数理解析研究所講究録 1431 巻 2005 年 37-44

WALTER'S METHOD APPLIED TO FUCHSIAN PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We will establish the unique existence of the solution to Fuchsian partial differential equations by means of the Banach Fixed Point Theorem. This method was used by W. Walter [6] to produce a simple and elegant proof of the Cauchy-Kowalevsky Theorem. This method also has the advantage that the right-hand-side function, the solution and the coefficients of the equation all have the same domain of definition. The resulting theorem thus improves the one obtained by the author in [3].

1. DEFINITIONS AND MAIN RESULT

We consider the existence and uniqueness of the solution to the singular linear partial differential equation

$$Pu := \big(t\partial_t\big)^m u + \sum_{\substack{j<m \\ j+|\alpha|\le m}} a_{j,\alpha}(t,z)\big(\mu(t)\partial_z\big)^\alpha \big(t\partial_t\big)^j u = f \tag{1.1}$$

in the space of functions continuous in $t \in \mathbb{R}$ and holomorphic in $z \in \mathbb{C}^n$. The partial differential operator P on the left-hand side of (1.1) is a slight generalization of the Fuchsian partial differential operator of weight zero introduced by Baouendi and Goulaouic in [1]. The coefficients of P are assumed to be continuous in t and holomorphic in z for each fixed t.

Associated with the Fuchsian operator P is a *characteristic polynomial* in λ with parameter z defined by

$$C(\lambda, z) = \lambda^m + \sum_{j<m} a_{j,0}(0,z)\lambda^j.$$

The roots $\lambda_1(z), \dots, \lambda_m(z)$ of this polynomial are called *characteristic exponents.* All throughout this paper, we will assume that there is a constant $c > 0$ such that for all z in the closure of the disc $D_R = \{z \in \mathbb{C}^n : |z_i| < R\ (1 \le i \le n)\}$, we have

$$\operatorname{Re}\lambda_j(z) \le -c \qquad \text{for } j = 1, 2, \dots, m. \tag{1.2}$$

The function $\mu(t)$ appearing in (1.1) is assumed to be continuous, positive and monotonically increasing on some interval $(0,T)$ and further satisfies $\int_0^T (\mu(t)/t)\,dt < \infty$. Such a function is referred to in [4] as a *weight function.* We can easily verify that t^κ, $1/(-\log t)^{\kappa+1}$ and $1/[(-\log t)\log(-\log t)^\kappa]$ are weight functions provided $\kappa > 0$. We can also see that all weight functions tend to zero as t tends to zero.

Under the above assumptions, the author proved in [2] the unique existence of the solution u of (1.1) that is also continuous in t and holomorphic in z. The unique solution also has the property that $(\partial_t t)^j u$ ($j = 1, \dots, m$) is continuous in t and holomorphic in z for each fixed t. This regularity result may be obtained *a priori* [1].

The author later offered in [3] a sharp version of unique solvability, sharp in the sense that the right-hand side function f and the solution u are defined on exactly the same domain. The coefficients of the operator P, however, were assumed to be defined on a larger domain.

In this paper, we will give another proof of the unique solvability of Equation (1.1) using the Banach Fixed Point Theorem. This approach was used by Walter in [6] to come up with a simple proof of the well-known Cauchy-Kowalevsky Theorem. As a by-product of this method, the right-hand side function f, the coefficients and the solution u all have the same domain of definition.

Let us now describe the said domain of definition using the weight function $\mu(t)$. First, we define the function $\varphi(t)$ by

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s}\,ds \quad (0 \le t \le T).$$

This definition is possible because $\mu(t)$ is a weight function. Using the function $\varphi(t)$ and a parameter $\eta > 0$, we define the conical domain

$$\Omega_{\eta,T} = \{(t,z) \in [0,T] \times D_R : \, 0 \le \frac{1}{\eta}\varphi(t) < R - |z|\}.$$

Here, the norm $|z|$ of the complex number $z \in \mathbb{C}^n$ is taken to be $\max_{1\le j\le n} |z_j|$. Note that the quantity $R - |z| - \varphi(t)/\eta$ is positive at all interior points (t,z) of $\Omega_{\eta,T}$ and tends to zero as (t,z) approaches the boundary.

