数理解析研究所講究録 1431 巻 2005 年 45-54

Radon transforms of Constructible functions on Grassmann manifolds

東京大学大学院数理科学研究科
(Graduate School of Mathematical Sciences, the University of Tokyo)
松井 優 (Yutaka Matsui)

Introduction

A constructible function ϕ on a real analytic or complex manifold X is a $\mathbb{Z}$-valued function which is constant along a stratification. We can choose a stratification according to the problem under consideration, so we work with subanalytic stratifications here.

In [6], P. Schapira defined Radon transforms of constructible functions. This is a kind of integral transformations. We consider the following diagram:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

Here X and Y are real analytic or complex manifolds, S is a locally closed subanalytic subset of $X \times Y$, and f and g are real or complex analytic maps, respectively. Then we can define Radon transform $\mathcal{R}_S(\phi)$ of a constructible function ϕ on X by

$$\mathcal{R}_S(\phi) = \int_g f^*\phi.$$

In [6], P. Schapira obtained a formula for $\mathcal{R}_S$ in general situation. This formula gives an inversion formula for Radon transforms of constructible functions from a real projective space to its dual in the case where the whole dimension is odd. Here inversion means left inverse. We can, that is, reconstruct a constructible function ϕ on the projective space from its Radon transform $\mathcal{R}_S(\phi)$. As a result, we can reconstruct the original subanalytic set K from the knowledge of the topological Euler numbers $\chi(K \cap H)$ for all affine hyperplane H.

In this lecture, we study Radon transforms of constructible functions from $X = F_{n+1}(p)$ to $Y = F_{n+1}(q)$. We denote it by $\mathcal{R}_{(n+1;p,q)}$. Here $F_{n+1}(p)$ is the Grassmann manifold, that is, the set of all the p dimensional subspaces in an $n+1$ dimensional vector space. We will construct inversion for Radon transformation. When p is not equal to 1, the assumptions of Schapira's formula are not satisfied. So our situation is more complicated than Schapira's situation.

We first recall constructible functions and Schapira's formula in the general case. Since the calculation in the proof of Schapira's formula is essentially important to study our situation, we prove Schapira's formula in this lecture.

In Section 2, we state only our results on Radon transforms of constructible functions on Grassmann manifolds. First we modify Schapira's formula in the general case under the almost same assumptions as Schapira. This gives an inversion formula for Radon transformation $\mathcal{R}_S$. We can apply this formula to Radon transformation $\mathcal{R}_{(n+1;1,q)}$. Second, we consider Radon transformation $\mathcal{R}_{(n+1;p,q)}$ for $p \neq 1$. We obtain an inversion formula for this by modifying the kernel function of this inversion transform under suitable conditions of p and q. Moreover we prove that Radon transformation $\mathcal{R}_{(n+1;p,n+1-p)}$ is the non-trivial isomorphism between $CF(F_{n+1}(p))$ and its dual $CF(F_{n+1}(n+1-p))$. Here $CF(X)$ is the set of constructible functions on X.

1 Preliminaries

1.1 Constructible functions

We recall the notation and results on constructible functions without proofs. For more details, we refer to [4].

For the simplicity, let X be a real analytic manifold.

Definition 1.1. A function $\phi : X \to \mathbb{Z}$ is set to be constructible if there exists a locally finite family of compact subanalytic contractible subsets $\{K_i\}_i$ of X such that

$$\phi = \sum_i c_i \mathbf{1}_{K_i}.$$

Here $c_i \in \mathbb{Z}$ and $\mathbf{1}_A$ is the characteristic function of the subset A.

We denote by $CF(X)$ the abelian group of all the constructible functions on X, and by $\mathscr{CF}_X$ the sheaf $U \mapsto CF(U)$ on X.

Example 1.2. Let $\mathbf{D}^b_{\mathbb{R}-c}(X)$ be the derived category of the category of complexes of $\mathbb{R}$-constructible sheaves and $F \in Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$ (the base ring is a field k with characteristic zero). Then its local Euler-Poincaré index

$$\chi(F)(x) = \sum_j (-1)^j \dim H^j(F)_x$$

is a constructible function.

From now on, χ denotes the local Euler-Poincaré index.

