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指数人口構造をもつ SIRS 伝染病モデルの解析
Analysis of an SIRS Epidemic Model with Exponential Demographic Structure

大阪府立大学大学院工学研究科 吉田 直樹 (Naoki Yoshida) [1]
原 惟行 (Tadayuki Hara) [2]
Graduate School of Engineering, Osaka Prefecture University

1 Introduction

It is one of basic and interesting problems to find periodic oscillations in epidemic models. Smith [11] found periodic oscillations in epidemic models with periodic parameters. Here we consider the existence of periodic solutions in an SIRS epidemic model with non-periodic parameters and with delay.

A constant population SIRS model with delay can exhibit periodic solutions for some parameter values, see Hethcote et al. [8]. By contrast, Cooke et al. [4] found some periodic solutions in a variable population SIRS model with delay and with exponential demographic structure. Here we incorporate a delay (i.e. a constant period of temporary immunity) into a few variable population disease model. Many disease models for a few variable population have been studied, see Cooke et al., Gao et al. and Hethcote et al. [5, 6, 9].

2 Model formulation

A population size $N(t)$ is divided into disjoint classes of individuals who are susceptible, infective and recovered with temporary immunity; with sizes denoted by $S(t)$, $I(t)$ and $R(t)$, respectively.

In our model all newborns are assumed susceptible, and the natural death rate constant is the same throughout the population. A constant disease-related death rate is included. We assume that the immune period is constant, denoted by τ. Thus the probability that an individual remains in the recovered class t units after becoming recovered (without dying) is given by the step function with value 1 for $t \leq \tau$, and 0 for $t > \tau$.

[1]Email: yoshida@ms.osakafu-u.ac.jp
[2]Email: hara@ms.osakafu-u.ac.jp

The flow of individuals is described in the transfer diagram:

$$\begin{array}{ccccccc} B(N)N\downarrow & & & & & & \\ S & \xrightarrow{\beta SI/N} & I & \xrightarrow{\lambda I} & R^{\tau} & \longrightarrow & S \\ \mu S\downarrow & & (\mu+\alpha)I\downarrow & & \mu R\downarrow & & \end{array}$$

Here $B(N)N$ is a birth rate function with $B(N)$ satisfying the following assumptions for $N \in (0, \infty)$:

(A1) $B(N) > 0$;

(A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;

(A3) $B(0^+) > \mu + \alpha$ and $\mu > B(+\infty)$.

Note that (A2) and (A3) imply that $B^{-1}(N)$ exists for $N \in (B(\infty), B(0^+))$, and (A3) assures that N does not go to extinction and cannot blow up.

The parameter $\mu > 0$ is the natural death rate constant, $\alpha \geq 0$ is the disease-related death rate constant, and $\lambda \geq 0$ is the rate constant for recovery. The force of infection is assumed to be of standard type, namely $\beta I/N$, with $\beta > 0$, the effective per capita contact rate constant of infective individuals.

Our model thus take the following form:

$$N(t) = S(t) + I(t) + R(t), \tag{2.1}$$

$$S'(t) = B(N(t))N(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \lambda I(t-\tau)e^{-\mu\tau}, \tag{2.2}$$

$$I'(t) = \frac{\beta S(t)I(t)}{N(t)} - (\mu + \lambda + \alpha)I(t), \tag{2.3}$$

$$R(t) = \int_{t-\tau}^{t} \lambda I(u)e^{-\mu(t-u)}du, \tag{2.4}$$

for $t > \tau$. It is convenient to shift time by τ, so that (2.1)–(2.4) hold for the new time $t > 0$, with the following initial conditions:

$$S(\theta) > 0,\ I(\theta) > 0,\ R(\theta) > 0 \text{ on } [-\tau, 0]. \tag{2.5}$$

Differentiating (2.4) gives

$$R'(t) = \lambda I(t) - \lambda I(t-\tau)e^{-\mu\tau} - \mu R(t). \tag{2.6}$$

Theorem 2.1. *a) A solution of the integro-differential system (2.2)–(2.4) with $N(t)$ given by (2.1) satisfies (2.6). b) Conversely, let $S(t)$, $I(t)$, $R(t)$ be a solution of the delay differential system (2.2), (2.3), (2.6) with $N(t)$ given by (2.1), and initial conditions given*

on the interval as stated above. In addition, suppose that

$$R(0) = \int_{-\tau}^{0} \lambda I(u) e^{\mu u} du. \tag{2.7}$$

Then this solution satisfies the integro-differential system (2.2)–(2.4). c) Moreover, for all $t \geq 0$, the solution exists, is unique and has $S(t) > 0$, $I(t) > 0$, $R(t) > 0$.