Let p be a fixed positive number. We denote by $X_p(\Omega_{\eta,T})$ the space of all functions that are continuous on $\Omega_{\eta,T}$ and holomorphic in z for each fixed t for which the quantity

$$\|u\|_p = \sup_{\Omega_{\eta,T}} |u(t,z)| \left(R - |z| - \frac{1}{\eta}\varphi(t)\right)^p$$

is finite. This space is a Banach space with the above norm [6, 5].

We now state the main result of this paper.

Theorem 1.1. *Assume that (1.2) holds. Then there exists $\eta > 0$ and $T > 0$ such that if the coefficients $a_{j,\alpha}$ are continuous and bounded on $\Omega_{\eta,T}$ and are holomorphic in z for each t, then for any f in $X_p(\Omega_{\eta,T})$, Equation (1.1) has a unique solution u defined in $\Omega_{\eta,T}$, and this solution satisfies $(\partial_t t)^j u \in X_p(\Omega_{\eta,T})$ for $j = 0, 1, \dots, m$.*

Remark 1.2. In [2] and [3], the right-hand side function f is assumed to be continuous in t, holomorphic in z, and bounded on $\Omega_{\eta,T}$. If such is the case, then f belongs in $X_p(\Omega_{\eta,T})$ for all $p > 0$. (We only have to note that the quantity $R - |z| - \frac{1}{\eta}\varphi(t)$ is most R in $\Omega_{\eta,T}$.) Thus, the above theorem is an improvement of the author's previous results.

2. Preliminaries

2.1. **Basic facts on Fuchsian equations.** We state some basic facts on ordinary differential equations of Fuchs type. We begin by considering the operator

$$P_m := (t\partial_t)^m + \sum_{j<m} a_{j,0}(0,z)(t\partial_t)^j. \tag{2.1}$$

This ordinary differential operator with parameter z is known as the Fuchsian principal part of the operator P given in (1.1). The following proposition is stated as Lemma 3 in [1].

Proposition 2.1. *Assume (1.2). Given any $f \in X_p(\Omega_{\eta,T})$, the equation $P_m u = f$ has a unique solution $u \in X_p(\Omega_{\eta,T})$. This unique solution is given by*

$$u(t,z) = \frac{1}{m!}\sum_{\sigma\in S_m}\int_0^1\cdots\int_0^1 \xi_1^{-\lambda_{\sigma(1)}-1}\cdots\xi_m^{-\lambda_{\sigma(m)}-1} f(\xi_1\cdots\xi_m t, z)\,d\xi_1\cdots d\xi_m,$$

where S_m denotes the group of permutations of $\{1,2,\ldots,m\}$.

The uniqueness of the solution of $P_m u = f$ allows us to define a linear operator $H_m : X_p(\Omega_{\eta,T}) \to X_p(\Omega_{\eta,T})$ by assigning to $f \in X_p(\Omega_{\eta,T})$ the uniquely obtained solution $u \in X_p(\Omega_{\eta,T})$, i.e., $H_m[f](t,z) = u(t,z)$.

Next, we consider the ordinary differential operator $(\partial_t t)^m \equiv (t\partial_t+1)^m$. The characteristic exponents of this operator are all equal to -1, hence the previous proposition applies.

Corollary 2.2. *Assume (1.2). Then for any $g \in X_p(\Omega_{\eta,T})$, the equation $(\partial_t t)^m u \equiv (t\partial_t + 1)^m u = g$ has a unique solution $u \in X_p(\Omega_{\eta,T})$, and this unique solution is given by*

$$u(t,z) \quad = \quad \int_0^1\cdots\int_0^1 g(\xi_1\cdots\xi_m t, z)\,d\xi_1\cdots d\xi_m. \tag{2.2}$$

As before, we appeal to the unique existence of the solution of $(\partial_t t)^m u = g$ to similarly define the operator $\mathcal{H}_m : X_p(\Omega_{\eta,T}) \to X_p(\Omega_{\eta,T})$ by $\mathcal{H}_m[g](t,x) := u(t,x)$. Note that for each $j = 1,\ldots,m-1$, we can also define an operator $\mathcal{H}_j : X_p(\Omega_{\eta,T}) \to X_p(\Omega_{\eta,T})$, which is nothing but the inverse of the operator $(\partial_t t)^j$. Note further that the integral representation of $\mathcal{H}_j[g]$ in the form (2.2) can be rewritten as

$$\mathcal{H}_j[g](t,z) = \frac{1}{t}\int_0^t \frac{1}{s_j}\int_0^{s_j}\cdots\frac{1}{s_2}\int_0^{s_2} g(s_1,z)\,ds_1 ds_2\cdots ds_j\,.$$

Finally, we state without proof a useful fact about the composition of the operators $(\partial_t t)^m$ and H_m. (See p. 465 of [1].)

