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WKB ANALYSIS TO NORMAL FORM THEORY OF VECTOR FIELDS

広島大学・理学研究科　　吉野　正史 (Masafumi Yoshino) [1]
Graduate School of Sciences
Hiroshima University

1. Introduction

In this note we shall study the relations between the exact asymptotic analysis of a so-called homology equation and the normal form theory of a singular vector field. A homology equation is a system of partial differential equations which appear in linearizing a singular vector field by the change of independent variables. We shall introduce a WKB solution of a homology equation which is a natural extension of the one introduced by Aoki-Kawai- Takei for the Painlevé equation. We then give a new unexpected connection between Poincaré series and the WKB solution via resummation procedure.

2. Homology equation

Let $x=(x_1,\dots,x_n)\in\mathbb{C}^n$, $n\geq 2$ be the variable in $\mathbb{C}^n$. We consider a singular vector field near the origin of $\mathbb{C}^n$

$$\mathcal{X}=\sum_{j=1}^{n}a_j(x)\frac{\partial}{\partial x_j},\quad a_j(0)=0,\quad j=1,\dots,n,$$

where $a_j(x)$ $(j=1,2,\dots,n)$ are holomorphic in some neighborhood of the origin. We set

$$X(x)=(a_1(x),\dots,a_n(x)),\quad \frac{\partial}{\partial x}=(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}),$$

and write

$$\mathcal{X}=X(x)\cdot\frac{\partial}{\partial x},\quad X(x)=\Lambda x+R(x),$$
$$R(x)=(R_1(x),\dots,R_n(x)),\quad R(x)=O(|x|^2),$$

where Λ is an n-square constant matrix.

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[1] e-mail: yoshino@math.sci.hiroshima-u.ac.jp,
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We want to linearize $\mathcal{X}$ by the change of variables,

$$(T), \qquad x = u(y), \quad u = (u_1, \dots, u_n),$$

namely,

$$X(u(y))\frac{\partial y}{\partial x}\frac{\partial}{\partial y} = X(u(y))\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial}{\partial y} = \Lambda y\frac{\partial}{\partial y}.$$

It follows that u satisfies the equation

$$X(u(y))\left(\frac{\partial u}{\partial y}\right)^{-1} = \Lambda y,$$

that is

$$\Lambda u + R(u) = \Lambda y\frac{\partial u}{\partial y}.$$

Hence, the vector field $\mathcal{X}$ is linearized by (T) iff u satisfies the following homology equation

$$\mathcal{L}u \equiv \Lambda y\frac{\partial u}{\partial y} = \Lambda u + R(u).$$

For simplicity, we rewrite the variable y as x, and we assume that Λ is a diagonal matrix with diagonal components given by $\lambda_i, i = 1, \dots, n$ in the following. Then $\mathcal{L}$ is given by

$$\mathcal{L} = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}.$$

Hence the homology equation is written in the following form

$$\mathcal{L}u_j = \lambda_j u_j + R_j(u), \quad j = 1, \dots, n.$$

3. WKB SOLUTION OF A HOMOLOGY EQUATION

Introduction of a large parameter
The natural way of introducing a large parameter in the symmetric form of a Painlevé equation is the following

$$\begin{aligned} \eta^{-1}U_1' &= \lambda_1 + U_1(U_2 - U_3) \\ \eta^{-1}U_2' &= \lambda_2 + U_2(U_3 - U_1) \\ \eta^{-1}U_3' &= \lambda_3 + U_3(U_1 - U_2). \end{aligned}$$

This is identical with the one introduced by Aoki-Kawai- Takei from the viewpoint of a monodromy preserving deformation apart from some minor constant. In view of the similarity of the homology equation to the symmetric form of a Painlevé equation, we introduce the large parameter in the homology equation in the following way

$$\eta^{-1}\mathcal{L}U_j = \eta^{-1}\mathcal{L}(\log u_j) = \lambda_j + \frac{R_j(u)}{u_j}, \quad j = 1, \dots, n,$$

where $U_j = \log u_j$.

A WKB solution (0 - instanton solution)

For the sake of simplicity we set $u(x) = x + v(x)$ in the original homology equation and we introduce a large parameter η by the above argument. The resultant equation is

$$(HG)_\eta \qquad \eta^{-1}\mathcal{L}v_j = \lambda_j v_j + R_j(x + v(x)), \quad j = 1, \dots, n.$$

Definition (WKB solution). A WKB solution (0 - instanton solution) $v(x, \eta)$ of $(HG)_\eta$ is a formal power series solution of $(HG)_\eta$ in the form

$$(3.1) \qquad v(x, \eta) = \sum_{\nu=0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1} v_1(x) + \cdots ,$$

where the series is a formal power series in η with coefficients $v_\nu(x)$ holomorphic vector functions in x in some open set in $\mathbb{C}^n$ independent of ν.