Proof. Assertions a) and b) are clear. For assertion c), note that the usual local existence, uniqueness and continuation results are applied to [7]. Let $T = \inf\{t > 0 | S(t)I(t)/N(t) = 0\}$ and suppose that T is finite. $I(t) > 0$ on $[0, T]$ by (2.3). From (2.4) it is clear that $R(t) > 0$ on $[0, T]$. The assumptions in the model imply that $S'(t) \geq -\beta SI/N - \mu S \geq -(\beta + \mu)S$, so that $S(T) \geq S(0)e^{-(\beta+\mu)T} > 0$. This contradicts the supposition that T is finite, so T must be infinite. Hence $S(t) > 0$, $I(t) > 0$, $R(t) > 0$. Add equations (2.2), (2.3), (2.6), and use (2.1) to obtain

$$N' = (B(N) - \mu)N - \alpha I. \tag{2.8}$$

Thus $N(t)$ doed not go to zero and cannot blow up to ∞. Consequently, the solution exists globally for all $t > 0$ and is unique. □

3 Disease-free equilibrium

Stability of disease-free equilibrium is stated in terms of a key threshold parameter

$$\mathcal{R}_0 = \frac{\beta}{\mu + \lambda + \alpha}. \tag{3.1}$$

A linear analysis shows the following theorem. The proof of Theorem 3.1 is omitted.

Theorem 3.1. *The system (2.2)–(2.4) with (2.1) always has the disease free equilibrium $(S(t), I(t), R(t)) = (B^{-1}(\mu), 0, 0)$. If $\mathcal{R}_0 < 1$, then it is locally asymptotically stable; if $\mathcal{R}_0 > 1$, then it is unstable.*

A global stability result can be given as follows.

Theorem 3.2. *For $\mathcal{R}_0 < 1$ all solutions of the system (2.2)–(2.4) with (2.1) approach the disease free equilibrium as $t \to \infty$.*

Proof. By (2.3), we have $I' \leq (\beta - \mu - \lambda - \alpha)I$, hence $I(t)$ has limit zero as $t \to \infty$ if $\beta - \mu - \lambda - \alpha < 0$. Then $R(t) \to 0$ from (2.4). Since (2.8) has the limit equation $N' = (B(N) - \mu)N$ (see [12]), $N(t) \to B^{-1}(\mu)$ as $t \to \infty$. Hence $S(t) \to B^{-1}(\mu)$ as $t \to \infty$. □

4 Endemic equilibrium

Let (S^*, I^*, R^*) be an endemic equilibrium. Then it must satisfy

$$\begin{aligned} N^* &= S^* + I^* + R^* \\ 0 &= B(N^*)N^* - \mu N^* - \alpha I^* \\ 0 &= \beta S^* \frac{I^*}{N^*} - (\mu + \lambda + \alpha) I^* \\ 0 &= \lambda I^* - \lambda I^* e^{-\mu\tau} - \mu R^*. \end{aligned}$$

Solving these equations, the following theorem holds.

Theorem 4.1. *If $\mathcal{R}_0 > 1$, then the system (2.2)-(2.4) with (2.1) has a unique endemic equilibrium $(S(t), I(t), R(t)) = (S^*, I^*, R^*)$ where*

$$S^* = \frac{\mu+\lambda+\alpha}{\beta} N^*, I^* = \left(1 - \frac{\mu+\lambda+\alpha}{\beta}\right) N^* \Big/ \left(1 + \frac{\lambda(1-e^{-\mu\tau})}{\mu}\right), R^* = \frac{\lambda(1-e^{-\mu\tau})}{\mu} I^*$$

$$\text{and} \quad N^* = B^{-1}\left(\mu + \alpha\left(1 - \frac{\mu+\lambda+\alpha}{\beta}\right) \Big/ \left(1 + \frac{\lambda(1-e^{-\mu\tau})}{\mu}\right)\right).$$

If $\mathcal{R}_0 \leq 1$, then there is no endemic equilibrium.

Hereafter we call that the disease persists in the population if $\liminf_{t\to\infty} I(t) > 0$.