Proposition 2.3. *There exists a constant $A > 0$ such that for any $g \in X_p(\Omega_{\eta,T})$, we have*

$$|(\partial_t t)^m H_m[g](t,z)| \leq A\,|g(t,z)| \quad \textit{for all } (t,z) \in \Omega_{\eta,T}\,.$$

This means that $(\partial_t t)^m H_m$ is a bounded operator on $X_p(\Omega_{\eta,T})$. This fact will be very handy in proving the main theorem.

2.2. **Some lemmata concerning functions in $X_p(\Omega_{\eta,T})$.** We state here some lemmata that will later serve as tools in establishing some estimates in the proof of the main theorem. The estimates in the proofs are analogous to those found in [6] and [5].

Lemma 2.4. *Let $g \in X_p(\Omega_{\eta,T})$. If the function $a(t,z)$ is continuous in t, holomorphic in z and bounded by A in $\Omega_{\eta,T}$, then the product ag is again in $X_p(\Omega_{\eta,T})$. Moreover,*

$$\|ag\|_p \leq A\,\|g\|_p \,.$$

Proof. This is evident from the definition of the norm in $X(\Omega_{\eta,T})$. □

Lemma 2.5. *Let $g \in X_p(\Omega_{\eta,T})$. Then for any integer $j \geq 1$, the function $\mathcal{H}_j[g](t,z)$ is again in $X_p(\Omega_{\eta,T})$. Moreover,*

$$\|\mathcal{H}_j[g]\|_p \leq \|g\|_p \,.$$

Proof. The continuity in t and holomorphy in z is clear, so we only need to show that the norm is finite. This may be seen using the integral representation of $\mathcal{H}_j(g)$. We have

$$\begin{aligned} |\mathcal{H}_j[g](t,z)| &\leq \int_0^1 \cdots \int_0^1 |g(\xi_1 \cdots \xi_j\, t, z)|\; d\xi_1 \cdots d\xi_j \\ &\leq \int_0^1 \cdots \int_0^1 \frac{\|g\|_p}{\left(R - |z| - \frac{1}{\eta}\varphi(\xi_1 \cdots \xi_j\, t)\right)^p}\, d\xi_1 \cdots d\xi_j \,. \end{aligned}$$

We then observe that the integrand is an increasing function of the ξ_i's. □

The following lemma is due to Nagumo. It gives a bound for $\partial_{z_i} g$ using the norm of g in $X_p(\Omega_{\eta,T})$.

Lemma 2.6 (Nagumo). *Let $g \in X_p(\Omega_{\eta,T})$. Then for all $(t,z) \in \Omega_{\eta,T}$ and $i = 1, \ldots, m$, we have*

$$|\partial_{z_i} g(t,z)| \leq \frac{C_p}{\left(R - |z| - \frac{1}{\eta}\varphi(t)\right)^{p+1}}\, \|g\|_p \,,$$

where the constant C_p is equal to $(p+1)(1+1/p)^p$.

For a proof of this estimate, see, e.g., [6].

Lemma 2.7. *Let $g \in X_p(\Omega_{\eta,T})$ and $1 \leq i \leq m$. If T is sufficiently small, then for any integers j, k with $j \geq k \geq 1$, the function $(\mu(t)\partial_{z_i})^k\, \mathcal{H}_j[g](t,z)$ is again in $X_p(\Omega_{\eta,T})$. Moreover, we have the estimate*

$$\|(\mu(t)\partial_{z_i})^k\, \mathcal{H}_j[g]\|_p \leq \eta^k\, \|g\|_p \,.$$

Proof. Again, the continuity in t and holomorphy in z is clear, so we only need to show that the norm is finite. In view Lemma 2.5, it is sufficient to consider the case when $j = k$.

