WKB ANALYSIS TO NORMAL FORM THEORY OF VECTOR FIELDS (Microlocal Analysis and Related Topics)

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WKB ANALYSIS TO NORMAL FORM THEORY OF VECTOR FIELDS

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1. INTRODUCTION

In this note we shall study the relations between the exact asymptotic analysis of a so-called homology equation and the normal form theory of a singular vector field. A homology equation is a system of partial differential equations which appear in linearizing a singular vector field by the change of independent variables. We shall introduce a WKB solution of a homology equation which is a natural extension of the one introduced by Aoki-Kawai-Takei for the Painlevé equation. We then give a new unexpected connection between Poincaré series and the WKB solution via resummation procedure.

2. HOMOLOGY EQUATION

Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $n \geq 2$ be the variable in $\mathbb{C}^n$. We consider a singular vector field near the origin of $\mathbb{C}^n$

$$\mathcal{X} = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}, \quad a_j(0) = 0, \quad j = 1, \ldots, n,$$

where $a_j(x)$ ($j = 1, 2, \ldots, n$) are holomorphic in some neighborhood of the origin. We set

$$X(x) = (a_1(x), \ldots, a_n(x)), \quad \frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}),$$

and write

$$\mathcal{X} = X(x) \cdot \frac{\partial}{\partial x}, \quad X(x) = \Lambda x + R(x),$$

$$R(x) = (R_1(x), \ldots, R_n(x)), \quad R(x) = O(|x|^2),$$

where $\Lambda$ is an $n$-square constant matrix.
We want to linearize $\mathcal{X}$ by the change of variables, 
\[ (T), \quad x = u(y), \quad u = (u_1, \ldots, u_n), \]

namely,
\[
X(u(y)) \frac{\partial y}{\partial x} \frac{\partial}{\partial y} = X(u(y)) \left( \frac{\partial x}{\partial y} \right)^{-1} \frac{\partial}{\partial y} = \Lambda y \frac{\partial}{\partial y}.
\]

It follows that $u$ satisfies the equation
\[
X(u(y)) \left( \frac{\partial u}{\partial y} \right)^{-1} = \Lambda y,
\]
that is
\[
\Lambda u + R(u) = \Lambda y \frac{\partial u}{\partial y}.
\]

Hence, the vector field $\mathcal{X}$ is linearized by $(T)$ iff $u$ satisfies the following homology equation
\[
\mathcal{L}u \equiv \Lambda y \frac{\partial u}{\partial y} = \Lambda u + R(u).
\]

For simplicity, we rewrite the variable $y$ as $x$, and we assume that $\Lambda$ is a diagonal matrix with diagonal components given by $\lambda_i, i = 1, \ldots, n$ in the following. Then $\mathcal{L}$ is given by
\[
\mathcal{L} = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}.
\]

Hence the homology equation is written in the following form
\[
\mathcal{L}u_j = \lambda_j u_j + R_j(u), \quad j = 1, \ldots, n.
\]

3. WKB solution of a homology equation

Introduction of a large parameter

The natural way of introducing a large parameter in the symmetric form of a Painlevé equation is the following
\[
\eta^{-1} U'_1 = \lambda_1 + U_1(U_2 - U_3) \\
\eta^{-1} U'_2 = \lambda_2 + U_2(U_3 - U_1) \\
\eta^{-1} U'_3 = \lambda_3 + U_3(U_1 - U_2).
\]

This is identical with the one introduced by Aoki-Kawai-Takei from the viewpoint of a monodromy preserving deformation apart from some minor constant. In view of the similarity of the homology equation to
the symmetric form of a Painlevé equation, we introduce the large parameter in the homology equation in the following way

\[ \eta^{-1} \mathcal{L} U_j = \eta^{-1} \mathcal{L}(\log u_j) = \lambda_j + \frac{R_j(u)}{u_j}, \quad j = 1, \ldots, n, \]

where \( U_j = \log u_j \).

A WKB solution (0 - instanton solution)

For the sake of simplicity we set \( u(x) = x + v(x) \) in the original homology equation and we introduce a large parameter \( \eta \) by the above argument. The resultant equation is

\[ (HG)_\eta \quad \eta^{-1} \mathcal{L} v_j = \lambda_j v_j + R_j(x + v(x)), \quad j = 1, \ldots, n. \]

Definition (WKB solution). A WKB solution (0 - instanton solution) \( v(x, \eta) \) of \( (HG)_\eta \) is a formal power series solution of \( (HG)_\eta \) in the form

\[ v(x, \eta) = \sum_{\nu = 0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1} v_1(x) + \cdots, \tag{3.1} \]

where the series is a formal power series in \( \eta \) with coefficients \( v_\nu(x) \) holomorphic vector functions in \( x \) in some open set in \( \mathbb{C}^n \) independent of \( \nu \).

