数理解析研究所講究録 1431 巻 2005 年 30-36

Hyperbolic equations with non analytic coefficients well posed in all Gevrey classes

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1 Introduction

This paper is devoted to the initial value problem (IVP) in the strip $[0,T]\times R_x$ for higher order, homogeneous, *hyperbolic* equations

$$\begin{cases} \partial_t^m u \;=\; \Big[a_1(t)\partial_x\partial_t^{m-1} + a_2(t)\partial_x^2\partial_t^{m-2} + \cdots + a_m(t)\partial_x^m\Big]u\,, \\ \partial_t^h u\,|_{t=0} \;=\; u_h(x)\,, \qquad h=0,\dots,m-1\,, \end{cases} \tag{1}$$

The coefficients $a_j(t)$ are smooth, real functions on $[0,T]$. The hyperbolicity means that

$$z^m - \sum_{j=1}^m a_j(t)z^{m-j} \;=\; \prod_{j=1}^m (z-\lambda_j(t)) \qquad \text{with} \quad \lambda_1(t)\le\lambda_2(t)\le\cdots\le\lambda_m(t)\,.$$

We say that (1) is *well posed* in a given functional space $\mathcal{X}$ if it admits a unique solution $u\in C^m([0,T],\mathcal{X})$ for all data $u_0,\dots,u_{m-1}\in\mathcal{X}$.

We denote by $\gamma^s = \gamma^s(R)$, where $s \geq 1$, the class of *Gevrey functions* of order s, i.e., the smooth functions φ satisfying

$$|\partial_x^h \varphi(x)| \leq \Lambda_K^{h+1}\, h!^s, \qquad \forall h \in N, \quad \forall x \in K, \quad \forall K \subset\subset R,$$

and we define

$$\gamma^\infty = \bigcup_{s\geq 1} \gamma^s.$$

Hence we have, denoting by $C^\omega = C^\omega(R)$ the class of real analytic functions,

$$C^\omega = \gamma^1 \subset \gamma^s \subset \gamma^{s'} \subset \gamma^\infty \subset C^\infty, \qquad \text{if} \quad 1 < s < s'.$$

When (1) is well posed in γ^s for all $s \geq 1$, we say that it is well posed in γ^∞. Note that the well-posedness in γ^∞ is very close to that in C^∞.

If the coefficients a_j are constant, we know that (1) is well posed in $C^\infty(R_x)$, but this result fails in case of variable coefficients. In any case, we have the well-posedness in some Gevrey classes; indeed we know, by Bronshtein's theorem ([B]), that (1), with smooth coefficients $a_j(t)$, is well posed in γ^s for

$$1 \leq s < 1 + \frac{1}{m-1}. \tag{2}$$

However, the upper bound in (2) can be sensibly improved for the special class of equations

$$u_{tt} = a(t)\, u_{xx}, \qquad a(t) \geq 0. \tag{3}$$

Indeed, we know that (see [CJS])

i) if $a(t) \in C^\omega$, the IVP for (3) is well posed in C^∞,

ii) if $a(t) \in C^\infty$, the IVP for (3) is well posed in γ^∞,

and, more precisely,

iii) if $a(t) \in C^k$, we have the well-posedness in γ^s for $1 \leq s < 1 + k/2$.

These results are optimal.

It is natural to try to extend this kind of results to higher order equations. In 1999, F. Colombini and N. Orrú found the following criterion of well-posedness for the equations (1), in case of *analytic coefficients*,

$$\lambda_i(t)^2 + \lambda_j(t)^2 \leq M\,(\lambda_i(t) - \lambda_j(t))^2, \qquad 1 \leq i < j \leq m, \tag{4}$$

where M is a uniform constant, by proving

Theorem [CO] *Let $a_j(t) \in C^\omega([0,T])$ with $a_j(0) = 0$ $(1 \leq j \leq m)$. Therefore, (1) is well posed in C^∞ if and only if its characteristic roots satisfy (4).*

The main purpose of this paper is to release the assumption of analyticity on the coefficients, and to prove that, even if these are C^∞, condition (4) ensures the well-posedness in γ^∞.

