_{n} < k_{n-1},$ interchange $n$ and $\overline{n},$ and $k_{n}$ and $k_{n-1}$ in the above configurations.

Next we put a configuration of the form

$$
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 2 & 3 & 3 \\
k_{2} & k_{2} & k_{3} & k_{n-2} \\
k_{n-2} & k_{n-3} & k_{n-2} & k_{n-3} \\
\end{array}
$$

on the left, and a configuration of the form

$$
\begin{array}{cccc}
\overline{n-2} & \overline{n-2} & \overline{n-3} & \overline{n-3} \\
\overline{n-3} & \overline{n-3} & \overline{n-4} & \overline{n-4} \\
k_{2} & k_{2} & k_{2} & k_{2} \\
k_{n-2} & k_{n-3} & k_{n-2} & k_{n-3} \\
\end{array}
$$

on the right.

We now use Lecouvey $D$ equivalence as in [9] or type $D$ sliding as in [16] to change this tableau into a skew tableau of shape $(s - k_{0}, s - k_{0} - k_{1})/(k_{1}).$ If $k_{1} > s - k_{0} - k_{1}$ (i.e., $k_{1} - (s - k_{0} - k_{1}) = 2k_{1} + k_{0} - s > 0$), place a configuration of the following form in the empty spaces in the middle of this skew tableau:

$$
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 \\
k_{2k_{1}+k_{0}-s} & k_{2k_{1}+k_{0}-s} & k_{2k_{1}+k_{0}-s} & k_{2k_{1}+k_{0}-s} \\
\end{array}
$$

where the number of $1$'s equals the number of $\bar{1}$'s and the number of $2$'s equals the number of $\bar{2}$'s.

If $s - k_{0}$ is odd, the middle column of the tableau constructed so far is $\frac{a}{a}$ for $1 \leq a \leq n$ or $\frac{n}{n}.$ Whatever it is, simply insert $k_{0}$ of this column into the tableau next to the middle column (cf. Section 3). If $s - k_{0}$ is even, the middle two columns are of the form $\frac{a}{b} \frac{b}{a}$ for some letters $a$ and $b$ (it is possible that $b$ is barred, in which case $b$ is the corresponding unbarred letter). In this case, simply add $k_{0}$ columns of the form $\frac{a}{a}$ between these columns.
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We provide a few examples with details of the construction of the tableaux, followed by examples with less detail in Table 1.

**Example 6.1.** Let \( n = 6 \), and consider the weight \( \Lambda_0 + 2\Lambda_1 + \Lambda_2 + 2\Lambda_4 + \Lambda_5 \). This weight has level \( 1 + 2 + 2 \cdot 1 + 2 + 1 = 8 \), so our procedure will result in a \( 2 \times 8 \) tableau, i.e., a minimal tableau in \( \tilde{B}^{2,8} \). Since \( k_4 + k_5 \) is odd and \( k_5 < k_4 \), we begin with
\[
\begin{array}{cccccccc}
3 & 5 & 4 & 2 \\
4 & 5 & 3 & 1
\end{array}
\]

To incorporate \( \Lambda_2 \), we amend this tableau to get
\[
\begin{array}{cccccccc}
1 & 3 & 5 & 4 & 2 \\
2 & 4 & 5 & 3 & 1
\end{array}
\]

Applying the type \( D \) sliding algorithm twice and inserting 1's and 11's gives us
\[
\begin{array}{cccccccc}
1 & 1 & 1 & 4 & 5 & 4 & 2 \\
2 & 4 & 5 & 4 & 1 & 1 & 1
\end{array}
\]

Finally, we insert one column in the middle, which yields
\[
\begin{array}{cccccccc}
1 & 1 & 1 & 4 & 4 & 5 & 4 & 2 \\
2 & 4 & 5 & 4 & 4 & 1 & 1 & 1
\end{array}
\]

**Example 6.2.** Let \( n = 6 \), and consider the weight \( \Lambda_0 + 2\Lambda_1 + \Lambda_2 + 2\Lambda_6 \). This weight has level \( 1 + 2 + 2 \cdot 1 + 2 = 7 \), so we will have a \( 2 \times 7 \) tableau at the end; i.e., a minimal tableau in \( \tilde{B}^{2,7} \). We begin with the tableau corresponding to \( 2\Lambda_6 \), which is
\[
\begin{array}{cccccccc}
5 & 5 & 6 & 6 & 5 \\
6 & 6 & 5 & 2
\end{array}
\]

and expand it thus an account of \( \Lambda_3 \):
\[
\begin{array}{cccccccc}
2 & 5 & 6 & 3 \\
3 & 6 & 5 & 2
\end{array}
\]

Type \( D \) sliding turns it into
\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 6 & 6 & 3 \\
3 & 6 & 6 & 2 & 1 & 1
\end{array}
\]

and inserting one column gives us
\[
\begin{array}{cccccccc}
1 & 1 & 2 & 2 & 6 & 6 & 3 \\
3 & 6 & 6 & 2 & 2 & 1 & 1
\end{array}
\]

**Example 6.3.** Table 1 shows several weights and the corresponding tableaux. The first 11 entries are all the level 2 weights for \( n = 4 \).

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**REFERENCES**


TYPE $D_{n}^{(1)}$ KIRILLOV-RESHETIKHIN CRYSTALS

<table>
<thead>
<tr>
<th>$n$</th>
<th>weight</th>
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<td>2</td>
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<td>$\Lambda_{2}$</td>
<td>2</td>
<td>4 4 4 4</td>
</tr>
<tr>
<td>7</td>
<td>$2\Lambda_{3} + 3\Lambda_{4}$</td>
<td>10</td>
<td>2 2 3 3 3 4 4 4 4 3 3</td>
</tr>
<tr>
<td>7</td>
<td>$2\Lambda_{0} + \Lambda_{2} + 2\Lambda_{4} + \Lambda_{5}$</td>
<td>10</td>
<td>1 3 3 4 4 4 5 4 4 4 2</td>
</tr>
<tr>
<td>7</td>
<td>$\Lambda_{4} + 2\Lambda_{6} + 7\Lambda_{7}$</td>
<td>11</td>
<td>3 5 5 6 6 7 7 7 8 8 8 8 8 8 8 8 8</td>
</tr>
</tbody>
</table>

TABLE 1. Weights and minimal tableaux


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