By setting \( v = (v^1, \ldots, v^n) \) we substitute the expansion (3.1) into \( (HG)_\eta \). First we note

\[ \mathcal{L} v^j = \sum_{\nu = 0}^{\infty} \mathcal{L} v^j_\nu(x) \eta^{-\nu}, \]

\[ R_j(x + v) = R_j(x + v_0 + v_1 \eta^{-1} + v_2 \eta^{-2} + \cdots) = R_j(x + v_0) + \eta^{-1} \sum_{k=1}^{n} \left( \frac{\partial R_j}{\partial x_k} \right) (x + v_0) v_1^k + O(\eta^{-2}). \]

By comparing the coefficients of \( \eta, \eta^0 = 1 \) and \( \eta^{-1} \) of both sides of \( (HG)_\eta \) we obtain

\[ \lambda_j v_0^j(x) + R_j(x_1 + v_0^1, \ldots, x_n + v_0^n) = 0, \quad j = 1, 2, \ldots, n, \tag{3.2} \]

\[ \mathcal{L} v_0^j = \lambda_j v_1^j + \sum_{k=1}^{n} \left( \frac{\partial R_j}{\partial x_k} \right) (x + v_0) v_1^k, \quad j = 1, 2, \ldots, n. \tag{3.3} \]
In order to determine \( v_\nu(x) \) \((\nu \geq 2)\) we compare the coefficients of \( \eta^{-\nu}\). We obtain

\[
(3.4) \quad \mathcal{L}v_{\nu - 1}^j = \lambda_j v_\nu^j + \sum_{k=1}^{n} \left( \frac{\partial R_j}{\partial z_k} \right) (x + v_0)v_k^j
\]

+ (terms consisting of \( v_k^j, k \leq \nu - 1 \) and \( j = 1, \ldots, n \)).

In order to determine \( v_\nu \) from the above recurrence relations we need a definition. Let \( \Lambda \) be the diagonal matrix with diagonal components given by \( \lambda_1, \ldots, \lambda_n \) in this order.

**Definition (turning point).** The point \( x \) such that

\[
(3.5) \quad \det (\Lambda + (\partial R/\partial z)(x + v_0)) = 0
\]

is called a *turning point* of the equation \((HG)_{\eta}\).

**Assumption.** We assume

\[
(A.1) \quad \lambda_j \neq 0, \quad j = 1, \ldots, n.
\]

Note that the origin \( x = 0 \) is not a turning point of \((HG)_{\eta}\) for any holomorphic \( v_0(x) = O(|x|^2) \), because \( \det \Lambda \neq 0 \).

Then, we have

**Proposition** Assume that \( \det \Lambda \neq 0 \). Then every coefficient \( v_\nu(x) \) of a WKB solution is uniquely determined as a holomorphic function in some neighborhood of the origin \( x = 0 \) independent of \( \nu \).

**Proof.** The function \( v_0^j(x) \) is holomorphic at the origin \( x = 0 \) and satisfies that \( v_0^j(x) = O(|x|^2) \). Hence it is uniquely determined by \((3.2)\) in view of the implicit function theorem. Then the functions

\[ v_k^j(x), \quad k = 1, 2, \ldots, j = 1, \ldots, n \]

can be uniquely determined by \((3.4)\) as holomorphic functions in some neighborhood of the origin by the assumption because the origin \( x = 0 \) is not a turning point of the equation. We note that \( v_k^j(x) \) are determined recursively by differentiation and algebraic manipulations. This implies that all \( v_k^j(x) \) are holomorphic in some neighborhood of the origin independent of \( \nu \). \( \square \)

**Definition (Resonance condition).** We say that \( \eta \) is *resonant*, if

\[
(3.6) \quad \sum_{i=1}^{n} \lambda_i \alpha_i - \eta \lambda_j = 0,
\]

for some \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n, |\alpha| \geq 2 \) and \( j, 1 \leq j \leq n \). If \( \eta \) is not resonant, then we say that \( \eta \) is *nonresonant*. 
**Definition (Poincaré condition)** We say that a homology equation satisfies a Poincaré condition, if the convex hull of \( \lambda_j, (j = 1, \ldots, n) \) in the complex plane does not contain the origin.

If a Poincaré condition is not verified, then we assume the following condition

\[
\lambda_j \in \mathbb{R}, \quad j = 1, \ldots, n.
\]

In this case, there are two important cases, namely, a Diophantine case and Liouville case. In the former case, either a Siegel condition or a Bruno (type) Diophantine condition is verified among \( \lambda_j, j = 1, \ldots, n \).

If no such conditions are satisfied, then we say that we are in a Liouville domain under our assumption.

We note that, if a Poincaré condition is verified, then the number of resonance is finite, while in a Siegel case, the number of resonance is, in general, infinite. Moreover the resonance may be a dense subset of a real line.

