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Formal power series solutions of nonlinear partial differential equations and their multisummability

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Abstract

Let $\hat{u}(t,x)=\sum_{n=1}^{\infty}u_n(x)t^n$, $(t,x)\in\mathbb{C}\times\mathbb{C}^d$, be a formal power series solution of a nonlinear partial differential equation in a complex domain. We study the multisummability of $\hat{u}(t,x)$, which implies existence of a genuine solution $u(t,x)$ with $u(t,x)\sim\hat{u}(t,x)$ as $t\to 0$ in strong sense. This article is a continuation of [7] in which linear partial differential equations were studied.
Key words: Asymptotic expansion, Formal power series solutions, Multisummablility

0 Introduction

In this article we study formal solutions of some nonlinear partial differential equations in the complex domain. It is an important problem to study the existence of genuine (true) solutions with given formal solutios. This problem was studied in [6] for general nonlinear partial differential equations, where the existence of genuine solutions were obtained. Recently we have the theory of multisummability of formal power series (see [1]). Multisummability of a formal power series $\hat{\Phi}(t,x)=\sum_{n=1}^{\infty}\phi_n(x)t^n$ means the existence of a holomorphic function $\Phi(t,x)$ on a sectorial region with $\Phi(t,x)\sim\hat{\Phi}(t,x)$ in much stronger sense. It is shown in [2], [3], [4] and [5] that formal power series solutions of ordinary differential equations are multisummable. The multisummability of formal solution was not studied in [6], because equations studied were more general. As for formal series solutions of partial differential equations, it is shown in [7] that they are multisummable for some class of linear partial differential equations. We generalize this result for nonlinear partial differential equations. The details will be published elsewhere ([9]).

1 Borel and Laplace transforms

In order to introduce the notion of multi-summability of formal power series we first define Laplace transform, Borel transform and their formal theory. For more detailed results of this topic we refer to [1]. The coordinates of $\mathbb{C}^{d+1}$ is denoted by $(t,x) = (t, x_1, \cdots, x_d) \in \mathbb{C} \times \mathbb{C}^d$. For a region Ω, $\mathcal{O}(\Omega)$ is the set of all holomorphic functions on Ω. We often use the following notations about sectorial regions. For $\theta \in \mathbb{R}$ and $\delta, \rho > 0$ set $S(\theta, \delta, \rho) = \{0 < |t| < \rho; |\arg t - \theta| < \delta\}$ ($S^*(\theta, \delta, \rho) = \{0 < |\xi| < \rho; |\arg \xi - \theta| < \delta\}$). $S(\theta, \delta) = S(\theta, \delta, \infty)$ ($S^*(\theta, \delta) = S^*(\theta, \delta, \infty)$) is an infinite sector in t-space (*resp.* ξ-space). θ is often called a direction. We also put $S_{\{0\}}(\theta, \delta) = \{t \in S(\theta, \delta); 0 < |t| < \omega(\arg t)\}$ ($S^*_{\{0\}}(\theta, \delta) = \{\xi \in S^*(\theta, \delta); 0 < |\xi| < \omega(\arg \xi)\}$) called a sectorial neighborhood of $t = 0$ (*resp.* $\xi = 0$), where $\omega(\cdot) > 0$ is a positive continuous function on $(\theta - \delta, \theta + \delta)$.

Let $U \subset \mathbb{C}^d$ be an open polydisk with center $x = 0$.

Definition 1.1. *Let $\gamma > 0$. $Exp_{\{\gamma\}}(S^* \times U)$ is the set of all $\phi(\xi, x) \in \mathcal{O}(S^* \times U)$ such that for $(\xi, x) \in (S^* \cap \{|\xi| \geq 1\}) \times U$*

$$|\phi(\xi, x)| \leq C \exp(c|\xi|^\gamma) \tag{1.1}$$

for some constants C and c.