By setting $v = (v^1, \dots, v^n)$ we substitute the expansion (3.1) into $(HG)_\eta$. First we note

$$\mathcal{L}v^j = \sum_{\nu=0}^{\infty} \mathcal{L}v_\nu^j(x)\eta^{-\nu},$$

$$R_j(x + v) = R_j(x + v_0 + v_1\eta^{-1} + v_2\eta^{-2} + \cdots)$$

$$= R_j(x + v_0) + \eta^{-1} \sum_{k=1}^{n} \left(\frac{\partial R_j}{\partial z_k}\right)(x + v_0)v_1^k + O(\eta^{-2}).$$

By comparing the coefficients of η, $\eta^0 = 1$ and η^{-1} of both sides of $(HG)_\eta$ we obtain

$$(3.2) \qquad \lambda_j v_0^j(x) + R_j(x_1 + v_0^1, \dots, x_n + v_0^n) = 0, \quad j = 1, 2, \dots, n,$$

$$(3.3) \qquad \mathcal{L}v_0^j = \lambda_j v_1^j + \sum_{k=1}^{n} \left(\frac{\partial R_j}{\partial z_k}\right)(x + v_0)v_1^k, \quad j = 1, 2, \dots, n.$$

In order to determine $v_\nu(x)$ $(\nu \geq 2)$ we compare the coefficients of $\eta^{-\nu}$. We obtain

$$(3.4) \qquad \mathcal{L}v^j_{\nu-1} = \lambda_j v^j_\nu + \sum_{k=1}^{n} \left(\frac{\partial R_j}{\partial z_k}\right)(x+v_0)v^k_\nu$$
$$+ \quad (\text{terms consisting of } v^j_k,\ k \leq \nu - 1 \text{ and } j = 1, \dots, n).$$

In order to determine v_ν from the above recurrence relations we need a definition. Let Λ be the diagonal matrix with diagonal components given by $\lambda_1, \dots, \lambda_n$ in this order.

Definition (turning point). The point x such that

$$(3.5) \qquad \det(\Lambda + (\partial R/\partial z)(x + v_0)) = 0$$

is called a *turning point* of the equation $(HG)_\eta$.

Assumption. *We assume*

$$(\text{A.1}) \qquad \lambda_j \neq 0, \qquad j = 1, \dots, n.$$

Note that the origin $x = 0$ is not a turning point of $(HG)_\eta$ for any holomorphic $v_0(x) = O(|x|^2)$, because $\det \Lambda \neq 0$.

Then, we have

Proposition *Assume that* $\det \Lambda \neq 0$. *Then every coefficient* $v_\nu(x)$ *of a WKB solution is uniquely determined as a holomorphic function in some neighborhood of the origin* $x = 0$ *independent of* ν.

Proof. The function $v^j_0(x)$ is holomorhic at the origin $x = 0$ and satisfies that $v^j_0(x) = O(|x|^2)$. Hence it is uniquely determined by (3.2) in view of the implicit function theorem. Then the functions

$$v^j_k(x), \quad k = 1, 2, \dots, j = 1, \dots, n$$

can be uniquely determined by (3.4) as holomorphic functions in some neighborhood of the origin by the assumption because the origin $x = 0$ is not a turning point of the equation. We note that $v^j_k(x)$ are determined recursively by differentiation and algebraic manupulations. This implies that all $v^j_k(x)$ are holomorphic in some neighborhood of the origin independent of ν. □

Definition (Resonance condition). We say that η is *resonant*, if

$$(3.6) \qquad \sum_{i=1}^{n} \lambda_i \alpha_i - \eta \lambda_j = 0,$$

for some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$, $|\alpha| \geq 2$ and $j, 1 \leq j \leq n$. If η is not resonant, then we say that η is *nonresonant*.

Definition (Poincaré condition) *We say that a homolgy equation satisfies a Poincaré condition, if the convex hull of $\lambda_j, (j = 1, \dots, n)$ in the complex plane does not contain the origin.*

If a Poincaré condition is not verified, then we assume the following condition

$$\lambda_j \in \mathbb{R}, \quad j = 1, \dots, n.$$

In this case, there are two important cases, namely, a Diophantine case and Liouville case. In the former case, either a Siegel condition or a Bruno (type) Diophantine condition is verified among λ_j, $j = 1, \dots, n$. If no such conditions are satisfied, then we say that we are in a Liouville domain under our assumption.

We note that, if a Poincaré condition is verified, then the number of resonance is finite, while in a Siegel case, the number of resonance is, in general, infinite. Moreover the resonance may be a dense subset of a real line.

