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The solution formula of the elastic equation in plane-stratified media and its applications for inverse problems

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1 Introduction and main result

Our problem originates from a simplified model of an experiment conducted by geophysicists. We cannot directly observe the structure inside the earth. Then, for example, we perform the following experiment in order to guess it: We create an artificial explosion at a certain point near the earth's surface. Waves generated by the explosion travel in the earth. We observe the waves on the earth's surface, and determine the structure inside the earth from the observation data.

We consider this problem, in particular, in the case when the earth consists of some layers. This problem has been studied by Bartoloni-Lodovici-Zirilli [1], Fatone-Maponi-Pignotti-Zirilli [2], Hansen [3], for instance. They deal with the wave equation as the equation which describes the behavior of waves in media. The wave equation is the most fundamental equation in equations which describe the behavior of the waves. However, the earthquake waves are described by the elastic equation rather than the wave equation. Then, we deal with the elastic equation. On the other hand, we consider the situation that the earth consists of two layers as the simplest case. Therefore, our problem is as follows.

Assume that two media, Medium 1 and Medium 2, are laying in a half-space, and the interface wall is parallel to the boundary of the half-space (see Figure 1). We assume that the speeds of the (primary and shear) waves and the density of the medium in Medium 1 are known, but the width of Medium 1, the speeds of the waves and the density of the medium in

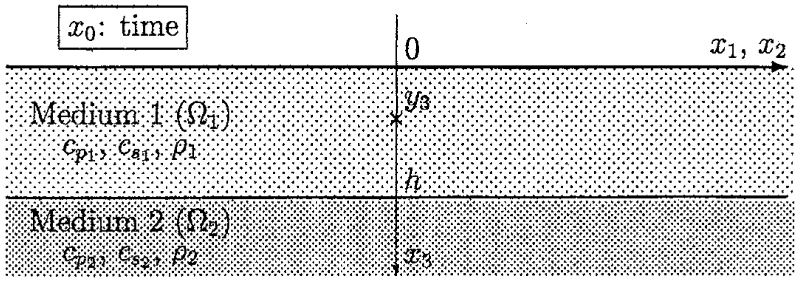

Figure 1: The situation which we consider.

Medium 2 are unknown. Under this situation, we try to identify these unknown data by using the known data or the data which can be observed near the boundary.

Now, we introduce the notations and formulate the problem above. Let us write $x' = (x_0, x_1, x_2)$, $x'' = (x_1, x_2, x_3)$ and $x''' = (x_1, x_2)$ for the coordinate $x = (x_0, x_1, x_2, x_3)$ in $\mathbb{R}^4$. The variable x_0 plays the role of the time and x'' the physical space. We introduce x' for short notation when we apply the Fourier-Laplace transformations with respect to (x_0, x_1, x_2).

Let $h > 0$, and $\Omega_1 := \{x'' \in \mathbb{R}^3 : 0 < x_3 < h\}$, $\Omega_2 := \{x'' \in \mathbb{R}^3 : x_3 > h\}$. The constant h describes the width of Medium 1, and Ω_k Medium k for $k = 1, 2$. We set $D_{x_j} := (1/i)(\partial/\partial x_j)$, $\nabla_{x''} = (D_{x_1}, D_{x_2}, D_{x_3})$, $\Delta_{x''} = D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2$. Let c_{p_k}, c_{s_k}, and ρ_k be positive real numbers and set

$$P_k(D_x)u := -D_{x_0}^2 u + (c_{p_k}^2 - c_{s_k}^2)\nabla_{x''}(\nabla_{x''} \cdot u) + c_{s_k}^2 \Delta_{x''} u,$$

$$B_k(D_x) := i\rho_k \begin{bmatrix} c_{s_k}^2 D_{x_3} & 0 & c_{s_k}^2 D_{x_1} \\ 0 & c_{s_k}^2 D_{x_3} & c_{s_k}^2 D_{x_2} \\ (c_{p_k}^2 - 2c_{s_k}^2) D_{x_1} & (c_{p_k}^2 - 2c_{s_k}^2) D_{x_2} & c_{p_k}^2 D_{x_3} \end{bmatrix}$$

for $k = 1, 2$. We assume $c_{p_k} > c_{s_k}$ for $k = 1$, 2. The positive number c_{p_k} describes the speed of the primary waves, c_{s_k} the speed of the shear waves, ρ_k the density of the medium in Ω_k. Suppose $0 < y_3 < h$. Set $y'' := (0, 0, y_3) \in \mathbb{R}^3_{x''}$, and $y := (0, y'') \in \mathbb{R}^4$.