Theorem 1.3 (([4, Theorem 9.7.11])). *For any $\phi \in CF(X)$, there exists an $F \in Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$ such that $\phi = \chi(F)$.*

Next, we recall operations on constructible functions [4]. These operations are induced by operations of $\mathbf{K}_{\mathbb{R}-c}(X)$ through the Euler-Poincaré index χ.

Definition 1.4. Let X and Y be two real analytic manifolds, and $f : Y \to X$ a real analytic map.

(i) The inverse image : Let $\phi \in CF(X)$. We set

$$f^*\phi(y) = \phi(f(y)).$$

Note that if $\phi = \chi(F)$, then $f^*\phi = \chi(f^{-1}F)$.

(ii) The integral : Let $\phi \in CF(X)$. Assume that ϕ is represented as $\phi = \chi(F) = \sum_i c_i \mathbf{1}_{K_i}$. Here $F \in Ob(\mathrm{D}^b_{\mathbb{R}-c}(X))$, and $\{K_i\}$ is a locally finite family of compact subanalytic contractible subsets. Assume moreover that ϕ has compact support. Then we set

$$\int_X \phi = \sum_i c_i = \chi(R\Gamma(X; F)).$$

(iii) The direct image : Let $\psi \in CF(Y)$. Assume that $f\colon \mathrm{supp}(\psi) \to X$ is proper. Here $\mathrm{supp}(\psi)$ denotes a support of ψ. We set

$$\left(\int_f \psi\right)(x) = \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}).$$

Note that if $\psi = \chi(G)$ and f is proper on $\mathrm{supp}(G)$, then $\displaystyle\int_f \psi = \chi(Rf_! G)$.

Remark 1.5. Let A be a locally closet subset of a manifold X. Then the integral $\displaystyle\int_X \mathbf{1}_A$ is not the usual integral, but a kind of topological integrals. By Theorem 1.3 and the definition, we have the following equalities:

$$\int_X \mathbf{1}_A = \chi(R\Gamma(X; k_A)) = \chi(R\Gamma(X; i_! i^{-1} k_X)) = \chi(R\Gamma_c(A; k_A)) = \chi_c(A).$$

Here k is $\mathbb{R}$ or $\mathbb{C}$, $i : A \to X$ is an inclusion morphism and χ_c is the topological Euler-Poincaré index with compact supports.

By the additivity of χ_c, we have some examples;

$$\int_{\mathbb{R}} \mathbf{1}_{[0,1]} = 1, \qquad \int_{\mathbb{R}} \mathbf{1}_{[0,1)} = 0, \qquad \int_{\mathbb{R}} \mathbf{1}_{(0,1)} = -1.$$

Proposition 1.6.

(i) *Inverse and direct images have functorial properties. Precisely, if $f\colon Y \to X$ and $g\colon Z \to Y$ are real analytic maps, then we have;*

(a) $g^* \circ f^* = (f \circ g)^*$, (b) $\displaystyle\int_{f\circ g} = \int_f \circ \int_g$.

(ii) *Consider a Cartesian diagram of morphisms of real analytic manifolds:*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

Let $\psi \in CF(Y)$. Suppose that f is proper on $\mathrm{supp}\psi$. *Then we have*

$$g^* \int_f \psi = \int_{f'} (h^*\psi).$$

1.2 Radon transforms of constructible functions and Schapira's formula

We recall the definition of Radon transforms of constructible functions and Schapira's formula ([6]).

Let X and Y be two real analytic manifolds, and S a locally closed subanalytic subset of $X \times Y$. Denote by p_1 and p_2 the first and second projections defined on $X \times Y$, and by f and g the restrictions of p_1 and p_2 to S respectively:

$$\begin{tikzcd} & X \times Y \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\ & \cup \; S \arrow[dl, "f"] \arrow[dr, "g"'] & \\ X & & Y. \end{tikzcd} \tag{1}$$

We assume;

$$p_2 \text{ is proper on } \bar{S} \text{ (the closure of } S \text{ in } X \times Y). \tag{2}$$

Definition 1.7. For a $\phi \in CF(X)$, we set

$$\mathcal{R}_S(\phi) = \int_g f^*\phi = \int_{p_2} 1_S(p_1^*\phi).$$

We call $\mathcal{R}_S(\phi)$ the Radon transform of ϕ.

Let $S' \subset Y \times X$ be another locally closed subanalytic subset. We denote again by p_2 and p_1 the first and second projections defined on $Y \times X$, by f' and g' the restrictions of p_1 and p_2 to S', and by r the projection $S \underset{Y}{\times} S' \to X \times X$.

