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Uniqueness and non-uniqueness of generalized functions with microlocal regularity

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Abstract

In this note, we study generalized unique continuity of subclasses of hyperfunctions, which was originally posed by M.Sato.

1 Introduction

In this note, discussion is given on a problem of the uniqueness for generalized functions which are microlocally regular.

Definition 1.1. Let N be an open neighborhood of $0 \in \mathbb{R}^n$, f be a function defined on N and $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. A function f is called to contain x_n as *a real analytic parameter at* $x = 0$ if and only if

$$(0; (0, \xi_n)) \notin WF_A(f), \tag{1}$$

where $WF_A(f)$ is the analytic wave front set of f, $(0, \xi_n) \in \mathbb{R}^n$, $\xi_n \in \mathbb{R}$.

The following problem is discussed in this note.

Problem 1.1. *Let f be a generalized function defined on a neighborhood N of $0 \in \mathbb{R}^n$, which contains x_n as an analytic parameter at $x = 0$. In this case, do the data*

$$\partial_{x_n}^k f|_S = 0 \quad \text{for all } k \in \mathbb{N} \cup \{0\}, \tag{2}$$

where $S = N \cap \{x_n = 0\}$, imply that $f = 0$ in some neighborhood of $x = 0$?

M.Sato proved that the answer to this problem is negative for hyperfunctions and conjectured that the data $J(D_n)f(x)|_{x_n=0} = 0$ for any local operator in x_n with constant coefficients would be sufficient for $f \equiv 0$ near the origin, which was proved by A.Kaneko in 1978. It is also known that the answer to Problem 1.1 is affirmative for non-quasi-analytic ultradistributions and that it is not for quasi-analytic ultradistributions. In this note, we give a reasonable sufficient condition for quasi-analytic ultradistributions to vanish near the origin, which is a counterpart of A.Kaneko's theorem for hyperfunctions.

2 Ultradistributions

In this section, we review the definition of ultradistributions. Let $M_p, p = 0, 1, \cdots$, be a sequence of positive numbers.

For non-quasi-analytic classes, the following conditions are imposed on M_p.

$(M.0)$ (normalization)

$$M_0 = M_1 = 1.$$

$(M.1)$ (logarithmic convexity)

$$M_p^2 \le M_{p-1}M_{p+1}, \quad p = 1, 2, \cdots .$$

$(M.2)$ (stability under ultradifferential operators)

$$\exists G, \quad \exists H \text{ such that } M_p \le GH^p \min_{0 \le q \le p} M_q M_{p-q}, \quad p = 0, 1, \cdots .$$

$(M.3)$ (strong non-quasi-analyticity)

$$\exists G \text{ such that } \sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \le Gp \frac{M_p}{M_{p+1}}, \quad p = 1, 2, \cdots .$$

$(M.2)$ and $(M.3)$ are often replaced by the following weaker conditions, respectively;

$(M.2)'$ (stability under differential operators)

$$\exists G, \quad \exists H \text{ such that } M_{p+1} \le GH^p M_p, \quad p = 0, 1, \cdots .$$

$(M.3)'$ (non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

We note that if $\sigma > 1$ then the Gevrey sequences

$$M_p = (p!)^\sigma \tag{3}$$

satisfy all the above conditions.

In this note, quasi-analytic ultradistributions are studied. The following conditions, (QA) and (NA), are imposed on M_p, instead of $(M.3)$ or $(M.3)'$;

(QA) (quasi-analyticity)

$$M_p \ge p!, \; p = 0, 1, \cdots, \quad \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} = \infty.$$

Let M_p be a sequence of positive numbers satisfying (QA). If

$$\liminf_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} > 0 \tag{4}$$

then $\mathcal{E}^{\{M_p\}}$ is the class of analytic functions. The following condition (NA) is imposed so that M_p does not define the analytic class.

(NA) (non-analyticity)

$$\lim_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} = 0.$$

Definition 2.1. $f \in \mathcal{E}(\Omega) = C^\infty(\Omega)$ is called *an ultradifferentiable function* of class (M_p) (resp. $\{M_p\}$) if and only if for compact subset $K \subset \Omega$ and for any $h > 0$ there exists some C (resp. there exist constants h and C) such that

$$\sup_{x\in K} |D^\alpha \varphi(x)| \leq Ch^{|\alpha|}M_{|\alpha|} \quad \text{for all } \alpha \tag{5}$$

holds.