We discuss the following equations:

$$P_1(D_x)G(x) = \delta(x-y)I, \quad x_0 \in \mathbb{R},\ x'' \in \Omega_1, \tag{1}$$
$$P_2(D_x)G(x) = 0, \quad x_0 \in \mathbb{R},\ x'' \in \Omega_2, \tag{2}$$
$$B_1(D_x)G(x)|_{x_3=+0} = 0, \quad x' \in \mathbb{R}^3, \tag{3}$$
$$G(x)|_{x_3=h-0} = G(x)|_{x_3=h+0}, \quad x' \in \mathbb{R}^3, \tag{4}$$
$$B_1(D_x)G(x)|_{x_3=h-0} = B_2(D_x)G(x)|_{x_3=h+0}, \quad x' \in \mathbb{R}^3, \tag{5}$$

where I is the identity matrix of order 3. These equations describe the situation that the initial data are $(\delta, 0, 0)$, $(0, \delta, 0)$, $(0, 0, \delta)$ at a point $y'' \in \Omega_1$ at time $x_0 = 0$ with the boundary condition (3) and the interface or transmission conditions (4) and (5). The equation (4) expresses the continuity of the displacement of waves on the interface wall, and (5) the continuity of the stress. We remark that we solve the equations (1)–(5) by the Fourier-Laplace transformations (see Section 2). In this regard, in order not to vanish the Lopatinski's determinant, we assume some inequalities for the constants c_{s_k}, c_{p_k}, ρ_k (see Nagayasu [6]).

The following main result says that except the special case we can reconstruct the width h of Ω_1, the speeds c_{p_2}, c_{s_2} of the waves and the density ρ_2 of the medium in Ω_2 from the observation data $G(x)|_{x_3=0}$ when the speeds c_{p_1}, c_{s_1} of the waves and the density ρ_1 of the medium in Ω_1 are known.

Main result. *Let c_{p_1}, c_{s_1}, ρ_1 and y_3 be given. Assume that the observation data $G(x)|_{x_3=+0}$ are given, where $G(x)$ denotes the solution of the equations (1)–(5). Then the constants c_{p_2}, c_{s_2}, ρ_2 are expressed with the known data. Moreover, the constant h is expressed with the known data unless $G(x)|_{x_3=+0} \equiv \widetilde{G}(x)|_{x_3=+0}$. Here $\widetilde{G}(x)$ describes the waves in the situation that only one medium Medium 1 is laying in the half-space, that is, $\widetilde{G}(x)$ is the solution of*

$$\begin{cases} P_1(D_x)\widetilde{G}(x) = \delta(x-y)I, & x_0 \in \mathbb{R},\ x'' \in \mathbb{R}^3, \\ B_1(D_x)\widetilde{G}(x)|_{x_3=+0} = 0, & x_0 \in \mathbb{R},\ (x_1, x_2) \in \mathbb{R}^2. \end{cases}$$

On the other hand, if $G(x)|_{x_3=+0} \equiv \widetilde{G}(x)|_{x_3=+0}$ then h is not identified.

We remark that this main result is the result in Nagayasu [6]. Some of the proof in Nagayasu [6] leave the proof in Nagayasu [5]. Then, in this paper, by mixing the proofs in [5] and [6], we prove shortly in most cases.

2 The solution formula

In this section, we solve the mixed problem (1)–(5). We mainly refer to Matsumura [4], Sakamoto [7], and Shimizu [8].

We first rewrite these equations. We define $E_1(x)$ by the fundamental solution of the forward Cauchy problem for $P_1(D_x)$ in the whole physical space $\mathbb{R}^3_{x''}$, namely, the inverse Fourier-Laplace transform of $P_1(\xi + i\eta)^{-1}$ in the sense of distribution:

$$E_1(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4_\xi} e^{ix\cdot(\xi+i\eta)} P_1(\xi + i\eta)^{-1} d\xi,$$

where we determine η so as to be able to define $E_1(x)$ as the distribution (see Shimizu [8]).