Let $\phi(\xi, x) \in Exp_{\{\gamma\}}(S^* \times U)$ such that for $(\xi, x) \in (S^* \cap \{0 < |\xi| \leq 1\}) \times U$

$$|\phi(\xi, x)| \leq C|\xi|^{\varepsilon - \gamma} \quad (\varepsilon > 0). \tag{1.2}$$

Then γ-Laplace transform $(\mathcal{L}_{\gamma,\theta}\phi)(t,x)$ is defined by

$$(\mathcal{L}_{\gamma,\theta}\phi)(t,x) = \int_0^{\infty e^{i\theta}} (\exp(-(\frac{\xi}{t})^\gamma)\phi(\xi, x) d\xi^\gamma, \tag{1.3}$$

$d\xi^\gamma = \gamma \xi^{\gamma - 1} d\xi$, which is holomorphic on $S_{\{0\}}(\theta, \pi/2\gamma + \delta) \times U$. Let $\psi(t,x)$ be a holomorphic function in $S_{\{0\}}(\theta, \pi/2\gamma + \delta) \times U$ with $|\psi(t,x)| \leq C|t|^\varepsilon$ ($\varepsilon > 0$). Let $\xi \neq 0$ with $|\arg \xi - \theta| < \delta$ and $\mathcal{C}$ be a contour in $S_{\{0\}}(\theta, \pi/2\gamma + \delta)$ from $0 \exp(i(\theta' + \arg \xi))$ to $0 \exp(i(-\theta' + \arg \xi))$ with $\pi/2\gamma < \theta' < \pi/2\gamma + \min\{\theta + \delta - \arg \xi,\ \arg \xi - \theta + \delta\}$. Then γ-Borel transform $(\mathcal{B}_{\gamma,\theta}\psi)(\xi, x)$ is defined by

$$(\mathcal{B}_{\gamma,\theta}\psi)(\xi, x) = \frac{1}{2\pi i} \int_{\mathcal{C}} \exp(\frac{\xi}{t})^\gamma \psi(t,x) dt^{-\gamma}. \tag{1.4}$$

Let $\phi_i(x,\xi) \in \mathcal{O}(U \times S^*_{\{0\}}(\theta,\delta))$ $(i = 1,2)$ satisfying $|\phi_i(x,\xi)| \le C|\xi|^{\varepsilon-\gamma}$ $(\varepsilon > 0)$. Then γ-convolution of $\phi_1(x,\xi)$ and $\phi_2(x,\xi)$ is defined by

$$(\phi_1 \underset{\gamma}{*} \phi_2)(x,\xi) = \int_0^\xi \phi_1((\xi^\gamma - \eta^\gamma)^{1/\gamma}, x)\phi_2(\eta,x)d\eta^\gamma \quad \xi \in S^*_{\{0\}}(\theta,\delta). \tag{1.5}$$

Let $0 < \gamma < \gamma'$ and $\kappa^{-1} = \gamma^{-1} - (\gamma')^{-1}$. Set

$$\mathcal{A}_{\gamma',\gamma,\theta} := \mathcal{B}_{\gamma',\theta}\mathcal{L}_{\gamma,\theta}, \tag{1.6}$$

which is called (γ',γ)-acceleration in the direction θ. It was introduced by Ecalle and shown that $\mathcal{A}_{\gamma',\gamma,\theta}$ can be extended to $\phi(\xi,x) \in Exp_{\{\kappa\}}(S^* \times U)$ with (1.2) and $(\mathcal{A}_{\gamma',\gamma,\theta}\phi)(\xi,x)$ is holomorphic in $S^*_{\{0\}}(\theta,\pi/2\kappa+\delta) \times U$. We have the following basic relations

Lemma 1.2. *(1). Let $\phi_i(\xi,x) \in Exp_{\{\gamma\}}(S^* \times U)$ $(i = 0,1,2)$ with* (1.2). *Then*

$$\mathcal{B}_{\gamma,\theta}\mathcal{L}_{\gamma,\theta}\phi_0 = \phi_0, \tag{1.7}$$

$$(\mathcal{L}_{\gamma,\theta}\phi_1)(\mathcal{L}_{\gamma,\theta}\phi_2) = \mathcal{L}_{\gamma,\theta}(\phi_1 \underset{\gamma}{*} \phi_2). \tag{1.8}$$

(2). Let $\phi_i(\xi,x) \in Exp_{\{\kappa\}}(S^* \times U)$ $(i = 1,2)$ with (1.2). *Then*

$$(\mathcal{A}_{\gamma',\gamma,\theta}\phi_1) \underset{\gamma'}{*} (\mathcal{A}_{\gamma',\gamma,\theta}\phi_2) = \mathcal{A}_{\gamma',\gamma,\theta}(\phi_1 \underset{\gamma}{*} \phi_2). \tag{1.9}$$

The preceeding theory is analytical. For our aim let us define formal γ-Borel transform.