Theorem 1 *Assume that the characteristic roots of* (1) *satisfy* (4), *and the coefficients* $a_j(t)$ *are* C^∞ *functions on* $[0,T]$. *Then,* (1) *is well posed in* γ^∞. *More precisely, if* $a_j \in C^k([0,T])$ *for some* $k \geq 2$, (1) *is well posed in* γ^s *for*

$$1 \leq s < 1 + \frac{k}{2\,(m-1)}\,.$$

When the a_j*'s belong to* $C^\omega([0,T])$, (1) *is well posed in* C^∞.

Remark 1 If we introduce the *discriminant* of the equation (1), i.e.,

$$\Delta(t) \;=\; \prod_{i<j} (\lambda_i(t) - \lambda_j(t))^2\,, \tag{5}$$

we can write (4) in the equivalent form

$$\sum_{1\leq i<j\leq m} \Big[(\lambda_i(t)^2 + \lambda_j(t)^2) \prod_{\substack{1\leq h<k\leq m \\ (h,k)\neq(i,j)}} (\lambda_h(t) - \lambda_k(t))^2\Big] \;\leq\; M'\,\Delta(t)\,. \tag{6}$$

Now, the left side of (6) is a symmetric polynomial in the roots λ_i, hence, by Newton's theorem, it is also a polynomial in the coefficients $\{a_1(t), \cdots, a_m(t)\}$. Thus (6), hence also (4), can be explicitly written in terms of the a_j's, instead of the λ_j's. For instance, for the second order equations $u_{tt} \;=\; a(t)\,u_{xx} + b(t)\,u_{tx}$, we have $\lambda_1^2 + \lambda_2^2 = \;\Delta/2 + b^2/2$, hence the condition (4) is equivalent to

$$\Delta(t) \;\equiv\; b(t)^2 + 4a(t) \;\geq\; c\,b(t)^2 \qquad (c>0)\,.$$

2 The energy estimates

The well-posedness of (1) in the analytic class γ^1 is well known, by the theorem of Bony and Shapira. Thus, we shall prove Theorem 1 only for $s > 1$.

Setting $U = (\partial_x^{m-1}u, \cdots, \partial_x^{m-j}\partial_t^{j-1}\,u, \cdots, \partial_t^{m-1}u)^t$, and

$$A(t) \;=\; \begin{pmatrix} 0 & 1 & 0 & \dots & \dots \\ & 0 & 1 & & \vdots \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_m(t) & \dots & & a_2(t) & a_1(t) \end{pmatrix}, \tag{7}$$

we transform the scalar equation (1) into the system

$$U_t = A(t)\, U_x\,, \qquad U(0,x) = U_0(x)\,, \tag{8}$$

which becomes, by Fourier transform $V(t,\xi) = \mathcal{F}_x U(t,x)$,

$$V'(t,\xi) = i\,\xi A(t)\, V(t,\xi)\,, \qquad V(0,\xi) = V_0(\xi)\,. \tag{9}$$

In [DS] it was constructed, for any hyperbolic matrix (7), a *quasi-symmetrizer* $Q_\varepsilon(t) = Q_\varepsilon(t)^*$, $0 < \varepsilon < 1$, with the following properties:

$$\begin{cases} Q_\varepsilon(t) = Q_0(t) + \varepsilon^2\, Q_1(t) + \ldots + \varepsilon^{2(m-1)}\, Q_{m-1}(t)\,, \quad \text{where} \\ Q_0(t),\, Q_1(t),\, \ldots, Q_{m-1}(t) \ \text{ have the same regularity in } t \text{ of A(t)}, \end{cases} \tag{10}$$

$$\varepsilon^{2(m-1)}\, |V|^2 \le (Q_\varepsilon(t)V, V) \le C\,|V|^2, \tag{11}$$

$$|\,(\{Q_\varepsilon(t)A(t) - A^*(t)Q_\varepsilon(t)\}\, V,\, V\,)\,| \le C\,\varepsilon\,(Q_\varepsilon(t)V, V)\,. \tag{12}$$