4. **Summability of a WKB solution in a Poincaré domain**

For the direction \( \xi, (0 \leq \xi < 2\pi) \) and the opening \( \theta > 0 \) we define the sector \( S_{\xi,\theta} \) by

\[
S_{\xi,\theta} = \left\{ \eta \in \mathbb{C}; \left| \text{Arg} \, \eta - \xi \right| < \frac{\theta}{2} \right\},
\]

where the branch of the argument is the principal value. Then we have

**Theorem 1. (Resummation)** Suppose that

\[
(C) \quad | \text{Arg} \, \lambda_j | < \frac{\pi}{4}, \quad j = 1, \ldots, n.
\]

Then, there exist a direction \( \xi \), an opening \( \theta > \pi \), a neighborhood \( U \) of the origin \( x = 0 \) and \( V(x, \eta) \) such that \( V(x, \eta) \) is holomorphic in \((x, \eta) \in U \times S_{\xi,\theta}\) and satisfies \((HG)_\eta\). The function \( V(x, \eta) \) is a Borel sum of the WKB solution \( v(x, \eta) \) in \( U \times S_{\xi,\theta} \) when \( \eta \to \infty \). Namely, for every \( N \geq 1 \) and \( R > 0 \), there exist \( C > 0 \) and \( K > 0 \) such that

\[
(4.2) \quad \left| V(x, \eta) - \sum_{\nu=0}^{N} \eta^{-\nu} v_\nu(x) \right| \leq CK^N N! |\eta|^{-N-1},
\]

\( \forall (x, \eta) \in U \times S_{\xi,\theta}, \, |\eta| \geq R. \)

**Remark.** The condition \((C)\) implies the Poincaré condition.
5. RECONSTRUCTION OF A POINCARE SOLUTION VIA ANALYTIC CONTINUATION OF A WKB SOLUTION

We shall make an analytic continuation (with respect to $\eta$) of a resummed WKB solution to the right half plane. We note that there exist an infinite number of resonances on the right-half plane $\text{Re} \eta > 0$ which accumulate only at infinity. The solution may be singular with respect to $\eta$ at the resonances. We have

**Theorem 2.** Suppose that (C) is verified. Then the resummed WKB solution is analytically continued to the right half plane as a single-valued function except for resonances. If the nonresonance condition holds, then the analytic continuation of a resummed WKB solution to $\eta = 1$ coincides with a classical Poincaré solution of a homology equation.

Next we consider the case where a Poincaré condition is verified, while the condition (C) is not satisfied. The essential difference in this case is that there is not a unique correspondence between the WKB solution and the Poincaré solution.

**Theorem 3.** Suppose that the Poincaré condition is verified. Then, there exist a direction $\xi$, an opening $\theta > 0$, a neighborhood $U$ of the origin $x = 0$ and $V(x, \eta)$ such that $V(x, \eta)$ is holomorphic in $(x, \eta) \in U \times S_{\xi, \theta}$ and satisfies $(HG).$ The WKB solution $v(x, \eta)$ is a Gevrey 2 asymptotic expansion of $V(x, \eta)$ in $U \times S_{\xi, \theta}$ when $\eta \to \infty$.

The function $V(x, \eta)$ is analytically continued with respect to $\eta$ to the right half plane as a single-valued function except for resonances. If the nonresonance condition is verified, then we can take $V(x, \eta)$ such that the analytic continuation of $V(x, \eta)$ to $\eta = 1$ coincides with a classical Poincaré solution of a homology equation with $\eta = 1$.

6. WKB SOLUTION IN A SIEGEL DOMAIN

In this section we assume that we are in a Siegel domain. Moreover, we assume, for the sake of simplicity

$\lambda_j \in \mathbb{R} (j = 1,2, \ldots, n)$ are linearly independent over $\mathbb{Q}$. Then the set of all resonances is dense on $\mathbb{R}$. We have

**Theorem 4.** Under the above conditions, there exist a direction $\xi$, an opening $\theta > 0$, a neighborhood $U$ of the origin $x = 0$ and $V(x, \eta)$ such that $V(x, \eta)$ is holomorphic in $(x, \eta) \in U \times S_{\xi, \theta}$ and satisfies $(HG)$. The WKB solution $v(x, \eta)$ is an asymptotic expansion of the function $V(x, \eta)$ in $U \times S_{\xi, \theta}$ when $\eta \to \infty$.
The function $V(x, \eta)$ is analytically continued with respect to $\eta$ to the upper (respectively lower) half plane as a single-valued function. If the nonresonance condition is verified, then we can take $V(x, \eta)$ such that

\[
\lim_{\pm \eta \to 1} V(x, \eta)
\]

does not exist as a formal power series and they coincide with a Siegel solution of a homology equation as a formal power series solution.

Remark. i) We do not know whether the WKB solution $v(x, \eta)$ is a Gevrey asymptotic expansion of $V(x, \eta)$ in $U \times S_{\xi, \theta}$ when $\eta \to \infty$.

ii) On the real line $\mathbb{R}$, $V(x, \eta)$ has dense singularities in $\eta$. Hence, $V(x, \eta)$ cannot be continued analytically to the point $\eta = 1$.

References


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