Definition 1.3. *Let $\hat{v}(t,x) = \sum_{n=0}^\infty v_n(x)t^n \in \mathcal{O}(U)[[t]]$.*
We say that $\hat{v}(t,x)$ has Gevrey order s in t, if there are positive constants A and B such that

$$\sup_{x\in U}|v_n(x)| \le AB^n\Gamma(sn+1). \tag{1.10}$$

The totality of such formal series is denoted by $\mathcal{O}(U)[[t]]_s$.

Let $\hat{v}(t,x) = \sum_{n=1}^\infty v_n(x)t^n \in t\mathcal{O}(U)[[t]]$. Then formal γ-Borel transform $(\hat{\mathcal{B}}_\gamma\hat{v})(\xi,x)$ is defined by

$$(\hat{\mathcal{B}}_\gamma\hat{v})(\xi,x) := \sum_{n=1}^\infty \frac{v_n(x)\xi^{n-\gamma}}{\Gamma(\frac{n}{\gamma})}. \tag{1.11}$$

In general $(\hat{\mathcal{B}}_\gamma \hat{v})(\xi,x)$ is a formal series in ξ. But if $\hat{v}(t,x) \in t\mathcal{O}(U)[[t]]_{\frac{1}{\gamma}}$, then $(\hat{\mathcal{B}}_\gamma \hat{v})(\xi,x)$ is converges, hence, it is holomorphic in $\{0 < |\xi| < \rho_0\}$ for some $\rho_0 > 0$.

2 Multisummability of formal series

Now let us proceed to define multisummability of $\hat{v}(t,x) = \sum_{n=1}^{\infty} v_n(x)t^n \in t\mathcal{O}(U)[[t]]$. Let $0 < k_r < k_{r-1} < \cdots < k_1 < k_0 = +\infty$ and define κ_i by $\kappa_i^{-1} = k_i^{-1} - k_{i-1}^{-1}$ for $1 \le i \le r$. Let $\{\theta_i\}_{i=1}^r$ be real constants such that $|\theta_i - \theta_{i-1}| \le \pi/2\kappa_i$. Set $\mathbf{k} = (k_1, \cdots, k_r)$ and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_r)$. We call $\boldsymbol{\theta}$ a multidirection.

Then $\hat{v}(t,x) \in t\mathcal{O}(U)[[t]]$ is $\mathbf{k}$-summable in multidirection $\boldsymbol{\theta}$, if the following conditions are satisfied.

(1) $\hat{v}(t,x) \in \mathcal{O}(U)[[t]]_{\frac{1}{k_r}}$. Then $v^r(\xi,x) := (\hat{\mathcal{B}}_{k_r}\hat{v})(\xi,x)$ converges uniformly on $\{0 < |\xi| < \rho_0\} \times U$ for some $\rho_0 > 0$.

(2) Let $i \in \{1, 2, \cdots, r-1, r\}$. $v^i(\xi,x)$ has the holomorphic prolongation to $S_i^* := S^*(\theta_i, \delta_i)$ for some $\delta_i > 0$ with exponential growth of order κ_i,

$$|v^i(\xi,x)| \le C\exp(c|\xi|^{\kappa_i}) \quad \text{on } (S_i^* \cap \{|\xi| \ge 1\}) \times U. \tag{2.1}$$

If $i \neq 1$, define $v^{i-1}(\xi,x) := (\mathcal{A}_{k_{i-1},k_i,\theta_i} v^i)(\xi,x)$, which is holomorphic in $S_{\{0\}}^*(\theta_i, \pi/2\kappa_i + \delta_i) \times U$.

Then $\mathbf{k}$-sum of $\hat{v}(t,x)$ in multidirection $\boldsymbol{\theta}$ is defined by $(\mathcal{L}_{k_1,\theta_1} v^1)(t,x) \in \mathcal{O}(S_{1,\{0\}} \times U)$, $S_1 = S(\theta_1, \pi/2\gamma_1 + \delta_1)$, and denoted by $v(t,x)$. It holds that

$$\begin{aligned} v^r(\xi,x) &= \sum_{n=1}^{\infty} \frac{v_n(x)\xi^{n-k_r}}{\Gamma(\frac{n}{k_r})} \quad \text{in} \quad \{0 < |\xi| < \rho_0\} \times U \\ v^{i-1}(\xi,x) &\sim \sum_{n=1}^{\infty} \frac{v_n(x)\xi^{n-k_{i-1}}}{\Gamma(\frac{n}{k_{i-1}})} \quad \text{in} \quad S_{\{0\}}^*(\theta_i, \pi/2\kappa_i + \delta_i) \times U \end{aligned} \tag{2.2}$$