From the existence of a $Q_\varepsilon(t)$ satisfying (10)–(12), we easily derive (see [DS]) the γ^s well-posedness of (8) for $s < 1 + 1/(m-1)$, i.e., the Bronshtein's bound, as soon as $A(t)$ is hyperbolic and of class $\mathcal{C}^2$ in a neighborhood of $[0,T]$. We recall shortly the proof. Putting

$$E_\varepsilon(t,\xi) = (Q_\varepsilon(t)V(t,\xi), V(t,\xi))\,,$$

$$K_\varepsilon(t,\xi) = \frac{|\,(Q'_\varepsilon(t)V(t,\xi), V(t,\xi))\,|}{(Q_\varepsilon(t)V(t,\xi), V(t,\xi))}\,,$$

we get, by (9),

$$E'_\varepsilon = (Q'_\varepsilon(t)V, V) + i\,\xi\,((Q_\varepsilon A - A^* Q_\varepsilon)V, V) \le \Big\{K_\varepsilon(t,\xi) + C\varepsilon\,|\xi|\Big\}\, E_\varepsilon\,. \tag{13}$$

To estimate K_ε, we apply the *Glaeser's inequality* (where $f(t) \ge 0$ in a neighborhood of $[0,T]$)

$$|f'(t)| \le C(\delta)\, f(t)^{1/2} \|f\|^{1/2}_{\mathcal{C}^2[-\delta, T+\delta]}\,, \qquad \forall \delta > 0\,,$$

to the functions $t \mapsto (Q_\varepsilon(t)V, V)$, for fixed $V \in \mathcal{C}^m$. Note that, by (10), these are nonnegative and equi-bounded in $\mathcal{C}^2$ on a neighborhood $\widetilde{I}$ of $[0,T]$.

Thus we get, for each vector $V \neq 0$,

$$\frac{|(Q'_\varepsilon(t)V,V)|}{(Q_\varepsilon(t)V,V)^{1/2}\,|V|} \leq C_0 \|Q_\varepsilon\|^{1/2}_{C^2(\tilde{I})} \leq C_1 \qquad \forall t \in [0,T],$$

where C_0, C_1 are constants independent of V and ε. Replacing V by $V(t,\xi)$ and recalling (11), we obtain

$$K_\varepsilon(t,\xi) \leq C_1\,\varepsilon^{-(m-1)}, \qquad \forall t \in [0,T], \tag{14}$$

whence, going back to (13),

$$E'_\varepsilon(t,\xi) \leq \left\{C_1\,\varepsilon^{-(m-1)} + \varepsilon\,|\xi|\right\} E_\varepsilon(t,\xi).$$

If we choose $\varepsilon = |\xi|^{1/m}$, by Gronwall's inequality and (11), we derive the apriori estimate

$$|V(t,\xi)| \leq C\,|\xi|^{1/\sigma} e^{C\,|\xi|^{1/\sigma}}\,|V_0(\xi)|, \qquad \text{with} \quad \sigma = 1 + \frac{1}{m-1}. \tag{15}$$

This ensures the γ^s well-posedness for $1 < s < 1 + 1/(m-1)$. Indeed, for any compact supported $U(x)$, we see that $U \in \gamma^s$ if and only if, for some positive constants C, δ, ν,

$$|\hat{U}(\xi)| \leq C\,|\xi|^{\nu}\,e^{-\delta\,|\xi|^{1/s}}. \qquad \square$$

To overpass, in (15), the bound $s < 1 + 1/(m-1)$ we must improve (14). To this end we need, first of all, that $A(t)$ is more regular than C^2; moreover, the quasi-symmetrizer $Q_\varepsilon(t)$ must behave like a diagonal matrix.

Definition 1 *A family $\mathcal{Q}$ of nonnegative matrices will be called nearly diagonal (ND) if there is a constant $c_0 > 0$ such that, for all $Q \in \mathcal{Q}$,*

$$(QV,V) \geq c_0\,(\Lambda_Q V,V), \quad \forall\, V \in C^m, \tag{16}$$

where

$$\Lambda_Q = \begin{pmatrix} q_{11} & & 0 \\ & \ddots & \\ 0 & & q_{mm} \end{pmatrix}.$$

For the nearly diagonal quasi-symmetrizer $Q_\varepsilon(t)$, we get the following:

Proposition 1 *Assume that $A(t)$ admits a quasi-symmetrizer $Q_\varepsilon(t)$ which satisfies* (10)–(12) *and* (16). *Therefore:*

i) if $A(t) \in \mathcal{C}^k([0,T])$, $k \geq 2$, (8) is γ^s well posed for $1 < s < 1 + k/[2(m-1)]$,

ii) if $A(t) \in \mathcal{C}^\omega([0,T])$, (8) is well posed in $\mathcal{C}^\infty$.