Denote the set of the ultradifferentiable functions of class $*$ on Ω by $\mathcal{E}^*(\Omega)$ and denote by $\mathcal{D}^*(\Omega)$ the set of all functions in $\mathcal{E}^*(\Omega)$ with support compact in Ω, where $* = (M_p)$ or $\{M_p\}$. For a compact subset $K \subset \Omega$ let

$$\mathcal{D}_K^* = \{\varphi \in \mathcal{D}^*(\mathbb{R}^n) \ ; \ \mathrm{supp} f \subset K\}, \tag{6}$$

and define

$$\mathcal{D}_K^{\{M_p\},h} = \{\varphi \in \mathcal{D}_K^{\{M_p\}} \ ; \ \exists C, \ \sup_{x\in K} |D^\alpha \varphi(x)| \leq Ch^{|\alpha|}M_{|\alpha|}, \text{ for all } \alpha\}. \tag{7}$$

These spaces are endowed with natural structure of locally convex spaces. For a positive sequence M_p satisfying $(M.0)$, $(M.1)$ and (NA), define its *associated functions* by

$$\widetilde{M}(t) := \sup_k \frac{t^k}{M_k},$$

$$\overline{M}(t) := \sup_k \log \frac{t^k k!}{M_k},$$

for $t > 0$.

Definition 2.2. Let $K \subset \mathbb{R}^n$ be a compact set and M_p satisfy $(M.1)$. ${\mathcal{E}_K^*}'$ is defined as the strong dual of $\mathcal{E}^*[K]$, where $* = (M_p)$ or $\{M_p\}$, and is called the set of *ultradistributions of the class $*$ supported by K.*

For more details on non-quasi-analytic ultradifferentiable functions and non-quasi-analytic ultradistributions, confer [7, 8]. Let us define the sheaf of quasi-analytic ultradistributions.

Definition 2.3. Let a sequence M_p of positive numbers satisfy $(M.0)$, $(M.1)$, $(M.2)'$, and

$$\limsup_{p\to\infty} \sqrt[p]{\frac{p!}{M_p}} < \infty. \tag{8}$$

We define a presheaf F^* on $\mathbb{R}^n$ by

$$F^*(\Omega) := {\mathcal{E}_{\mathbb{R}^n}^*}' / {\mathcal{E}_{\mathbb{R}^n\setminus\Omega}^*}', \tag{9}$$

where Ω is any open set in $\mathbb{R}^n$ and $* = (M_p)$ or $\{M_p\}$. We denote the corresponding sheaf by $\mathcal{F}^*$. If M_p satisfies $(M.3)'$, $\mathcal{F}^* = \mathcal{D}^{*\prime}$ If M_p satisfies (QA) and (NA), then we call $\mathcal{F}^*$ *the sheaf of quasi-analytic ultradistributions of class $*$.*

In [4], L.Hörmander proved that it is $\mathcal{F}^*$ that is the natural sheaf satisfying $\mathcal{F}_K^* = \mathcal{E}^{*\prime}$ for any compact set K. He also proved that the sheaf $\mathcal{F}^*$ is flabby if and only if the sequence M_p satisfies the condition (QA). For more details, confer [4].

Definition 2.4. Let M_p satisfy $(M.2)$. The differential operator

$$P(D) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$$

of infinite order is called an *ultradifferential operator of class* (M_p) *(resp.* $\{M_p\}$*)* if and only if the coefficients satisfy the estimate

$$|a_\alpha| \leq CL^{|\alpha|}/M_{|\alpha|}$$

for some L and C (resp. for any $L > 0$ and some C).

3 Known results on Problem 1.1

In this section, known results on Problem 1.1 and their applications are introduced. As was mentioned in Introduction, M.Sato proved that for $f \in \mathcal{B}$, the answer to Problem 1.1 is negative (cf [6]). In 1978, A.Kaneko [5] gave a sufficient condition for local uniqueness of hyperfunctions with analytic parameters.