We have, by considering the behavior at $\xi = 0$ and (2.1)

$$|v^i(\xi,x)| \le A|\xi|^{1-k_i}\exp(c|\xi|^{\kappa_i}) \quad \text{on } S_i^* \times U. \tag{2.3}$$

Let $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}(U)[[t]]$. Set $\hat{w}(t,x) := (\hat{v}(t,x) - v_0(x)) \in t\mathcal{O}(U)[[t]]$. If $\hat{w}(t,x)$ is $\mathbf{k}$-summable in multidirection $\boldsymbol{\theta}$, we say that $\hat{v}(t,x)$ is $\mathbf{k}$-summable in multidirection $\boldsymbol{\theta}$. Set $\hat{w}^r(\xi,x) = (\hat{\mathcal{B}}_{k_r}\hat{w})(\xi,x)$ and $w^{i-1}(\xi,x) = (\mathcal{A}_{k_{i-1},k_i,\theta_i}w^i)(\xi,x)$ for $2 \le i \le r$. $\mathbf{k}$-sum of $\hat{v}(t,x)$ is defined by $v_0(x) + (\mathcal{L}_{k_1,\theta_1}w^1)(t,x)$ and denoted by $v(t,x)$. $v(t,x)$ is holomorphic in $S_{1,\{0\}} \times U$ and $v(t,x) \sim \hat{v}(t,x)$ as $t \to 0$ in $S_{1,\{0\}} \times U$. There are other equivalent definitions of $\mathbf{k}$-summabilty multidirection $\boldsymbol{\theta}$. We refer [1] for this topics.

3 Formal power series solutions of nonlinear partial differential equations

First we introduce notations about partial differential equations. As before $(t,x) = (t,x_1,\cdots,x_d) \in \mathbb{C} \times \mathbb{C}^d$. For multi-indices $\alpha = (\alpha_0,\alpha_1,\cdots,\alpha_d) = (\alpha_0,\alpha') \in \mathbb{N}^{d+1}$, $|\alpha| = \sum_{i=0}^d \alpha_i$ and $\vartheta^{\alpha_0}\partial^{\alpha'} = (t\partial_t)^{\alpha_0}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$. Let M be the cardinal number of the set $\boldsymbol{\Delta}_m = \{\alpha \in \mathbb{N}^{d+1}; |\alpha| \le m\}$. For $A = (A_\alpha;\alpha \in \boldsymbol{\Delta}_m) \in \mathbb{N}^M$ and $Z = (Z_\alpha;\alpha \in \boldsymbol{\Delta}_m)$ define

$$|A| = \sum_{\alpha\in\boldsymbol{\Delta}_m} A_\alpha,\ Z^A = \prod_{\alpha\in\boldsymbol{\Delta}_m} Z_\alpha^{A_\alpha}.$$

So Z^A is a monomial in $\{Z_\alpha\}_{\alpha\in\Delta(m)}$ with degree $|A|$. Set $\mathbb{N}^{M*} := \mathbb{N}^M - \{0\} = \{A \in \mathbb{N}^M; |A| \ge 1\}$. For $A \in \mathbb{N}^{M*}$, set $m_A = max\{|\alpha|; A_\alpha \ne 0\}$.

Let $U_0 \subset \mathbb{C}$ ($U \subset \mathbb{C}^d$) be an open polydisk with center $t = 0$ (*resp.* $x = 0$) and $\Omega \subset \mathbb{C}^M$ be a neighborhood of $Z = 0$. Set $\vartheta = t\partial_t$. Let $L(u) = L(t,x,\vartheta^{\alpha_0}\partial_x^{\alpha'}u)$ be a nonlinear partial differential equation with order m of the form

$$L(u) = \sum_{A\in\mathbb{N}^{M*}} c_A(t,x) \prod_{\alpha\in\boldsymbol{\Delta}_m} (\vartheta^{\alpha_0}\partial^{\alpha'}u)^{A_\alpha} + f(t,x), \tag{3.1}$$