Then Schapira posed the following assumptions:

$$p_1' \text{ is proper on } \bar{S}' \text{ (the closure of } S' \text{ in } Y \times X), \tag{3}$$

$$\exists \lambda, \mu \in \mathbb{Z} \text{ s.t. } \lambda \neq \mu \text{ and } \chi(r^{-1}(x,x')) = \begin{cases} \lambda & (x \neq x'), \\ \mu & (x = x'), \end{cases} \tag{4}$$

where χ is the topological Euler-Poincaré index. We use the same symbol χ as the local Euler-Poincaré index.

In this lecture, we refer to the assumption (4) as Schapira's condition.

Under the notation above, Schapira's formula state as follows.

Theorem 1.8 (([6, Theorem 3.1])). *Assume* (2), (3) *and* (4). *Then, for any* $\phi \in CF(X)$, *we have*

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = (\mu - \lambda)\phi + \left(\int_X \lambda\phi\right) 1_X.$$

Proof. For the convenience of readers, we recall the proof of this theorem. Denote by h and h' the projections from $S \underset{Y}{\times} S'$ to S and S' respectively.

Consider the following diagram:

$$\begin{tikzcd} & & S \underset{Y}{\times} S' \arrow[dl, "h"'] \arrow[d, "r"] \arrow[dr, "h'"] & & \\ & S \arrow[dl, "f"'] \arrow[dr, "g"] & X \times X \arrow[dll, "q_1"] \arrow[drr, "q_2"'] & S' \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\ X & & Y & & X. \end{tikzcd}$$

Since the square

$$\begin{tikzcd} S \underset{Y}{\times} S' \arrow[r, "h'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & S' \arrow[d, "g'"] \\ S \arrow[r, "g"] & Y \end{tikzcd}$$

is of Cartesian, we have

$$\begin{aligned} \mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) &= \int_{f'} (g'^* \int_g (f^*\phi)) = \int_{f' \circ h'} ((f \circ h)^* \phi) = \int_{q_2} \int_r r^* q_1^* \phi \\ &= \int_{q_2} k(x, x') q_1^* \phi. \end{aligned}$$

Here we have

$$k(x, x') = \int_r r^* \mathbf{1}_{X \times X} = \int_r \mathbf{1}_{S \underset{Y}{\times} S'}.$$

By Schapira's condition (4), we have

$$\int_r \mathbf{1}_{S \underset{Y}{\times} S'} = \mu \mathbf{1}_{\Delta_X} + \lambda \mathbf{1}_{X \times X \setminus \Delta_X} = (\mu - \lambda) \mathbf{1}_{\Delta_X} + \lambda \mathbf{1}_{X \times X},$$

where Δ_X is the diagonal of $X \times X$.

Since $\int_{q_2} \mathbf{1}_{\Delta_X}\ q_1^* \phi = \phi$ and $\int_{q_2} \mathbf{1}_{X \times X}\ q_1^* \phi = \int_X \phi$, we obtain the result. □

In [6], Schapira applied this formula to correspondences of real flag manifolds; that is, we consider the following diagram called the correspondence;

$$\begin{tikzcd} & F_{n+1}(1, q) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ F_{n+1}(1) & & F_{n+1}(q), \end{tikzcd} \tag{5}$$

where f and g are projections.

We set $\mathcal{R}_{(n+1;1,q)} = \mathcal{R}_S$ and $\mathcal{R}_{(n+1;q,1)} = \mathcal{R}_{S'}$, where $S = F_{n+1}(1, q)$ and $S' = F_{n+1}(q, 1)$. Then this situation satisfies the assumptions of Schapira's formula, because we have

$$r^{-1}(x, x') \simeq \begin{cases} F_{n-1}(q-2) & (x \neq x'), \\ F_n(q-1) & (x = x'). \end{cases}$$

Therefore we can apply Theorem 1.8 to this case.

Proposition 1.9 (([6, Proposition4.1])). *Consider the correspondence* (5). *For any $\phi \in CF(F_{n+1}(1))$, we have*

$$\begin{aligned} &\mathcal{R}_{(n+1;q,1)} \circ \mathcal{R}_{(n+1;1,q)}(\phi) \\ &= (\mu_n(q-1) - \mu_{n-1}(q-2))\phi + \mu_{n-1}(q-2) \left(\int_{F_{n+1}(1)} \phi \right) \mathbf{1}_{F_{n+1}(1)}. \end{aligned}$$

In particular, if n is odd, we obtain an inversion formula for $\mathcal{R}_{(n+1;1,n)}$.