Theorem 3.1. *Let $f(x)$ be a hyperfunction defined on $U' \times \{|x_n| < \delta\}$, where $U' \subset \mathbb{R}^{n-1}$ is open. Assume that $f(x)$ contains x_n as a real analytic parameter in $U' \times \{|x_n| < \delta\}$ and that for any local operator $J(D_n)$ with constant coefficients in the normal derivative D_n in x_n, there holds*

$$J(D_n)f(x)|_{x_n=0} = 0, \tag{10}$$

where the symbol of $J(D_n)$ is given by

$$J(\xi_n) = \sum_{k=0}^{\infty} b_k \xi_n^k, \quad \text{with} \quad \lim_{k\to\infty} \sqrt[k]{|b_k| k!} = 0.$$

Then $f(x) \equiv 0$ in a neighborhood of $U' \times \{0\}$.

In 1992, J.Boman [1] proved that Problem 1.1 is affirmatively solved for distributions.

Theorem 3.2. *Let f be a distribution defined on a neighborhood N of $x = 0$. Assume*

$$(0; (0, \xi_n)) \notin WF_A(f),$$

and that the restrictions to $\{x_n = 0\} \cap N$ of f and all its derivatives in x_n vanish;

$$\partial_{x_n}^k f|_{\{x_n=0\}\cap N} = 0 \quad \text{for all } k \in \mathbb{N} \cup \{0\}.$$

Then $f = 0$ in some neighborhood of $x = 0$.

In 1993, S.Tanabe and the author [10] extended this theorem and proved that the answer to Problem 1.1 is also affirmative for non-quasi-analytic ultradistributions.

Theorem 3.3. *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$ and $(M.3)'$, f be an ultradistribution of class $*$ defined on a neighborhood N of $x = 0$, where $* = (M_p)$ or $\{M_p\}$. Assume that*

$$(0; (0, \xi_n)) \notin WF_A(f),$$

and that the restrictions to $\{x_n = 0\} \cap N$ of f and all its derivatives in x_n vanish;

$$\partial_{x_n}^k f|_{\{x_n=0\}\cap N} = 0 \quad \text{for all } k \in \mathbb{N} \cup \{0\}.$$

Then $f = 0$ in some neighborhood of $x = 0$.

In the same paper [10], they also proved unique solvability in an overdetermined Cauchy problem for the wave equation as an application of Theorem 3.3. Consider the following overdetermined Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^\alpha u|_{x=x_0} = u_\alpha(t) \quad \text{for any } \alpha. \end{cases} \tag{11}$$

Local uniqueness of this problem is considered. Since all conormals to the line $\{(t, x_0) \mid t \in \mathbb{R}\}$ in $\mathbb{R}_t \times \mathbb{R}_x^n$ are non-characteristic, u contains x as real analytic parameters on $x = x_0$. Therefore application of Theorem 3.3 proves local uniqueness in (11).

For hyperfunctions, however, uniqueness in (11) does not hold, which was proved by A.Kaneko in 1993. Let us introduce his counterexample. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ u|_{x_1=0} = \varphi(x', t), \quad \frac{\partial u}{\partial x_1}|_{x_1=0} = \psi(x', t) = 0, \end{cases} \tag{12}$$

where $x = (x_1, x') \in \mathbb{R}^n$ and φ is M.Sato's counterexample with x' as analytic parameters. It turns out that the Cauchy problem (12) allows a local hyperfunction solution satisfying

$$\partial_x^\alpha u|_{x=0} = 0,$$

for any α.

In 1995, J.Boman [2] proved that Problem 1.1 is affirmatively solved for non-quasi-analytic ultradistributions, even if the regularity condition in the parameter x_n is replaced by quasi-analytic one.

In 2000, he also proved, by modifying M.Sato's counterexample, that the answer to Problem 1.1 is negative for quasi-analytic ultradistributions (cf. [3]).

In 2002, the author modified A.Kaneko's counterexample by applying J.Boman's modification of Sato's counterexample in order to prove that uniqueness in (11) does not hold in the class of quasi-analytic ultradistributions.