We put $F_1(x)$ and $F_2(x)$ by

$$F_1(x) := E_1(x-y) - G(x), \quad x'' \in \Omega_1, \tag{6}$$

$$F_2(x) := G(x), \quad x'' \in \Omega_2, \tag{7}$$

respectively. Since the distribution $E_1(x-y)$ describes the first propagation of the waves due to a point source, the distribution $F_1(x)$ describes the propagation in Ω_1 of the second waves caused by the first waves, the boundary wall $\{x_3 = 0\}$, and the interface wall $\{x_3 = h\}$. By (6) and (7), we can rewrite the equations (1)–(5) to the equations in $F_1(x)$ and $F_2(x)$. For example, the equation (1) is rewritten as follows:

$$P_1(D_x)F_1(x) = 0, \quad x_0 \in \mathbb{R},\ x'' \in \Omega_1. \tag{8}$$

Next, we take the Fourier-Laplace transformations with respect to x' for the equation (8) and so on. In order not to vanish the Lopatinski's determinant, we take the Fourier-Laplace transformations along $S_m := \{(\chi(\xi'), \xi_1, \xi_2) : \xi' \in \mathbb{R}^3_{\xi'}\}$, where $\chi(\xi') := \xi_0 - im\log(2 + |\xi'|)$, and m is a positive real large enough. The Lopatinski's determinant does not vanish by the assumptions for the constants c_{s_k}, c_{p_k}, ρ_k, and Tarski-Seidenberg theorem and so on in the same way to Matsumura [4]. Then we obtain

$$P_1(\zeta', D_{x_3})\widehat{F}_1(\zeta', x_3) = 0, \quad 0 < x_3 < h \tag{9}$$

and so on, where $\zeta' = (\chi(\xi'), \xi_1, \xi_2) \in S_m$. Hereafter, for $\xi' \in \mathbb{R}^3$ we define ζ' by $(\chi(\xi'), \xi_1, \xi_2)$ in this way. Moreover, we equate ξ_3 to ζ_3.

Last, we apply $V(\zeta''')^{-1}$ on the left and $V(\zeta''')$ on the right to the equation (9) and so on, where

$$V(\zeta''') = \frac{1}{|\zeta'''|}\begin{bmatrix} \zeta_1 & 0 & -\zeta_2 \\ \zeta_2 & 0 & \zeta_1 \\ 0 & |\zeta'''| & 0 \end{bmatrix}.$$

Since

$$P_k(\zeta', D_{x_3}) = -V(\zeta''')\begin{bmatrix} P_{k1}(\zeta', D_{x_3}) & 0 \\ 0 & P_{k2}(\zeta', D_{x_3}) \end{bmatrix} V(\zeta''')^{-1},$$
$$B_k(\zeta', D_{x_3}) = V(\zeta''')\begin{bmatrix} B_{k1}(\zeta', D_{x_3}) & 0 \\ 0 & B_{k2}(\zeta', D_{x_3}) \end{bmatrix} V(\zeta''')^{-1}$$

hold, we have

$$P_{11}(\zeta', D_{x_3})\varphi_1(\zeta', x_3) = 0, \quad 0 < x_3 < h, \tag{10}$$
$$P_{21}(\zeta', D_{x_3})\varphi_2(\zeta', x_3) = 0, \quad x_3 > h, \tag{11}$$
$$B_{11}(\zeta', D_{x_3})\varphi_1(\zeta', x_3)|_{x_3=0} = -\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3} B_{11}(\zeta)P_{11}(\zeta)^{-1} d\xi_3, \tag{12}$$
$$[\varphi_1(\zeta', x_3) + \varphi_2(\zeta', x_3)]|_{x_3=h} = -\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} P_{11}(\zeta)^{-1} d\xi_3, \tag{13}$$
$$\begin{aligned}[B_{11}(\zeta', D_{x_3})\varphi_1(\zeta', x_3) + B_{21}(\zeta', D_{x_3})\varphi_2(\zeta', x_3)]|_{x_3=h} \\ = -\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} B_{11}(\zeta)P_{11}(\zeta)^{-1} d\xi_3,\end{aligned} \tag{14}$$

and

$$P_{12}(\zeta', D_{x_3})\psi_1(\zeta', x_3) = 0, \quad 0 < x_3 < h, \tag{15}$$
$$P_{22}(\zeta', D_{x_3})\psi_2(\zeta', x_3) = 0, \quad x_3 > h, \tag{16}$$
$$B_{12}(\zeta', D_{x_3})\psi_1(\zeta', x_3)|_{x_3=0} = -\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3} B_{12}(\zeta)P_{12}(\zeta)^{-1} d\xi_3, \tag{17}$$
$$[\psi_1(\zeta', x_3) + \psi_2(\zeta', x_3)]|_{x_3=h} = -\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} P_{12}(\zeta)^{-1} d\xi_3, \tag{18}$$
$$\begin{aligned}[B_{12}(\zeta', D_{x_3})\psi_1(\zeta', x_3) + B_{22}(\zeta', D_{x_3})\psi_2(\zeta', x_3)]|_{x_3=h} \\ = -\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} B_{12}(\zeta)P_{12}(\zeta)^{-1} d\xi_3,\end{aligned} \tag{19}$$