Theorem 4.2. *For the system (2.2)-(2.4) with (2.1), the disease persists in the population if $\mathcal{R}_0 > 1$.*

Proof. By (2.8), there are positive constants k and K such that $k \leq N(t) \leq K$ for large t since $I(t) \leq N(t)$. Let $X = \{(S, I, R) \in \mathbb{R}^3_+ | k \leq S + I + R \leq K\}$. Then the space of functions $C = C([-\tau, 0], X)$ is the complete metric space with supnorm $||\varphi|| = \sup_{s\in[-\tau,0]} |\varphi(s)|$. A subset Y of C with the Lipschitz condition is compact (see [2, p. 170]). By taking the Lipschitz constant large enough, Y is forward invariant and attractive.

Therefore we restrict the analysis to Y. Let $\mathbf{x} = (S(\theta), I(\theta), R(\theta))$ with $\theta \in [-\tau, 0]$ be a point in Y. We can show that $\mathbf{S} = \{\mathbf{x} \in Y : I(0) = 0\}$ is a forward invariant compact subset of Y. Let $P : Y \to \mathbb{R}^+$ (a continuously differentiable functional) be defined by $P(\mathbf{x}) = I(0)$ and let 'dot' denote differentialtion along a solution. Then $\dot{P}/P = \beta S(0)/N(0) - (\mu+\lambda+\alpha)$ for solutions starting in $Y\backslash\mathbf{S}$. Since solutions starting in $\mathbf{S}$ approach $(B^{-1}(\mu), 0, 0) \in \mathbf{S}$, by applying the theorem on average Liapunov functions (see [3, 10]), it follows that $\mathbf{S}$ is a uniform repeller. Hence $\liminf_{t\to\infty} I(t) > 0$. □

If there is no disease related death (i.e. $\alpha = 0$), then local stability of the unique endemic equilibrium is governed by the characteristic equation

$$(z - H + \mu)\left\{z^2 + \left(\mu + \beta\frac{I^*}{N^*}\right)z + \beta\frac{I^*}{N^*}(\mu + \lambda) - \beta\frac{I^*}{N^*}\lambda e^{-(\mu+z)\tau}\right\} = 0 \qquad (4.1)$$

where $H = \frac{d}{dN}B(N)N\Big|_{N=N^*}$. Applying a stability switch criterion [1] to (4.1), it follows that this equation can have purely imaginary roots for some parameter values. Thus for some $\tau > 0$, arising by a Hopf bifurcation, periodic solutions are possible.

If the set of parameters $\beta = 0.2 \times 365$, $\lambda = 365/7$, $\mu = 1/80$, $\alpha = 0$, $B(N) = 0.012 + 1/(90N)$ satisfying $\mathcal{R}_0 > 1$ are given, for $\tau \in (0.5, 2.7 \times 10^2)$, the endemic equilibrium is unstable whereas it is asymptotically stable for $0 \leq \tau < 0.5$ and for any $\tau > 2.7 \times 10^2$. These results are in agreement with our computer simulations using MATHEMATICA. Numerically solutions of the system (2.2)–(2.4) oscillate for $\tau = 1.0$ (Fig. 1).

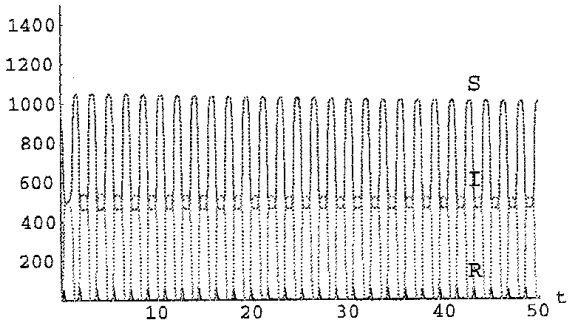

Figure 1: Numerical solutions for the SIRS model (2.2)–(2.4) with initial conditions given by $S(t) = 1000, I(t) = 1, R(t) = \tau e^{-\mu\tau}$ and with $\tau = 1.0$.

5 Conclusions

In this paper, we have formulated a few variable population SIRS disease transmission model with a constant immune period. We have identified an important threshold parameter $\mathcal{R}_0$ in (3.1). The disease dies out if $\mathcal{R}_0 < 1$, because infective individuals tend to zero. If $\mathcal{R}_0 > 1$, the disease can persist in the population. Local stability of the endemic equilibrium is analyzed under the restriction that $\alpha = 0$. For some parameter values, intermediate delays show a destabilizing effect on the endemic equilibrium, and yield periodic solutions. An immune period is regarded as zero in SIS models, while it is regarded as infinite (permanent immune) in SIR models. Our simple model suggests that a finite and some extent of immune period brings a different qualitative feature from classical SIS & SIR epidemic models.

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