Theorem 3.4. (cf. [9]) *Assume that the sequence M_p satisfies $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA). There exists such a quasi-analytic ultradistribution $u(t,x)$ of class (M_p) or $\{M_p\}$ satisfying*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^\alpha u|_{x=x_0} = 0 \quad \textit{for any } \alpha. \end{cases} \tag{13}$$

that $u(t,x) \not\equiv 0$ in any neighborhood of $x = x_0$.

In view of these known results, it is left open to give sufficient data, as a counterpart of (10), for local uniqueness of quasi-analytic ultradistributions with an analytic parameter.

Problem 3.1. *Let $f(x)$ be a quasi-analytic ultradistribution defined on a neighborhood N of $0 \in \mathbb{R}^n$, which contains x_n as an analytic parameter at $x = 0$. Give a sufficient data with respect to the derivatives in x_n for uniqueness of f in a neighborhood of $x = 0$.*

Of course, we obtain uniqueness of a quasi-analytic ultradistribution in a neighborhood of $x = 0$ f we assume the same assumption as Theorem 3.1, however, the condition (10) may be too sufficient for quasi-analytic ultradistributions. By Problem 3.1, we mean that we would like to give a 'reasonable' sufficient condition for quasi-analytic ultradistributions as a counterpart of (10).

4 Main theorem

It is our main purpose in this note to solve Problem 3.1.

Theorem 4.1. *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA), f be an ultradistribution of class $*$ defined on a neighborhood N of $x = 0$, where $* = (M_p)$ or $\{M_p\}$. Assume that $f(x)$ contains x_n as a real analytic parameter at $x = 0$ and that for any ultradifferential operator $P(D_n)$ of the class $*$ with constant coefficients in the normal derivative D_n in x_n, there holds*

$$P(D_n)f(x)|_S = 0, \tag{14}$$

where $S = N \cap \{x_n = 0\}$. Then $f(x) \equiv 0$ in some neighborhood of $x = 0$.

Let us introduce sketch of the proof of Theorem 4.1. The idea to prove the main theorem is modification of A.Kaneko's proof of Theorem 3.1.

Definition 4.1. Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)$ (QA) and (NA), $* = (M_p)$ or $\{M_p\}$. For a compact set K, define $\mathcal{E}^{*P}(K)$ as the space $\mathcal{E}^*(K)$ with the topology defined by the semi-norms

$$\|f\|_P := \sup_{x \in K} |P(D)f(x)|, \tag{15}$$

where $P(D)$ runs over the ultradifferential operators of class $*$ with constant coefficients.

Proposition 4.1. *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA), $* = (M_p)$ or $\{M_p\}$. For a compact set K, $\mathcal{E}^{*P}(K) \to \mathcal{E}^*(K)$ is sequentially continuous and $\mathcal{E}^{*P}(K)$ is sequentially complete.*

To prove this proposition, the structure theorem for non-analytic ultradistributions (including all the quasi-analytic ones) play an important role.

Theorem 4.2. *Let N_p satisfy $(M.0)$, $(M.1)$, $(M.2)$ and (NA). Assume that $f \in \mathcal{E}^{*\prime}_K$, where $K \subset \mathbb{R}^n$ is a compact set and $* = (M_p)$ or $\{M_p\}$. Then for any class $\dagger$ satisfying $* < \dagger$ there exist $g \in \mathcal{D}^\dagger(\mathbb{R}^n)$ and an ultradifferential operator $P(D)$ of class $*$ such that $f = P(D)g$, where $* < \dagger \Leftrightarrow \mathcal{E}^* \subsetneq \mathcal{E}^\dagger$.*

By virtue of Proposition 4.1, we obtain the following lemmas.

Lemma 4.1. *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA), $* = (M_p)$ or $\{M_p\}$ and $f \in \mathcal{F}^*$ in some neighborhood of $x = 0$. Assume that for any ultradifferential operator $P(D)$ with constant coefficients in the class $*$, $P(D)f$ is continuous at $x = 0$. Then $f \in \mathcal{E}^*(\{0\})$.*

Lemma 4.2. *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA), $* = (M_p)$ or $\{M_p\}$ and $f_k(x)$ be a sequence in $\mathcal{E}^*(\{0\})$. Assume that for any ultradifferential operator $P(D) = P_1(D_1)\cdots P_n(D_n)$ of the product type with constant coefficients in the class $*$, the numerical sequence $P_1(D_1)\cdots P_n(D_n)f_k(0)$ converge to a finite limit as $k \to \infty$. Then the sequence f_k converges in $\mathcal{E}^*(\{0\})$.*

Lemma 4.2 is a key to prove our main theorem, to prove which we modified A.Kaneko's idea in [5]. We also apply the following proposition.