where $P_{kj}(\zeta', D_{x_3})$ and $B_{kj}(\zeta', D_{x_3})$ are defined by

$$\begin{aligned}
&P_{k1}(\zeta', D_{x_3})\\
&= \begin{bmatrix} \zeta_0^2 - (c_{s_k}^2 D_{x_3}^2 + c_{p_k}^2 |\zeta'''|^2) & -(c_{p_k}^2 - c_{s_k}^2)|\zeta'''| D_{x_3} \\ -(c_{p_k}^2 - c_{s_k}^2)|\zeta'''| D_{x_3} & \zeta_0^2 - (c_{p_k}^2 D_{x_3}^2 + c_{s_k}^2 |\zeta'''|^2) \end{bmatrix},\\
&P_{k2}(\zeta', D_{x_3}) = \zeta_0^2 - c_{s_k}^2 (D_{x_3}^2 + |\zeta'''|^2),\\
&B_{k1}(\zeta', D_{x_3}) = i\rho_k \begin{bmatrix} c_{s_k}^2 D_{x_3} & c_{s_k}^2 |\zeta'''| \\ (c_{p_k}^2 - 2c_{s_k}^2)|\zeta'''| & c_{p_k}^2 D_{x_3} \end{bmatrix},\\
&B_{k2}(\zeta', D_{x_3}) = i\rho_k c_{s_k}^2 D_{x_3},
\end{aligned}$$

and $\varphi_k(\zeta', x_3)$ and $\psi_k(\zeta', x_3)$ are defined by

$$\begin{bmatrix} \varphi_k(\zeta', x_3) & 0 \\ 0 & \psi_k(\zeta', x_3) \end{bmatrix} = V(\zeta''')^{-1} \widehat{F}_k(\zeta', x_3) V(\zeta''') \tag{20}$$

for $k = 1, 2$. The equations (10)–(14) and (15)–(19) are the mixed problems for the ordinary differential equations in essence with respect to $\varphi_k(\zeta', x_3)$ and $\psi_k(\zeta', x_3)$, respectively. Hence we have

$$\psi_1(\zeta', x_3) = a_+(\zeta') e^{i\tau_{s_1}^+ x_3} + a_-(\zeta') e^{-i\tau_{s_1}^+ x_3}, \tag{21}$$
$$\psi_2(\zeta', x_3) = b_+(\zeta') e^{i\tau_{s_2}^+ x_3} \tag{22}$$

by the equations (15) and (16), and

$$\begin{aligned}
\begin{bmatrix} \varphi_{11l}(\zeta', x_3) \\ \varphi_{12l}(\zeta', x_3) \end{bmatrix} &= \alpha_{+pl} e^{i\tau_{p_1}^+ x_3} \begin{bmatrix} |\zeta'''| \\ \tau_{p_1}^+ \end{bmatrix} + \alpha_{-pl} e^{-i\tau_{p_1}^+ x_3} \begin{bmatrix} |\zeta'''| \\ -\tau_{p_1}^+ \end{bmatrix} \\
&\quad + \alpha_{+sl} e^{i\tau_{s_1}^+ x_3} \begin{bmatrix} \tau_{s_1}^+ \\ -|\zeta'''| \end{bmatrix} + \alpha_{-sl} e^{-i\tau_{s_1}^+ x_3} \begin{bmatrix} \tau_{s_1}^+ \\ |\zeta'''| \end{bmatrix},
\end{aligned} \tag{23}$$

$$\begin{bmatrix} \varphi_{21l}(\zeta', x_3) \\ \varphi_{22l}(\zeta', x_3) \end{bmatrix} = \beta_{+pl} e^{i\tau_{p_2}^+ x_3} \begin{bmatrix} |\zeta'''| \\ \tau_{p_2}^+ \end{bmatrix} + \beta_{+sl} e^{i\tau_{s_2}^+ x_3} \begin{bmatrix} \tau_{s_2}^+ \\ -|\zeta'''| \end{bmatrix} \tag{24}$$

for $l = 1, 2$ by the equations (10) and (11), where

$$\varphi_k(\zeta', x_3) = \begin{bmatrix} \varphi_{k11}(\zeta', x_3) & \varphi_{k12}(\zeta', x_3) \\ \varphi_{k21}(\zeta', x_3) & \varphi_{k22}(\zeta', x_3) \end{bmatrix} \tag{25}$$

and $\tau_{p_k}^+(\zeta')$ [resp. $\tau_{s_k}^+(\zeta')$] is the root with positive imaginary part of the equation in τ: $\zeta_0^2 - c_{p_k}^2(\zeta_1^2 + \zeta_2^2 + \tau^2) = 0$ [resp. $\zeta_0^2 - c_{s_k}^2(\zeta_1^2 + \zeta_2^2 + \tau^2) = 0$] for

$k = 1, 2$. We can determine $\alpha_{\pm * l}$ and β_{+*l} $(* = p,\ s)$ by the forms (23) and (24) and the equations (12)–(14). In the same way, we can also determine $a_\pm$ and b_+ by the forms (21) and (22) and the equations (17)–(19).