The topological meaning of this inversion formula is seemed to be interesting. This inversion formula means that we can reconstruct the original subanalytic set from the knowledge of the Euler-Poincaré indices of all its affine slices.

2 Inversion transforms of Radon transforms of constructible functions

We generalize (5); that is, we consider the following diagrams:

$$\begin{tikzcd}
 & F_{n+1}(p)\times F_{n+1}(q) \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\
 & F_{n+1}(p,q) \arrow[u, phantom, "\cup"] \arrow[dl, "f"] \arrow[dr, "g"'] & \\
F_{n+1}(p) & & F_{n+1}(q).
\end{tikzcd} \tag{6}$$

We set $X = F_{n+1}(p)$, $Y = F_{n+1}(q)$ and $S = F_{n+1}(p,q)$.
We consider the following problems;

(i) an inversion formula for $\mathcal{R}_{(n+1;1,q)}$ in the case where n is even or $q \neq n$,

(ii) an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ in the case where $1 < p$ and $1 < q$.

Namely, we consider the reconstruction of ϕ from $\mathcal{R}_S(\phi)$ on Grassmann manifolds.

We remark that Schapira already considered this diagram (6) in [6], but he could not obtain results for these problems.

2.1 A minor modification of Schapira's formula

First, we modify Schapira's formula. By modifying the kernel function of the transposed transformation $\mathcal{R}_{S'}$, we can obtain an inversion formula in general case under the almost same assumtions as Schapira's formula.

Definition 2.1. For a $\psi \in CF(Y)$, we set

$$\mathcal{R}^{-1}(\psi) = \int_{p_1} (\mu \mathbf{1}_{S'} - \lambda \mathbf{1}_{Y\times X})(p_2^*\psi).$$

Definition 2.2. We define the transposed set of S by

$${}^tS = \{(y,\ x) \in Y \times X \mid (x,\ y) \in S\ \}.$$

In this section, we assume Schapira's assumptions (2), (3), (4) and the following assumption:

$${}^tS = S'. \tag{7}$$

Proposition 2.3. *Let $\phi \in CF(X)$. Then we have*

$$\mathcal{R}^{-1} \circ \mathcal{R}_S(\phi) = \mu(\mu - \lambda)\phi.$$

In particular, if $\mu(\mu - \lambda)$ is not zero, we can reconstruct the original constructible function ϕ from its Radon transform $\mathcal{R}_S(\phi)$ by dividing the last term by this constant $\mu(\mu - \lambda)$.

We apply this result to the complex or real Grassmann manifolds. We recall the Euler-Poincaré index of the Grassmann manifold.

In the complex case, we have

$$\mu_n(p) = \binom{n}{p}. \tag{8}$$

In the real case, we have

$$\mu_n(p) = \begin{cases} 0 & (\text{if } p(n-p) \text{ is odd}), \\ \begin{pmatrix} E\left(\frac{n}{2}\right) \\ E\left(\frac{p}{2}\right) \end{pmatrix} & (\text{if } p(n-p) \text{ is even}). \end{cases} \tag{9}$$

Here $E\left(\frac{n}{2}\right)$ denotes the integral part of $\frac{n}{2}$, $\binom{a}{b}$ is the binomial coefficient.

We consider the correspondence (5). Then the assumptions (2), (3), (4) and (7) are satisfied. We remark that

$$\mu = \mu_n(q-1), \qquad \lambda = \mu_{n-1}(q-2).$$

We consider the conditions of q that $\mu(\mu-\lambda) \neq 0$ from (8) and (9).

Therefore we can apply Proposition 2.3 to the Grassmann cases.

Proposition 2.4. *We have $\mu(\mu-\lambda) \neq 0$ if either one of the following conditions are satisfied;*

(i) $q > 1$ *under the complex Grassmann case,*

(ii) q *is odd and* $1 < q < n+1$ *under the real Grassmann case.*

In particular, then we obtain an inversion formula for $\mathcal{R}_{(n+1;1,q)}$.