We first consider the case when $k = 1$. Using the definition of $\mathcal{H}_1$, Nagumo's Lemma and the fact that $\varphi'(t) = \mu(t)/t$, we have

$$\begin{aligned}
\left|\left(\mu(t)\partial_{z_i}\mathcal{H}[g]\right)(t,z)\right| &\leq \frac{\mu(t)}{t}\int_0^t \frac{\|g\|_p}{\left(R-|z|-\frac{1}{\eta}\varphi(s)\right)^{p+1}}\,ds \\
&= \|g\|_p \frac{d}{dt}\int_0^t \frac{\varphi(t)-\varphi(s)}{\left(R-|z|-\frac{1}{\eta}\varphi(s)\right)^{p+1}}\,ds\,. \qquad (2.3)
\end{aligned}$$

Define two non-negative, monotononically increasing functions on $[0,T]$ with parameter $|z|$ by

$$h_1(t) = \int_0^t \frac{\varphi(t)-\varphi(s)}{\left(R-|z|-\frac{1}{\eta}\varphi(s)\right)^{p+1}}\,ds \quad \text{and} \quad h_2(t) = \eta\int_0^t \frac{ds}{\left(R-|z|-\frac{1}{\eta}\varphi(s)\right)^{p}}\,.$$

Note that $h_1(0) = h_2(0) = 0$. The two functions are not only continuous on $[0,T]$, they are in fact continuously differentiable in $(0,T)$. Moreover, because $0 \leq \frac{1}{\eta}\varphi(t) < R-|z|$ for all (t,z) in $\Omega_{\eta,T}$, we see that $h_1(t)$ is strictly less than $h_2(t)$ on $(0,T]$. Since $h_2(t)$ is easily checked to possess a finite derivative from the right, $h_1(t)$ does as well. Appealing to the continuity of the derivative, we can choose T to be sufficiently small such that $h_1'(t) \leq h_2'(t)$ for all t in $[0,T]$. (The derivatives at the endpoints should be understood as one-sided derivatives.)

In summary, if T is chosen small enough, we have

$$\begin{aligned}
h_2'(t) - h_1'(t) &= \frac{\eta}{\left(R-|z|-\frac{1}{\eta}\varphi(t)\right)^{p}} - \frac{\mu(t)}{t}\int_0^t \frac{ds}{\left(R-|z|-\frac{1}{\eta}\varphi(s)\right)^{p+1}}\,. \\
&\geq 0\,.
\end{aligned}$$

Combining this with (2.3), we arrive at

$$\left|\left(\mu(t)\partial_{z_i}\mathcal{H}[g]\right)(t,z)\right| \leq \frac{\eta\,\|g\|_p}{\left(R-|z|-\frac{1}{\eta}\varphi(t)\right)^{p}}\,,$$

as claimed.

Let us now consider the case when $k \geq 2$. From Nagumo's Lemma, we know that

$$\begin{aligned}
\left|\left(\mu(t)^k\partial_{z_i}^k\mathcal{H}_k[g]\right)(t,z)\right| &\leq \mu(t)^k\int_0^1\cdots\int_0^1 \frac{\|g\|_p\,d\xi_1\cdots d\xi_k}{\left(R-|z|-\frac{1}{\eta}\varphi(\xi_1\cdots\xi_k\,t)\right)^{p+k}} \\
&\leq \|g\|_p\prod_{j=1}^k\left[\mu(t)\int_0^1 \frac{d\xi_j}{\left(R-|z|-\frac{1}{\eta}\varphi(\xi_j t)\right)^{p/k+1}}\right],
\end{aligned}$$

in view of the fact that the integrand is an increasing function of the ξ_j's. We can then apply the result for $k = 1$ to each of the terms of the product to obtain the desired result. $\square$

Remark 2.8. We can easily generalize the above lemma to show that if α is a multi-index with $|\alpha| = k \geq 1$ and $j \geq k$, then $(\mu(t)\partial_z)^\alpha\mathcal{H}_j[g]$ is again in $X_p(\Omega_{\eta,T})$, and we have the estimate $\|(\mu(t)\partial_z)^\alpha\mathcal{H}_j[g]\|_p \leq \eta^{|\alpha|}\|g\|_p$.