where $L(t,x,Z) = \sum_{A\in\mathbb{N}^{M*}} c_A(t,x)Z^A + f(t,x) \in \mathcal{O}(U_0\times U\times\Omega)$ and $f(t,x) = L(t,x,0)$. In this article for simplicity we assume $L(t,x,Z)$ is holomorphic on $U_0 \times U \times \Omega$. Let $e_A \in \mathbb{N}$ such that $c_A(t,x) = t^{e_A}b_A(t,x)$ with $b_A(0,x) \not\equiv 0$. The linear part of $L(u)$ denoted by $L_{lin}(t,x,\vartheta,\partial_x)u$, that is,

$$\begin{aligned} L_{lin}(t,x,\vartheta,\partial_x)u &= \sum_{\{A\in\mathbb{N}^{M*};\ |A|=1\}} c_A(t,x) \prod_{\alpha\in\boldsymbol{\Delta}_m} (\vartheta^{\alpha_0}\partial^{\alpha'}u)^{A_\alpha} \\ &= \sum_{\alpha\in\mathbb{N}^{d+1}} c_\alpha(t,x)\vartheta^{\alpha_0}\partial^{\alpha'}u. \end{aligned}$$

Further we extract from $L_{lin}(t,x,\vartheta,\partial_x)$ the terms of ordinary differential operator with respect to t. It is denoted by $L_{lin,\vartheta}(t,x,\vartheta)$ and is of the form

$$L_{lin,\vartheta}(t,x,\vartheta)=\sum_{h=0}^{m}c_h(t,x)\vartheta^h=\sum_{h=0}^{m}t^{e_h}b_h(t,x)\vartheta^h. \tag{3.2}$$

In the present article we consider nonlinear partial differential equations which are regarded as perturbations of ordinary differential operators in some sense. In order to explain the meaning of perturbations we define the characteristic polygon. We denote by $\lrcorner(a,b)$ an infinite rectangle with lower right corner (a,b), $\lrcorner(a,b):=\{(x,y)\in\mathbb{R}^2; x\le a, y\ge b\}$. Define a convex set $\Sigma_{L_{lin,\vartheta}}\subset\mathbb{R}^2$ by

$$\Sigma_{L_{lin,\vartheta}}=\textit{the convex hull of}\cup_{h=0}^{m}\lrcorner(h,e_h), \tag{3.3}$$

which is called the characteristic polygon of $L_{lin,\vartheta}$. The boundary of $\Sigma_{L_{lin,\vartheta}}$ consists of a vertical half line $\Sigma_{L_{lin,\vartheta}}(0)$, segments $\{\Sigma_{L_{lin,\vartheta}}(i)\}_{i=1}^{p^*-1}$ and a horizontal half line $\Sigma_{L_{lin,\vartheta}}(p^*)$. Let γ_i be the slope of $\Sigma_{L_{lin,\vartheta}}(i)$. Then $0=\gamma_{p^*}<\gamma_{p^*-1}<\cdots<\gamma_1<\gamma_0=+\infty$. Let $\{(m_i,e(i))\in\mathbb{R}^2; 0\le i\le p^*-1\}$ be the set of vertices of $\Sigma_{L_{lin,\vartheta}}$, where $e(i):=e_{m_i}$ and $0\le m_{p^*-1}<\cdots<m_1<m_0=m$. The endpoints of the segment $\Sigma_{L_{lin,\vartheta}}(i)$ are $(m_{i-1},e(i-1))$ and $(m_i,e(i))$. Set $\mathcal{I}_i=\{h\in\mathbb{N}; (h,e_h)\in\Sigma_{L_{lin,\vartheta}}(i)\}$ and define subsets $\mathfrak{N}_i$ of $\mathbb{N}^{M*}$ for $1\le i\le p^*$ by

$$\begin{aligned}\mathfrak{N}_i=&\{A=(A_\alpha)\in\mathbb{N}^{M*}; |A|=1 \text{ and } A_\alpha=1\\ &\text{for some } \alpha=(h,0,\cdots,0)\in\mathbb{N}^{d+1} \text{ with } h\in\mathcal{I}_i\}.\end{aligned} \tag{3.4}$$

We assume the following:

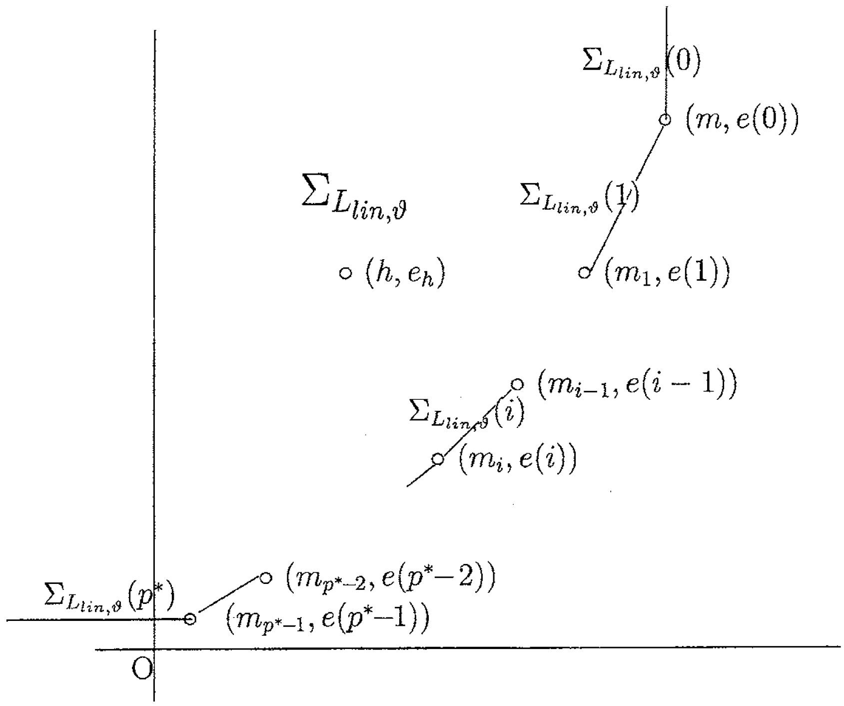
$$\textbf{(C.0)}\qquad b_{m_i}(0,0)\neq 0 \quad \text{for all}\quad 0\le i\le p^*-1. \tag{3.5}$$

Suppose (C.0) holds. Then there is $R>0$ such that for all $0\le i\le p^*-1$

$$b_{m_i}(0,x)\neq 0 \quad on \quad \{|x|\le R\}. \tag{3.6}$$

Set for $1\le i\le p^*-1$

$$\begin{cases} B_i(\xi,x)=\displaystyle\sum_{(h,0')\in\mathcal{I}_i}b_h(0,x)(\gamma_i\xi)^h,\\ B_i^0(\xi,x)=\displaystyle\sum_{(h,0')\in\mathcal{I}_i}b_h(0,x)(\gamma_i\xi)^{h-m_i}.\end{cases} \tag{3.7}$$



Then $B_i(\xi, x) = (\gamma_i \xi)^{m_i} B_i^0(\xi, x)$ and $B_i^0(\xi, x)$ is a polynomial in ξ with degree $(m_{i-1} - m_i)$ and $B_i^0(0, x) = b_{m_i}(0, x) \neq 0$ on $\{|x| \leq R\}$.

Definition 3.1. *Suppose* (C.0) *holds and* $p^* \geq 2$. *Let* $i \in \{1, 2, \cdots, p^* - 1\}$. *Set* $Z_i(r) = \bigcup_{|x| \leq r} \{\xi; B_i^0(\xi^{\gamma_i}, x) = 0\}$. *A singular direction of level* γ_i *on* $\{|x| \leq r\}$ *is an argument of an element of* $Z_i(r)$. *We denote by* $\Xi_i(r)$ *the totality of singular directions on* $\{|x| \leq r\}$ *of level* γ_i.

We give other conditions.

(C.1)

(1). *There exists a formal solution* $\hat{u}(t, x) = \sum_{n=\nu_0}^{\infty} u_n(x) t^n \in \mathcal{O}(U)[[t]]$ *of* $L(\hat{u}) = 0$ *with* $\nu_0 \geq 1$.

(2). *The following holds for all* $i \in \{1, 2, \cdots, p^*\}$. *For* $A \in \mathbb{N}^{M*} - \mathfrak{N}_i$

$$e_A + (|A| - 1)\nu_0 - e(i-1) > \gamma_i (m_A - m_{i-1}). \tag{3.8}$$

The condition **(C.1)**-(2) ((3.8)) means that $L(\cdot)$ is a perturbation of $L_{lim,\vartheta}$ in some sense. We note that $e_A + (|A|-1)\nu_0 - e(i-1) = \gamma_i(m_A - m_{i-1})$ holds if and only if $A \in \mathfrak{N}_i$, which means $e(h) - e(i-1) = \gamma_i(h - m_{i-1})$ for $h \in \mathcal{I}_i$.