2.2 Inversion formulas on Grassmann manifolds

For $p < q$, we consider the following diagram:

$$\begin{tikzcd}
 & F_{n+1}(p) \times F_{n+1}(q) \arrow[dddl, "p_1"'] \arrow[dddr, "p_2"] & \\
 & \cup & \\
 & F_{n+1}(p,q) \arrow[dl, "f"] \arrow[dr, "g"'] & \\
F_{n+1}(p) & & F_{n+1}(q).
\end{tikzcd}$$

We set $X = F_{n+1}(p)$, $Y = F_{n+1}(q)$ and $S = F_{n+1}(p,q)$.

We remark that Schapira's condition (4) is not satisfied if $1 < p$. This is because $r^{-1}(x_1, x_2)$ is $(p+1)$-th cases according to $\dim(x_1 \cap x_2)$.

We introduce new sets in order to construct to an inversion transformation for $\mathcal{R}_{(n+1;p,q)}$.

Definition 2.5. We set

(i) $S_i = \{(y,x) \in Y \times X \mid \dim(y \cap x) = i\}$ for $i = 0, 1, \cdots p$,

(ii) $Z_j = \{(x_1, x_2) \in X \times X \mid \dim(x_1 \cap x_2) = j\}$ for $j = 0, 1, \cdots p$.

Consider the following diagram:

$$\begin{tikzcd}
 & & S \underset{Y}{\times} S_i \arrow[dl, "h"'] \arrow[d, "r"] \arrow[dr, "h'"] & & \\
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & X \times X \arrow[dll, "q_1"] \arrow[drr, "q_2"'] & S_i \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\
X & & Y & & X.
\end{tikzcd}$$

Denote by h and h' the projections from $S \underset{Y}{\times} S_i$ to S and S' respectively.

Note that $Z_p = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ and we have $\int_{q_2} \mathbf{1}_{Z_p} q_1^* \phi = \phi$. In order to apply the same argument as in the proof of Theorem 1.8 and Section 2.1, we modify the kernel such that $\int_r$(kernel) is equal to $\mathbf{1}_{Z_p}$.

We can calculate $\int_r \mathbf{1}_{S \underset{Y}{\times} S_i}$ by Schubert calculus [3]. This calculation is complicated but elementary, so we omit this calculation in this lecture. We set

$$\left(\int_r \mathbf{1}_{S \underset{Y}{\times} S_i}\right)(x_1, x_2) = \sum_{j=0}^{p} \left(\int_{S \underset{Y}{\times} S_i} \mathbf{1}_{r^{-1}(x_1, x_2) \cap r^{-1}(Z_j)}\right) \cdot \mathbf{1}_{Z_j} =: \sum_{j=0}^{p} c_{ij} \mathbf{1}_{Z_j}.$$

Note that we can calculate these c_{ij} concretely.

Here, we denote by $C^{p,q}$ the square matrix $(c_{ij})_{0 \le i,j \le p}$ of size $(p+1)$. Since this is the lower triangular matrix, we have

$$|\det C^{p,q}| = \prod_{j=0}^{p} \mu_{n+1-2p+j}(q-p) \tag{10}$$

in both cases. In particular it is $\mathbb{Z}$-valued.

In the argument here after, we consider the case where $\det C^{p,q} \neq 0$. We derive the following conditions for $\det C^{p,q} \neq 0$ from (8) and (9):

(i) $p + q \le n + 1$ in the complex Grassmann case,

(ii) $p + q \le n + 1$ and $q - p$ is even in the real Grassmann case.

Under the preliminaries above, we define the kernel function of an inversion formula for $\mathcal{R}_{(n+1;p,q)}$.

We obtain the equation

$$C^{p,q} \begin{pmatrix} \mathbf{1}_{Z_0} \\ \mathbf{1}_{Z_1} \\ \vdots \\ \mathbf{1}_{Z_p} \end{pmatrix} = \begin{pmatrix} \int_r \mathbf{1}_{S \underset{Y}{\times} S_0} \\ \int_r \mathbf{1}_{S \underset{Y}{\times} S_1} \\ \vdots \\ \int_r \mathbf{1}_{S \underset{Y}{\times} S_p} \end{pmatrix}.$$