The above is the way of solving the equations (1)–(5). In the next section, we prove our main result by using these formulas.

3 The sketch of the proof

In this section, we prove the main theorem in most cases.

First, we remark that the condition that "$G(x)|_{x_3=0}$ is given" is equivalent to the condition that "$\varphi_1(\zeta', x_3)|_{x_3=0}$ and $\psi_1(\zeta', x_3)|_{x_3=0}$ are given" by (6) and (20).

Now, we prove the following lemma. This lemma says that we obtain the behavior of the waves in Medium 1 when the observation data are given.

Lemma 3.1. *Let c_{p_1}, c_{s_1}, ρ_1 and y_3 be given. Assume that the observation data $N_\varphi(\zeta') := \varphi_1(\zeta', x_3)|_{x_3=0}$ [resp. $N_\psi(\zeta') := \psi_1(\zeta', x_3)|_{x_3=0}$] are given. Then $\varphi_1(\zeta', x_3)$ [resp. $\psi_1(\zeta', x_3)$] is expressed with the known data.*

Proof. Here, we prove only the case of ψ_1. We can prove the case of φ_1 in the same way.

By the assumption and (21), we have

$$a_+(\zeta') + a_-(\zeta') = N_\psi(\zeta'). \tag{26}$$

On the other hand, we have

$$a_+(\zeta')\tau_{s_1}^+(\zeta') - a_-(\zeta')\tau_{s_1}^+(\zeta') = \frac{i}{2c_{s_1}^2} e^{i\tau_{s_1}^+(\zeta')y_3} \tag{27}$$

by (21) and (17). By (26) and (27), we have

$$\begin{bmatrix} a_+(\zeta') \\ a_-(\zeta') \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \tau_{s_1}^+(\zeta') & -\tau_{s_1}^+(\zeta') \end{bmatrix}^{-1} \begin{bmatrix} N_\psi(\zeta') \\ \dfrac{i}{2c_{s_1}^2} e^{i\tau_{s_1}^+(\zeta')y_3} \end{bmatrix},$$

that is, we can express $a_\pm(\zeta')$ with the known data. Therefore, we can express $\psi_1(\zeta')$ with the known data by (21). □

By Lemma 3.1, we obtain the behavior of the waves in Medium 1 when the observation data are given. Then, we define $\varphi_1^N(\zeta', x_3)$ [resp. $\psi_1^N(\zeta', x_3)$] by the behavior of the waves in Medium 1 for the observation data $N_\varphi(\zeta') := \varphi_1(\zeta', x_3)|_{x_3=0}$ [resp. $N_\psi(\zeta') := \psi_1(\zeta', x_3)|_{x_3=0}$]. Moreover, we also define $a_\pm^N(\zeta')$ and $\varphi_{1jl}^N(\zeta', x_3)$ $(j, l = 1, 2)$ as $a_\pm(\zeta')$ and $\varphi_{1jl}(\zeta', x_3)$ which are obtained in Lemma 3.1

Next, we prove a lemma needed in order to express the unknown constants with the known data.

Lemma 3.2. *Let c_{s_1}, ρ_1, y_3 be given. Assume that the observation data $N_\psi(\zeta')$ are given. Then we have*

$$\begin{aligned}\rho_2 c_{s_2}^2 \tau_{s_2}^+(\zeta') &\left\{ a_+^N(\zeta') e^{i\tau_{s_1}^+(\zeta')h} + a_-^N(\zeta') e^{-i\tau_{s_1}^+(\zeta')h} - \frac{ie^{i(h-y_3)\tau_{s_1}^+(\zeta')}}{2c_{s_1}^2 \tau_{s_1}^+(\zeta')} \right\} \\ = \rho_1 c_{s_1}^2 \tau_{s_1}^+(\zeta') &\left\{ a_+^N(\zeta') e^{i\tau_{s_1}^+(\zeta')h} - a_-^N(\zeta') e^{-i\tau_{s_1}^+(\zeta')h} - \frac{ie^{i(h-y_3)\tau_{s_1}^+(\zeta')}}{2c_{s_1}^2 \tau_{s_1}^+(\zeta')} \right\}.\end{aligned} \tag{28}$$