3. Proof of Main Theorem

We first write the operator P as $P = P_m + Q_0 + Q_1$, where P_m is the Fuchsian principal part of P,

$$Q_0 = \sum_{j<m} b_j(t,z)\big(\partial_t t\big)^j, \tag{3.1}$$

and

$$Q_1 = \sum_{\substack{j<m, |\alpha|\geq 1\\ j+|\alpha|\leq m}} b_{j,\alpha}(t,z)\big(\mu(t)\partial_z\big)^\alpha \big(\partial_t t\big)^j. \tag{3.2}$$

Note that each $b_j(t,z)$ in (3.1) is a linear combination of the functions $a_{l,0}(t,z) - a_{l,0}(0,z)$, where $l \geq j$, so that by continuity, its modulus on $\Omega_{\eta,T}$ can be made small by choosing T small enough. Similarly, each $b_{j,\alpha}(t,z)$ in (3.2) is a linear combination of the functions $a_{l,\alpha}(t,z)$, where $l \geq j$, and hence is also bounded in $\Omega_{\eta,T}$.

Now, since we know *a priori* that any solution u of $Pu = f$ has extra regularity in t, we will seek a solution of the form $u = \mathcal{H}_m[g]$, where the function $g(t,z)$ is continuous in t and holomorphic in z for each fixed t. Equation (1.1) can now be rewritten as

$$P_m\mathcal{H}_m(g) = f - Q_0\mathcal{H}_m[g] - Q_1\mathcal{H}_m[g],$$

or equivalently (by Proposition 2.1 and Corollary 2.2) as,

$$g = \big(\partial_t t\big)^m H_m\big[f - Q_0\mathcal{H}_m[g] - Q_1\mathcal{H}_m[g]\big]. \tag{3.3}$$

We now define an operator $\mathcal{T}$ on the space $X_p(\Omega_{\eta,T})$ using the right-hand side of (3.3), i.e., for $g \in X_p(\Omega_{\eta,T})$, we define

$$\mathcal{T}g = \big(\partial_t t\big)^m H_m\big[f - Q_0\mathcal{H}_m[g] - Q_1\mathcal{H}_m[g]\big]. \tag{3.4}$$

We then see that part of Theorem 1.1 is implied by the following result. For the following theorem claims that a solution defined in $\Omega_{\eta,T}$ exists and there is only one such solution in the space $X_p(\Omega_{\eta,T})$.

Theorem 3.1. *The operator $\mathcal{T}$ maps the Banach space $X_p(\Omega_{\eta,T})$ into itself. Moreover, if T and η are small enough, then $\mathcal{T}$ is a contraction.*

Proof. We first take an arbitrary $g \in X_p(\Omega_{\eta,T})$ and show that $\mathcal{T}g$ is again in $X_p(\Omega_{\eta,T})$. In view of Proposition 2.3, it is sufficient to show that $f - Q_0\mathcal{H}_m[g] - Q_1\mathcal{H}_m[g]$ is in $X_p(\Omega_{\eta,T})$.

Let us consider each of the three terms separately. The first one is obvious because f is assumed to be in $X_p(\Omega_{\eta,T})$. As for the second term, we use the definition of the operator $\mathcal{H}_j[g]$ to rewrite it as follows:

$$\begin{aligned} Q_0\mathcal{H}_m[g] &= \sum_{j<m} b_j(t,z)\big(\partial_t t\big)^j\mathcal{H}_m[g] \\ &= \sum_{j<m} b_j(t,z)\mathcal{H}_{m-j}[g]. \end{aligned}$$

Applying Lemmata 2.4 and 2.5, we see that $Q_0\mathcal{H}_m[g]$ is in $X_p(\Omega_{\eta,T})$. Finally, we consider the last term. We also rewrite it as

$$\begin{aligned} Q_1\mathcal{H}_m[g] &= \sum_{\substack{j<m,|\alpha|\geq 1\\ j+|\alpha|\leq m}} b_{j,\alpha}(t,z)\big(\mu(t)\partial_z\big)^\alpha\big(\partial_t t\big)^j\mathcal{H}_m[g] \\ &= \sum_{\substack{j<m,|\alpha|\geq 1\\ j+|\alpha|\leq m}} b_{j,\alpha}(t,z)\big(\mu(t)\partial_z\big)^\alpha\mathcal{H}_{m-j}[g]. \end{aligned}$$

Since we always have $|\alpha| \leq m-j$ and each $b_{j,\alpha}(t,z)$ is bounded in $\Omega_{\eta,T}$, we can apply Lemmata 2.4 and 2.7 to claim that if T is small enough, $Q_1\mathcal{H}_m[g]$ is again in $X_p(\Omega_{\eta,T})$.