Proposition 4.2. (cf. [4]) *Let a sequence M_p of positive numbers satisfy $(M.1)$, $(M.2)'$, and*

$$\limsup_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} < \infty,$$

Ω be an open set in $\mathbb{R}^n$. For any $f \in \mathcal{F}^*(\Omega)$, where $* = (M_p)$ or $\{M_p\}$, there exist $g_j \in \mathcal{E}^{*\prime}(\Omega)$ such that the supports of g_j's are locally finite and for any $K \subset\subset \Omega$ we have $f = \sum g_j$, the sum taken over the terms with support intersecting K. Conversely, every such sum defines an element in $\mathcal{F}^*(\Omega)$.

Sketch of the proof of Theorem 4.1. The first task for the proof is to prove that the function $f(x)$ is quasi-analytic (of class $*$) in some neighborhood of $x = 0$. By Proposition 4.2, there exists an ultradistribution $g(x)$ of class $*$ such that

(i) $f = g$ in some neighborhood of $x = 0$.

(ii) supp g is contained in the closure of V, where V is an open bounded neighborhood of the origin.

In order to prove that $f(x)$ is quasi-analytic on a neighborhood of the origin, it is sufficient to show that $g(x)$ is quasi-analytic on a neighborhood of the origin. Put

$$g(x,\varepsilon) := \int_{\mathbb{R}^{n-1}} g(y', x_n)E(x'-y',\varepsilon)dy', \tag{16}$$

where

$$E(x',\varepsilon) := \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'}e^{-\varepsilon|\xi'|}d\xi' = \frac{\Gamma(n/2)\varepsilon}{\pi^{n/2}(|x'|^2+\varepsilon^2)^{n/2}}.$$

It is proved that

$$g(x,\varepsilon) \to g(x) \quad \text{in} \quad \mathcal{E}'_{\overline{V'}},$$

where V' is some subset of V. It is also proved that for any ultradifferential operator $P(D) := P_1(D_1)\cdots P_n(D_n)$ of product type in class $*$, $P(D)g(x,\varepsilon)$ converges to some ultradifferential function of class $*$ on a neighborhood of the origin in $\mathbb{R}^n$ as $\varepsilon \to 0$. By virtue of Lemma 4.2, we can prove that $g(x,\varepsilon)$ converges to some ultradifferential function of class $*$ on a neighborhood of the origin in $\mathbb{R}^n$ as $\varepsilon \to 0$. This means that f is of class $*$ in some neighborhood of $x = 0$. By the assumption of the theorem and quasi-analyticity of f, there holds

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x)\Big|_{x=0} = 0, \quad \text{for any } \alpha. \tag{17}$$

Therefore, the theorem is proved by unique continuity property of the quasi-analytic functions. □

5 Conclusion and problems

In the previous section, we have succeeded in giving a reasonable sufficient condition for local uniqueness of quasi-analytic ultradistributions. There are several problems left unsolved for further development. Let us pose the following problems

Problem 5.1. (i) *Are the conditions given in Theorems 3.1 and 4.1 possibly the best ones?*

(ii) *Does Theorem 4.1 hold if the regularity condition on the parameter is replaced by the quasi-analytic one?*

The author thinks that the answer to any of these questions is affirmative. For solution of the first problem, it is required to construct a quasi-analytic ultradistribution $f(x) \not\equiv 0$ of class $*$ defined on a neighborhood N of $x = 0$ contains x_n as a real analytic parameter at $x = 0$ satisfying

$$P(D_n)f(x)|_S = 0,$$

for any ultradifferential operator $P(D_n)$ of the class † with $\mathcal{E}^* \subsetneq \mathcal{E}^\dagger$. In order to solve the second one, preparation on microfunction theory of quasi-analytic type are necessary. Anyway, for the time being, both of these problems are left open.

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