Remark 3.3. We remark that the equality (28) is equivalent to the following equality:

$$\begin{aligned}\left\{\rho_1 c_{s_1}^2 \tau_{s_1}^+(\zeta') - \rho_2 c_{s_2}^2 \tau_{s_2}^+(\zeta')\right\} &\left\{ a_+^N(\zeta') - \frac{ie^{-i\tau_{s_1}^+(\zeta')y_3}}{2c_{s_1}^2 \tau_{s_1}^+(\zeta')} \right\} \\ &= \left\{\rho_1 c_{s_1}^2 \tau_{s_1}^+(\zeta') + \rho_2 c_{s_2}^2 \tau_{s_2}^+(\zeta')\right\} a_-^N(\zeta') e^{-2i\tau_{s_1}^+(\zeta')h}.\end{aligned} \tag{29}$$

Proof of Lemma 3.2. By (21), (22) and (18), we have

$$a_+^N(\zeta') e^{i\tau_{s_1}^+(\zeta')h} + a_-^N(\zeta') e^{-i\tau_{s_1}^+(\zeta')h} + b_+(\zeta') e^{i\tau_{s_2}^+(\zeta')h} = \frac{ie^{i(h-y_3)\tau_{s_1}^+(\zeta')}}{2c_{s_1}^2 \tau_{s_1}^+(\zeta')}. \tag{30}$$

In a similar way, by (21), (22) and (19), we have

$$\begin{aligned}i\rho_1 c_{s_1}^2 \tau_{s_1}^+(\zeta') &\left\{ a_+^N(\zeta') e^{i\tau_{s_1}^+(\zeta')h} - a_-^N(\zeta') e^{-i\tau_{s_1}^+(\zeta')h} \right\} \\ &+ i\rho_2 c_{s_2}^2 \tau_{s_2}^+(\zeta') b_+(\zeta') e^{i\tau_{s_2}^+(\zeta')h} = -\frac{1}{2}\rho_1 e^{i(h-y_3)\tau_{s_1}^+(\zeta')}.\end{aligned} \tag{31}$$

Multiplying (30) by $i\rho_2 c_{s_2}^2 \tau_{s_2}^+(\zeta')$ and subtracting (31) from it, we have the equality (28). □

Now, we express the constants h, c_{s_2}, ρ_2 with the known data by using Lemma 3.2.

Theorem 3.4. *Let c_{s_1}, ρ_1, y_3 be given. Assume that the observation data $N_\psi(\zeta')$ are given. Then the constant h is expressed with the given data if $a_-^N(\chi_0(\xi_0), 0, 0) \not\equiv 0$, where we define $\chi_0(\xi_0) := \chi(\xi_0, 0, 0)$.*

Proof. We remark that $\tau_{s_k}^+(\chi_0(\xi_0), 0, 0) = -\chi_0(\xi_0)/c_{s_k}$. Substituting (29) into $\zeta' = (\chi_0(\xi_0), 0, 0)$ and simplifying it, we have

$$\kappa_1 e^{2i\chi_0(\xi_0)h/c_{s_1}} = K_1^N(\xi_0), \tag{32}$$

where we define

$$\begin{aligned}\kappa_1 &= \frac{\rho_1 c_{s_1} + \rho_2 c_{s_2}}{\rho_1 c_{s_1} - \rho_2 c_{s_2}},\\ K_1^N(\xi_0) &= \frac{a_+^N(\chi_0(\xi_0), 0, 0) + \left\{ie^{i\chi_0(\xi_0)y_3/c_{s_1}}/2c_{s_1}\chi_0(\xi_0)\right\}}{a_-^N(\chi_0(\xi_0), 0, 0)}.\end{aligned}$$

Applying D_{ξ_0} to the equality (32) and mixing it with (32) multiplied by $(2ih/c_{s_1})(D_{\xi_0}\chi_0)(\xi_0)K_1^N(\xi_0)$, we have the equality

$$h = \frac{c_{s_1}(D_{\xi_0}K_1^N)(\xi_0)}{2i(D_{\xi_0}\chi_0)(\xi_0)K_1^N(\xi_0)}. \tag{33}$$

$K_1^N(\xi_0)$ is the known data since $a_\pm^N(\zeta')$ are the known data. Hence we can express the constant h with the given data by (33). $\square$

Remark 3.5. Even if $a_-^N(\chi_0(\xi_0), 0, 0) \equiv 0$, if $a_-^N(\zeta') \not\equiv 0$ then the constants h, c_{s_2}, ρ_2 can be expressed with the given data, and in this case we obtain $\rho_1 c_{s_1} = \rho_2 c_{s_2}$ in particular. However we omit the proof of this case (see Nagayasu [5]).