Having shown that $\mathcal{T}$ maps $X_p(\Omega_{\eta,T})$ into itself, we now show that if T and η are small enough, then $\mathcal{T}$ is a contraction. Let us take any two functions $g_1, g_2 \in X_p(\Omega_{\eta,T})$ and consider $\mathcal{T}(g_1-g_2)$. From (3.4), we see that

$$\begin{aligned} \mathcal{T}(g_1-g_2) &= -\big(\partial_t t\big)^m H_m\big[Q_0\mathcal{H}_m[g_1-g_2]+Q_1\mathcal{H}_m[g_1-g_2]\big] \\ &= -\big(\partial_t t\big)^m H_m\big[Q_0\mathcal{H}_m[g_1-g_2]\big]-\big(\partial_t t\big)^m H_m\big[Q_1\mathcal{H}_m[g_1-g_2]\big]. \end{aligned}$$

Let us estimate the two terms separately. Let $B_0(T)$ be a bound for all the $b_j(t,z)$'s and B_1 be a bound for all the $b_{j,\alpha}(t,z)$'s. (Note that we have indicated the dependence of B_0 in T; we can make it as small as we please by choosing a smaller T.) We apply the estimates in Lemmata 2.4 and 2.5 to the first term to obtain

$$\big\|-\big(\partial_t t\big)^m H_m\big[Q_0\mathcal{H}_m[g_1-g_2]\big]\big\|_p \leq A\sum_{j<m} B_0(T)\|g_1-g_2\|_p.$$

Similarly, we apply the estimates in Lemmata 2.4 and 2.7 to the second term to obtain

$$\big\|-\big(\partial_t t\big)^m H_m\big[Q_1\mathcal{H}_m[g_1-g_2]\big]\big\|_p \;\leq\; A\sum_{\substack{j<m,|\alpha|\geq 1\\ j+|\alpha|\leq m}} B_1\eta^{|\alpha|}\|g_1-g_2\|_p$$

Combining these two estimates, we see that there exists a constant $C>0$ for which

$$\|\mathcal{T}(g_1-g_2)\|_p \leq \big(B_0(T)+\eta\big)C\|g_1-g_2\|_p.$$

It is then clear that for sufficiently small values of T and η, the operator $\mathcal{T}$ is a contraction map on $X_p(\Omega_{\eta,T})$. □

Since $\mathcal{T}$ is a contraction on $X_p(\Omega_{\eta,T})$, the Banach Fixed Point Theorem implies the existence of a unique fixed point. We have thus shown that there exists a unique u in $X_p(\Omega_{\eta,T})$ that satisfies (1.1).

Suppose there exists another function w that is defined also in $\Omega_{\eta,T}$, continuous in t, holomorphic in z (for each fixed t) and satisfies (1.1) in $\Omega_{\eta,T}$. Take an arbitrary point $(t_0,z_0)\in\Omega_{\eta,T}$ and choose suitable numbers R', T' and η' such that $\Omega_{\eta',T'}$ contains (t_0,z_0) but the closure of $\Omega_{\eta',T'}$ is contained in $\Omega_{\eta,T}$.

Since w is now a bounded function in $\Omega_{\eta',T'}$, it is in $X_p(\Omega_{\eta',T'})$. Obviously, so is the previously obtained solution u. By applying the arguments in the proof of Theorem 3.1 in the space $X_p(\Omega_{\eta',T'})$, we see that w and u must coincide in

$\Omega_{\eta',T'}$). In particular, they must coincide at the point (t_0, z_0). Since (t_0, z_0) was arbitrarily chosen, we see that $w \equiv u$ in the whole of $\Omega_{\eta,T}$. This completes the proof of Theorem 1.1.

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