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The role of virtual turning points in the deformation of higher order linear equations

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In this talk we discuss exact WKB analysis for Noumi-Yamada systems of type $A_2^{(1)}$ and $A_4^{(1)}$; the first equation, denoted by $(NY)_2$, is equivalent to traditional fourth Painlevé equation, and the second, denoted by $(NY)_4$, is a fourth order nonlinear ODE, which are given as follows ($m=1,2$; cf. [T]):

$$(NY)_{2m}: \frac{du_j}{dt} = \eta[u_j(u_{j+1}-u_{j+2}+\cdots-u_{j+2m})+\alpha_j] \quad (j=0,1,\cdots,2m), \tag{1}$$

$$\alpha_0+\cdots+\alpha_{2m}=\eta^{-1}, \quad u_0+\cdots+u_{2m}=t. \tag{2}$$

These equations are derived from the compatibility condition of a pair of linear system, which is called Lax pair, in just the same way as other higher order Painlevé equations discussed in [N], [KKNT] etc.. In the case of Noumi-Yamada systems, however, the size of the Lax pair is greater than two, so that we must consider some virtual turning points and new Stokes curves. The explicit form of $(L)_{2m}$, one of the Lax pair for $(NY)_{2m}$, is given as follows:

$$(L)_{2m}: \frac{\partial}{\partial x}\psi = \eta A\psi, \tag{3}$$

$$A = -\frac{1}{x}\begin{pmatrix} \epsilon_1 & u_1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \epsilon_{2m-1} & u_{2m-1} & 1 \\ x & & & \epsilon_{2m} & u_{2m} \\ xu_0 & x & & & \epsilon_{2m+1} \end{pmatrix}, \tag{4}$$

where ϵ_j $(j=1,\cdots,2m+1)$ are parameters related to α_j $(j=0,1,\cdots,2m)$.

We substitute the 0-parameter solution of $(NY)_{2m}$ into $(L)_{2m}$ and draw pictures of Stokes curves of $(L)_{2m}$ by using a computer.

If the parameter t is on a Stokes curve γ of $(NY)_{2m}$ and if it is sufficiently close to a turning point τ from which the Stokes curve emanates, then a double turning point and a simple turning point of $(L)_{2m}$ are connected by a Stokes curve of $(L)_{2m}$ as the general theory in [T] asserts. However, at a point on the Stokes curve γ far away from the turning point τ, we observe that no pair of ordinary turning points are connected. In fact, in this case at some point on γ another simple turning point comes across the Stokes curve of $(L)_{2m}$ connecting two turning points, and consequently on a portion of γ far away from τ an ordinary turning point and a virtual turning point are connected instead ([AKSST]). Moreover in the case of $(L)_4$, we also observe that two virtual turning points are connected by a new Stokes curve on some portion of γ sufficiently far away from τ.

These phenomena show that a "virtual" turning point is really a "real" object and strongly support the assertion that there is no distinction between virtual and ordinary turning points theoretically.

We also report a phenomenon which should be regarded as an extension of Nishikawa phenomena. When we study the change of the Stokes geometry for $(L)_4$ near a crossing point of Stokes curves of $(NY)_4$, we can observe that the Stokes geometry for $(L)_4$ becomes degenerate also at some point outside the (ordinary) Stokes curves of $(NY)_4$; many virtual turning points are concerned in this degeneracy. This fact shows that there exist new Stokes curves in the Stokes geometry for $(NY)_4$ as well as other higher order Painlevé equations discussed in [N], [KKNT] etc..

References

[AKSST] T. Aoki, T. Kawai, S. Sasaki, A. Shudo and Y. Takei, Virtual turning points and bifurcation of Stokes curves for higher ordinary differential equations, to be published.

[KKNT] T. Kawai, T. Koike, Y. Nishikawa and Y. Takei, On the Stokes geometry of higher order Painlevé equations, RIMS Preprint No.1443, 2004.

[N] 西川享宏, $P_{\mathrm{II}} - P_{\mathrm{IV}}$ hierarchy の WKB 解析, 数理解析研究所講究録 1316 「高階 Painlevé 方程式の Stokes 図形の西川現象」, 京都大学数理解析研究所, 2003, 19-103.

[T] Y. Takei, Toward the Exact WKB Analysis for Higher-Order Painlevé Equations – The Case of Noumi-Yamada Systems –, Publ. RIMS, Kyoto Univ., 40(2004), 709-730.

For the details and their further development, the reader is referred to my articles

> The role of virtual turning points in the deformation of higher order linear differntial equations

and

> The role of virtual turning points in the deformation of higher order linear differntial equations, II – On new Stokes curves of Noumi-Yamada systems –

(both in Japanese). These articles will soon appear in RIMS Kôkyûroku "Deformation of linear differential equations and virtual turning points".