By Theorem 3.4, we express the constant h with the known data and the observation data. Hence we can assume that the constant h is also given. Next, we express the constants c_{s_2} and ρ_2.

Theorem 3.6. *Let c_{s_1}, ρ_1, y_3 and h be given. Assume that the observation data $N_\psi(\zeta')$ are given. Then the constants c_{s_2} and ρ_2 are expressed with the given data.*

Proof. Put

$$K_2^N(\xi') = a_+^N(\zeta')e^{i\tau_{s_1}^+(\zeta')h} + a_-^N(\zeta')e^{-i\tau_{s_1}^+(\zeta')h} - \frac{ie^{i(h-y_3)\tau_{s_1}^+(\zeta')}}{2c_{s_1}^2\tau_{s_1}^+(\zeta')},$$
$$K_3^N(\xi') = a_+^N(\zeta')e^{i\tau_{s_1}^+(\zeta')h} - a_-^N(\zeta')e^{-i\tau_{s_1}^+(\zeta')h} - \frac{ie^{i(h-y_3)\tau_{s_1}^+(\zeta')}}{2c_{s_1}^2\tau_{s_1}^+(\zeta')}.$$

We remark that $K_2^N(\xi')$ and $K_3^N(\xi')$ are expressed with the given data. Moreover

$$\rho_2 c_{s_2}^2 \tau_{s_2}^+(\zeta')K_2^N(\xi') = \rho_1 c_{s_1}^2 \tau_{s_1}^+(\zeta')K_3^N(\xi') \tag{34}$$

holds by the equality (28). By (34), we have

$$\rho_2 c_{s_2}^2 \tau_{s_2}^+(\zeta') = K_4^N(\xi') \tag{35}$$

where $K_4^N(\xi') := \rho_1 c_{s_1}^2 \tau_{s_1}^+(\zeta')K_3^N(\xi')/K_2^N(\xi')$. We remark that $K_4^N(\xi')$ is expressed with the given data. Squaring the equality (35), we obtain

$$\rho_2^2 c_{s_2}^2 \left\{\chi(\xi')^2 - c_{s_2}^2(\xi_1^2+\xi_2^2)\right\} = K_4^N(\xi')^2. \tag{36}$$

Applying D_{ξ_0} to the equality (36) and simplifying it, we have

$$\rho_2^2 c_{s_2}^2 = \frac{K_4^N(\xi')(D_{\xi_0}K_4^N)(\xi')}{\chi(\xi')(D_{\xi_0}\chi)(\xi')}. \tag{37}$$

Hence we have

$$c_{s_2}^2 = \frac{\chi(\xi')\left\{\chi(\xi')(D_{\xi_0}K_4^N)(\xi') - (D_{\xi_0}\chi)(\xi')K_4^N(\xi')\right\}}{(D_{\xi_0}K_4^N)(\xi')(\xi_1^2+\xi_2^2)} \tag{38}$$

by substituting (37) into (36) and simplifying it. Since the right-hand side of (38) is expressed with only the given data and c_{s_2} is positive, the constant c_{s_2} is expressed with the given data by (38). Then by (37) we have

$$\rho_2^2 = \frac{K_4^N(\xi')(D_{\xi_0}K_4^N)(\xi')}{c_{s_2}^2\chi(\xi')(D_{\xi_0}\chi)(\xi')}. \tag{39}$$

We remark that the right-hand side of (39) is expressed with only the given data. Therefore we express also the constant ρ_2 with the given data because ρ_2 is positive. □

By Theorem 3.4 and 3.6, the unknown constant which is not expressed with the given data yet is only the constant c_{p_2}. The constant c_{p_2} does not appear in the equality (28). Then we determine c_{p_2} by using the equalities for $\varphi_k(\zeta', x_3)$.

Theorem 3.7. *Let c_{p_1}, c_{s_1}, ρ_1 and y_3 be given. Let c_{s_2}, ρ_2 and h be given. Assume that the observation data $N_\varphi(\zeta')$ are given. Then the constant c_{p_2} is expressed with the given data.*