4. Summability of a WKB solution in a Poincaré domain

For the direction ξ, $(0 \leq \xi < 2\pi)$ and the opening $\theta > 0$ we define the sector $S_{\xi,\theta}$ by

$$S_{\xi,\theta} = \left\{ \eta \in \mathbb{C};\ |\mathrm{Arg}\,\eta - \xi| < \frac{\theta}{2} \right\}, \tag{4.1}$$

where the branch of the argument is the principal value. Then we have

Theorem 1. (Resummation) *Suppose that*

$$|\ \mathrm{Arg}\ \lambda_j| < \frac{\pi}{4}, \quad j = 1, \dots, n. \tag{C}$$

Then, there exist a direction ξ, an opening $\theta > \pi$, a neighborhood U of the origin $x = 0$ and $V(x, \eta)$ such that $V(x, \eta)$ is holomorphic in $(x, \eta) \in U \times S_{\xi,\theta}$ and satisfies $(HG)_\eta$. The function $V(x, \eta)$ is a Borel sum of the WKB solution $v(x, \eta)$ in $U \times S_{\xi,\theta}$ when $\eta \to \infty$. Namely, for every $N \geq 1$ and $R > 0$, there exist $C > 0$ and $K > 0$ such that

$$\left| V(x,\eta) - \sum_{\nu=0}^{N} \eta^{-\nu} v_\nu(x) \right| \leq C K^N N! |\eta|^{-N-1}, \tag{4.2}$$
$$\forall (x, \eta) \in U \times S_{\xi,\theta},\ |\eta| \geq R.$$

Remark. The condition (C) implies the Poincaré condition.

5. Reconstruction of a Poincaré solution via analytic continuation of a WKB solution

We shall make an analytic continuation (with respect to η) of a resummed WKB solution to the right half plane. We note that there exist an infinte number of resonaces on the right-half plane $\mathrm{Re}\,\eta > 0$ which accumulate only at infinity. The solution may be singular with respect to η at the resonances. We have

Theorem 2. *Suppose that (C) is verified. Then the resummed WKB solution is analytically continued to the right half plane as a single-valued function except for resonances. If the nonresonance condition holds, then the analytic continuation of a resummed WKB solution to $\eta = 1$ coincides with a classical Poincaré solution of a homology equation.*

Next we consider the case where a Poincaré condition is verified, while the condition (C) is not satisfied. The essential difference in this case is that there is not a unique correspondence between the WKB solution and the Poincaré solution.

Theorem 3. *Suppose that the Poincaré condition is verified. Then, there exist a direction ξ, an opening $\theta > 0$, a neighborhood U of the origin $x = 0$ and $V(x,\eta)$ such that $V(x,\eta)$ is holomorphic in $(x,\eta) \in U \times S_{\xi,\theta}$ and satisfies $(HG)_\eta$. The WKB solution $v(x,\eta)$ is a Gevrey 2 asymptotic expansion of $V(x,\eta)$ in $U \times S_{\xi,\theta}$ when $\eta \to \infty$.*

The function $V(x,\eta)$ is analytically continued with respect to η to the right half plane as a single- valued function except for resonances. If the nonresonance condition is verified, then we can take $V(x,\eta)$ such that the analytic continuation of $V(x,\eta)$ to $\eta = 1$ coincides with a classical Poincaré solution of a homology equation with $\eta = 1$.

6. WKB solution in a Siegel domain

In this section we assume that we are in a Siegel domain. Moreover, we assume, for the sake of simplicity

$\lambda_j \in \mathbb{R}$ $(j = 1, 2, \dots, n)$ *are linearly independent over* $\mathbb{Q}$.

Then the set of all resonances is dense on $\mathbb{R}$. We have

Theorem 4. *Under the above conditions, there exist a direction ξ, an opening $\theta > 0$, a neighborhood U of the origin $x = 0$ and $V(x,\eta)$ such that $V(x,\eta)$ is holomorphic in $(x,\eta) \in U \times S_{\xi,\theta}$ and satisfies $(HG)_\eta$. The WKB solution $v(x,\eta)$ is an asymptotic expansion of the function $V(x,\eta)$ in $U \times S_{\xi,\theta}$ when $\eta \to \infty$.*

The function $V(x,\eta)$ is analytically continued with respect to η to the upper (respectively lower) half plane as a single-valued function. If the nonresonance condition is verified, then we can take $V(x,\eta)$ such that

$$\lim_{\pm\eta\to 1} V(x,\eta)$$

exists as a formal power series and they coincide with a Siegel solution of a homology equation as a formal power series solution.

Remark. i) We do not know whether the WKB solution $v(x,\eta)$ is a Gevrey asymptotic expansion of $V(x,\eta)$ in $U\times S_{\xi,\theta}$ when $\eta\to\infty$.

ii) On the real line $\mathbb{R}$, $V(x,\eta)$ has dense singularities in η. Hence, $V(x,\eta)$ cannot be continued analytically to the point $\eta=1$.

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Present adress: Graduate School of Sciences, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan. e-mail: yoshino@math.sci.hiroshima-u.ac.jp