When $\det C^{p,q} \neq 0$, we can solve this equation with respect to $\mathbf{1}_{Z_p}$ by Cramer's formula:

$$\det C^{p,q} \cdot \mathbf{1}_{Z_p} = \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \int_\gamma \mathbf{1}_{S \underset{Y}{\times} S_0} \\ c_{10} & c_{11} & \ddots & \vdots & \int_\gamma \mathbf{1}_{S \underset{Y}{\times} S_1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \int_\gamma \mathbf{1}_{S \underset{Y}{\times} S_{p-1}} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \int_\gamma \mathbf{1}_{S \underset{Y}{\times} S_p} \end{pmatrix}.$$

Definition 2.6. If $\det C^{p,q} \neq 0$, we set

$$K_{p,q} = \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \mathbf{1}_{S_0} \\ c_{10} & c_{11} & \ddots & \vdots & \mathbf{1}_{S_1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \mathbf{1}_{S_{p-1}} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \mathbf{1}_{S_p} \end{pmatrix}.$$

Then we can define $\mathcal{R}^{-1}(\psi)$ for a $\psi \in CF(F_{n+1}(q))$ by

$$\mathcal{R}^{-1}(\psi) = \int_{p_1} K_{p,q} \cdot (p_2^* \psi).$$

The main result is:

Theorem 2.7. *If $\det C^{p,q} \neq 0$, then for any $\phi \in CF(F_{n+1}(p))$ we have*

$$\mathcal{R}^{-1} \circ \mathcal{R}_{(n+1;p,q)}(\phi) = \det C^{p,q} \cdot \phi.$$

This means that we can reconstruct the original constructible function ϕ from its Radon transform $\mathcal{R}_{(n+1;p,q)}(\phi)$ by dividing the last term by the constant $\det C^{p,q}$. In particular, we obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ if either one of the following conditions are satisfied;

(i) $p+q \leq n+1$ *under the complex Grassmann case,*

(ii) $p+q \leq n+1$ *and $q-p$ is even under the real Grassmann case.*

Remark 2.8. In the argument above, we consider only the case where $p < q$. However, we obtain results in other cases.

When $p = q$, the inversion formula is clear because $\mathcal{R}_{(n+1;p,q)} = id_{CF(F_{n+1}(p))}$.

When $p > q$, by the duality, we can obtain

$$\begin{cases} (n+1-p) < (n+1-q), \\ F_{n+1}(n+1-p) \simeq F_{n+1}(p), \\ F_{n+1}(n+1-q) \simeq F_{n+1}(q). \end{cases}$$

Then we have only to consider the result in the case of $p < q$ by these dualities of Grassmann manifolds. Namely, we obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ if $p+q \geq n+1$ in the complex case or if $p+q \geq n+1$ and $p-q$ is even in the real case.

For general p and q, our inversion formulas are not always inverse transforms of Radon transformations. $\mathcal{R}^{-1}$ gives only left inverse transformation. However, we can show that $\mathcal{R}^{-1}$ gives right inverse transformation by the same calculation when $p+q=n+1$.

Theorem 2.9. *Let $p+q=n+1$ hold. The inversion transformation $\mathcal{R}^{-1}$ defined in Definition 2.6 gives the inverse transformation for Radon transformation $\mathcal{R}_{(n+1;p,q)}$. Namely, the Radon transformation $\mathcal{R}_{(n+1;p,q)}$ is the non-trivial isomorphism from $CF(F_{n+1}(p))$ to $CF(F_{n+1}(q))$ up to constant if either one of the following conditions are satisfied;*

(i) *the complex case,*

(ii) *$p+q=n+1$ and $q-p$ is even in the real case.*

References

[1] J.L. Brylinski. *Transforms canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes tregonométriques. Géométrie et analyse microlocales.* Astérisque No. 140-141 (1986), pp.3-134, 251.

[2] L. Ernström. *Topological Radon transforms and the local Euler obstruction.* Duke Math. Journal **76** (1994), pp. 1-21.

[3] W. Fulton. *Young Tableaux.* Cambridge University Press, 1997.

[4] M. Kashiwara and P. Schapira. *Sheaves on manifolds.* Grundlehren der Math. Wiss. **292**, Springer-Verlag, Berlin, 1990.

[5] C. Marastoni. *Grassmann duality for $\mathcal{D}$-modules.* Ann. Sci. École Norm. Sup. **31** (1998), no.4, pp.459-491.

[6] P. Schapira. *Tomography of constructible functions.* Lecture Notes Computer Science **948**, Springer Berlin (1995), pp. 427-435.