Remark 3.2. (1) *The condition* **(C.1)**-(2) *depends on* ν_0.

(2) *It is obvious that* (3.8) *holds for large* $|A|$. *Hence it is enough to expand* $c_A(t,x)$ *with respect to* t, $c_A(t,x) = t^{e'_A} b'_A(t,x)$ $(e'_A \le e_A)$, *so that* $e'_A + (|A|-1)\nu_0 - e(i-1) > \gamma_i(M_A - m_{i-1})$.

Consider nonlinear partial differential equation

$$L(u) := L(t,x,\vartheta^{\alpha_0}\partial_x^{\alpha'} u) = 0. \tag{EQ}$$

Then our main result is

Theorem 3.3. *Assume that* $p^* \ge 2$, (C.0) *and* (C.1) *hold. Let* $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_{p^*-1})$ *be a multidirection such that* $[\theta_i - \delta_i, \theta_i + \delta_i] \cap \overline{\Xi_i(R)} = \emptyset$ *for* $\delta_i > 0$. *Then* $\hat{u}(t,x) \in \mathcal{O}(U')[[t]]$ *is* $\gamma = (\gamma_1, \cdots, \gamma_{p^*-1})$*-summable in the multidirection* $\boldsymbol{\theta}$ *for a polydisk* $U' \subset U$.

Set $S_i^* = S^*(\theta_i, \delta_i)$. Then the assumption $[\theta_i - \delta_i, \theta_i + \delta_i] \cap \overline{\Xi_i(R)} = \emptyset$ means $B_i^0(\xi^{\gamma_i}, x)$ is invertible on $S_i^* \times \{|x| \le R\}$, which is used in the proof of Theorem 3.3.

We give an outline of the proof of Theorem 3.3. For the details of it we refer to [9]. First we show

Proposition 3.4. *Let* $\hat{u}(t,x) = \sum_{n=1}^{\infty} u_n(x) t^n$ *be a formal solution of* $L(\hat{u}) = 0$. *Then there are constants* A *and* B *such that*

$$|u_n(x)| \le AB^n \Gamma\Big(\frac{n}{\gamma_{p^*-1}}\Big) \tag{3.9}$$

in a neighborhood V *of* $x = 0$.

It follows from Proposition 3.4 that

$$(\hat{\mathcal{B}}_{\gamma_{p^*-1}} \hat{u})(\xi, x) := \sum_{n=1}^{\infty} \frac{u_n(x)\xi^{n-\gamma_{p^*-1}}}{\Gamma(\frac{n}{\gamma_{p^*-1}})} \tag{3.10}$$

is holomorphic in $\{\xi; 0 < |\xi| < \rho\} \times V$. Set $\phi(\xi, x) = (\hat{\mathcal{B}}_{\gamma_{p^*-1}} \hat{u})(\xi, x)$. As for holomorphic extension of $\phi(\xi, x)$ to an infinite sector with respect to ξ, we have

Proposition 3.5. *Let θ_{p^*-1} be a direction and $\delta_{p^*-1} > 0$ be a small constant such that $[\theta_{p^*-1}-\delta_{p^*-1}, \theta_{p^*-1}+\delta_{p^*-1}] \cap \bar{\Xi}_i(R) = \emptyset$. Set $S^*_{p^*-1} = S^*(\theta_{p^*-1}, \delta_{p^*-1})$. Then $\phi(\xi, x)$ is holomorphically extensible to $S^*_{p^*-1} \times W$ and $\phi \in Exp_{\{\kappa_{p^*-1}\}}(S^*_{p^*-1} \times W)$ for a neighborhood W of x=0.*

By Proposition 3.5, $(\mathcal{A}_{\gamma_{p^*-2},\gamma_{p^*-1},\theta_{p^*-1}} \hat{\mathcal{B}}_{\gamma_{p^*-1}} \hat{u})(\xi, x)$ is defined, and we can show that it belongs to $Exp_{\{\kappa_{p^*-2}\}}(S^*_{p^*-2} \times W')$, $S^*_{p^*-2} = S^*(\theta_{p^*-2}, \delta_{p^*-2})$, so $(\mathcal{A}_{\gamma_{p^*-3},\gamma_{p^*-3},\theta_{p^*-2}} \mathcal{A}_{\gamma_{p^*-2},\gamma_{p^*-1},\theta_{p^*-1}} \hat{\mathcal{B}}_{\gamma_{p^*-1}} \hat{u})(\xi, x)$ is defined. Consequently, by continuating these processes,