3 Construction of the quasi-symmetrizer

To define the quasi-symmetrizer, we first put:

Notation: $T^\sharp = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}^{m\times m}$, where $T \in \mathcal{M}^{k\times k}$ $(k < m)$.

Then, we define the $m \times m$ matrix $Q_\varepsilon^{(m)}(\lambda)$, by setting (inductively)

$$\begin{cases} Q_\varepsilon^{(1)}(\lambda) = 1\,, \\ Q_\varepsilon^{(m)}(\lambda) = Q_0^{(m)}(\lambda) + \varepsilon^2 \sum_{i=1}^{m} [\, Q_\varepsilon^{(m-1)}(\pi_i\lambda)\,]^\sharp . \end{cases}$$

Putting

$$\mathcal{S}_M = \left\{ \lambda \in R^m : \lambda_i^2 + \lambda_j^2 \le M\,(\lambda_i - \lambda_j)^2\,, \quad \text{for all } 1 \le i < j \le m \right\},$$

we get the following:

Proposition 2 *For each $M > 0$, the family of matrices*

$$\mathcal{Q}_M = \left\{ Q_\varepsilon^{(m)}(\lambda) : \; 0 < \varepsilon < 1\,, \; \lambda \in \mathcal{S}_M \right\}$$

is nearly diagonal in the sense of Definition 1.

The scalar problem (1) is equivalent to (8) where $A(t)$ is given by (7). Proposition 2 ensure that $Q_\varepsilon(t)$ satisfies (16), and we can apply Proposition 1 to reach the conclusion of Theorem 1.

References

[B] M.D. Bronshtein, *The Cauchy problem for hyperbolic operators with multiple variable characteristics*, Trudy Moskow Mat. Obsc., **41** (1980), 83-99; Trans. Moscow Math. Soc., **1** (1982), 87-103.

[CJS] F. Colombini, E. Jannelli and S. Spagnolo, *Well-posedness in the Gevrey classes of the Cauchy problem for a non-strictly hyperbolic equation with coefficients depending on time*, Ann. Scu. Norm. Sup. Pisa, **10** (1983), 291–312.

[CN] F. Colombini and T. Nishitani, *Two by two strongly hyperbolic systems and Gevrey classes* , Workshop on Partial Differential Equations (Ferrara, 1999). Ann. Univ. Ferrara Sez. VII (N.S.) **45** (1999), suppl., 79–108 (2000).

[CO] F. Colombini and N. Orrù, *Well-posedness in C^∞ for some weakly hyperbolic equations*, J. Math. Kyoto Univ., **39** (1999), 399–420.

[DKS] P. D'Ancona, T. Kinoshita and S. Spagnolo, *Weakly hyperbolic systems with Hölder continuous coefficients*, to appear on J. Diff. Equat.

[DS] P. D'Ancona and S. Spagnolo, *Quasi-symmetrization of hyperbolic systems and propagation of the analytic regularity*, Boll. Un. Mat. It., **1-B** (1998), 169–185.

[J1] E. Jannelli, *Linear Kovalewskian systems with time dependent coefficients*, Comm. PDE, **9** (1984), 1373-1406.

[J2] E. Jannelli, *On the symmetrization of the principal symbol of hyperbolic equations*, Comm. PDE, **14** (1989), 1617-1634.

[J3] E. Jannelli, *Sharp estimates about quasi-symmetrizers for scalar hyperbolic operators*, Lecture held in the Meeting on "Hyperbolic Equations", Venice, April 2002

[K] T. Kinoshita, *Gevrey wellposedness of the Cauchy problem for the hyperbolic equations of third order with coefficients depending only on time*, Publ. RIMS Kyoto Univ., **34** (1998), 249–270.

[S1] S. Spagnolo, *On the absolute continuity of the roots of some algebraic equations*, Ann. Univ. Ferrara, **25** (Suppl. 1999), 327–337.

[S2] S. Spagnolo, *Hyperbolic systems well posed in all Gevrey classes*, in Geometric Analysis of PDE and Several Complex Variables, S. Chanillo, P. Cordaro, N. Hanges, J. Hounie, and A. Meziani Eds., Contemporary Mathematics, to appear

[T] S. Tarama, *On the Lemma of Colombini, Jannelli and Spagnolo*, Mem. Fac. Engin. Osaka City Univ., **41** (2000), 111–115.