Proof. By the $(1,1)$-component of (13) and the $(2,1)$-component of (14), we have

$$\begin{bmatrix} |\zeta'''| & 1 \\ (\tau_{s_2}^+)^2 - |\zeta'''|^2 & -2|\zeta'''| \end{bmatrix} \begin{bmatrix} \beta_{+p1} e^{i\tau_{p_2}^+ h} \\ \beta_{+s1} e^{i\tau_{s_2}^+ h} \tau_{s_2}^+ \end{bmatrix} = \begin{bmatrix} K_5^N(\zeta') \\ K_6^N(\zeta') \end{bmatrix},$$

where $K_5^N(\zeta')$ and $K_6^N(\zeta')$ are defined by

$$K_5^N(\zeta') := -\varphi_{111}^N(\zeta', x_3)|_{x_3=h} + \frac{i}{2\zeta_0^2} \left\{ \frac{|\zeta'''|^2}{\tau_{p_1}^+} e^{i(h-y_3)\tau_{p_1}^+} + \tau_{s_1^+} e^{i(h-y_3)\tau_{s_1}^+} \right\},$$

$$\begin{aligned} K_6^N(\zeta') := &-\frac{\rho_1}{\rho_2 c_{s_2}^2} \{ (c_{p_1}^2 - 2c_{s_1}^2)|\zeta'''|\varphi_{111}^N(\zeta', x_3)|_{x_3=h} \\ &\qquad + c_{p_1}^2 (D_{x_3}\varphi_{111}^N)(\zeta', x_3)|_{x_3=h} \} \\ &+ \frac{i\rho_1 c_{s_1}^2}{2\zeta_0^2 \rho_2 c_{s_2}^2} \left\{ \frac{|\zeta'''|((\tau_{s_1}^+)^2 - |\zeta'''|^2)}{\tau_{p_1}^+} e^{i(h-y_3)\tau_{p_1}^+} - 2|\zeta'''|\tau_{s_1}^+ e^{i(h-y_3)\tau_{s_1}^+} \right\}. \end{aligned}$$

We remark that $K_5^N(\zeta')$ and $K_6^N(\zeta')$ are expressed with only the given data. Since

$$\begin{vmatrix} |\zeta'''| & 1 \\ (\tau_{s_2}^+)^2 - |\zeta'''|^2 & -2|\zeta'''| \end{vmatrix} = -\frac{\zeta_0^2}{c_{s_2}^2} \neq 0,$$

we can solve this equation, and obtain

$$\beta_{+p1} e^{i\tau_{p_2}^+ h} = \frac{c_{s_2}^2}{\zeta_0^2} K_7^N(\zeta') \tag{40}$$

in particular, where $K_7^N(\zeta')$ is defined by $K_7^N(\zeta') := 2|\zeta'''|K_5^N(\zeta') + K_6^N(\zeta')$, and this is expressed with the given data. In a similar way, by the $(2,1)$-component of (13) and the $(1,1)$-component of (14), we have

$$\beta_{+p1} e^{i\tau_{p_2}^+ h} \tau_{p_2}^+ = \frac{c_{s_2}^2}{\zeta_0^2} K_8^N(\zeta'), \tag{41}$$

where

$$
\begin{aligned}
&K_8^N(\zeta') \\
&:= ((\tau_{s_1}^+)^2 - |\zeta'''|^2) \\
&\quad \times \left[-\varphi_{121}^N(\zeta', x_3)|_{x_3=0} + \frac{i}{2\zeta_0^2} \left\{ |\zeta'''| e^{i(h-y_3)\tau_{p_1}^+} - |\zeta'''| e^{i(h-y_3)\tau_{s_1}^+} \right\} \right] \\
&\quad + |\zeta'''| \left[-\frac{\rho_1}{\rho_2 c_{s_2}^2} \left\{ c_{s_1}^2 (D_{x_3}\varphi_{111}^N)(\zeta', x_3)|_{x_3=h} + c_{s_1}^2 |\zeta'''| \varphi_{121}^N(\zeta', x_3)|_{x_3=h} \right\} \right. \\
&\qquad \left. + \frac{i\rho_1 c_{s_1}^2}{2\zeta_0^2 \rho_2 c_{s_2}^2} \left\{ 2|\zeta'''|^2 e^{i(h-y_3)\tau_{p_1}^+} + ((\tau_{s_1}^+)^2 - |\zeta'''|^2) e^{i(h-y_3)\tau_{s_1}^+} \right\} \right].
\end{aligned}
$$

We remark that $K_8^N(\zeta')$ is determined with the given data. By the equalities (40) and (41), we obtain

$$
\tau_{p_2}^+ = \frac{K_8^N(\zeta')}{K_7^N(\zeta')}. \tag{42}
$$

Squaring the equality (42) and simplifying it, we have the equality

$$
c_{p_2}^2 = \frac{\zeta_0^2}{(K_8^N(\zeta')/K_7^N(\zeta'))^2 + |\zeta'''|^2}. \tag{43}
$$

The constant c_{p_2} is expressed with the given data hence the right-hand side of (43) is expressed with the given data and c_{p_2} is positive. □

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