$$u(t, x) = (\mathcal{L}_{\gamma_1,\theta_1} \mathcal{A}_{\gamma_1,\gamma_2,\theta_2} \mathcal{A}_{\gamma_2,\gamma_3,\theta_3} \cdots \mathcal{A}_{\gamma_{p^*-2},\gamma_{p^*-1},\theta_{p^*-1}} \hat{\mathcal{B}}_{\gamma_{p^*-1}} \hat{u})(t, x) \qquad (3.11)$$

can be defined, hence, $\hat{u}(t, x)$ is $\boldsymbol{\gamma}$-summable in the direction $\boldsymbol{\theta}$, $u(t, x)$ is $\boldsymbol{\gamma}$-sum of $\hat{u}(t, x)$ and $L(u) = 0$ holds.

4 Remarks and Generalization

(1) In this article we assume for simplicity that $L(t, x, Z)$ is holomorphic in a full neighborhood $U_0 \times U \times \Omega$. We can show the similar result as Theorem 3.3 under the following condition:

$L(t, x, Z)$ is the $\boldsymbol{\gamma}$-sum of $\hat{L}(t, x, Z) \in \mathcal{O}(U \times \Omega)[[t]]$ which is $\boldsymbol{\gamma}$-summable in multidirection $\boldsymbol{\theta}$.

(2) We give comments about more general nonlinear partial differential equations to which we can apply the preceding results.

Let $K(t, x, \vartheta^{\alpha_0} \partial_x^{\alpha'} u) = 0$ be a nonlinear partial differential equation with order m, where $K(t, x, Z)$, $(t, x, Z) \in \mathbb{C} \times \mathbb{C}^d \times \mathbb{C}^M$, and be holomorphic in a neighborhood of $(t, x, Z) = (0, 0, 0)$ with $K(0, 0, 0) = 0$. Suppose that there exists a formal power series $\hat{u}(t, x) = \sum_{n=0}^{\infty} u_n(x) t^n \in \mathcal{O}(U)[[t]]$ satisfying $K(\hat{u}) = 0$. Let $\nu_0 \geq 1$ and $v_{\nu_0-1}(t, x) = \sum_{n=1}^{\nu_0-1} u_n(x) t^n$. Consider

$$L(\nu_0; u) := K\big(t, x, \vartheta^{\alpha_0} \partial_x^{\alpha'} (v_{\nu_0-1} + u)\big) = 0, \qquad (4.1)$$

which depends on $v_{\nu_0-1}(t, x)$ and has a formal power series solution $\hat{u}(t, x) = \sum_{n=\nu_0}^{\infty} u_n(x) t^n$. If $L(\nu_0; u)$ satisfies the assumptions of Theorem 3.3, then we have multi-summability of the formal power series solution, that is, there exists a solution $u(t, x)$ represented such as (3.11) and $u(t, x) \sim \hat{u}(t, x)$ as $t \to 0$ in some sector.

References

[1] W. Balser, Formal power series and linear systems of meromorphic ordinary differential equations, *Universitext Springer,* (1999).

[2] W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya, Multisummabilty of formal power series solutions of linear ordinary differential equations, *Asymptotic Analysis,* **5** (1991), 27-45.

[3] B.L.J. Braaksma, Multisummabilty and Stokes multipliers of linear meromorphic differential equations, *J. Differential Equations,* **92** (1991), 45-75.

[4] B.L.J. Braaksma, Multisummabilty of formal power series solutions of nonlinear meromorphic differential equations, *Ann. Inst. Fourier,* **42** (1992), 517-540.

[5] J.-P. Ramis and Y. Sibuya, A new proof of multisummability of formal solutions of nonlinear meromorphic differential equations, Ann. Inst. Fourier **44** (1994), 811–848.

[6] S. Ōuchi, Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations, *J. Math. Sci. Univ. Tokyo,* **2** (1995), 375-417.

[7] S.Ōuchi, Multisummability of Formal Solutions of Some Linear Partial Differential Equations, *J. Differential Equations,* **185** (2002), 513-549.

[8] S.Ōuchi, Borel summability of formal solutions of some first order singular partial differential equations and normal forms of vector fields, *to appear in J. Math. Soc. Japan* (2005).

[9] S.Ōuchi, Multisummability of formal power series solutions of nonlinear partial differential equations in complex domains, (*preprint*).