数理解析研究所講究録 1432 巻 2005 年 43-48

Blowup of solutions to some systems related to Keller-Segel system

(Keller-Segel 系に関係する方程式の解の爆発について)

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Keller and Segel introduced a parabolic system to describe the aggregation of cellular slime molds. They introduce the system according to the following hypothesis.

Cells sense the gradient of chemical concentration and move toward higher concentration. We refer to the phenomenon as chemotaxis. Cells produce the chemical substance.

Then, the chemical substance is an attractant.

The following system is so called Keller-Segel system.

$$(KS)\begin{cases} u_t = \nabla\cdot(\nabla u - u\nabla v) \text{ in } \Omega\times[0,\infty), \\ v_t = \Delta v - v + u \text{ in } \Omega\times[0,\infty), \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 \text{ on } \partial\Omega\times[0,\infty), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0 \text{ in } \Omega. \end{cases}$$

Here, $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $u_0(\not\equiv 0)$ and v_0 are smooth and nonnegative in $\overline{\Omega}$.

$u(x,t)$ represents the density of cells at (x,t) and $v(x,t)$ represents the chemical concentration at (x,t).

In this system, cells sense the gradient of chemical substance, move toward higher concentration and produce the chemical substance. Then, the direction of flow due to chemotaxis is almost opposite to one of diffusion.

If we can neglect the diffusion of cells and chemical substance. At a place, some cells exist and produce some chemical substance. Then, some cells sense the gradient of chemical concentration, and move toward the place. Then, at the place the density of cells increases and much chemical substance is produced. By repeating this story, the aggregation of cells occurs.

However, in this explanation the diffusion is neglected. Then, when the intensity of chemotaxis is more strong than one of diffusion, the aggregation occurs. I think that the blowup of solutions to mathematical model corresponds to the aggregation.

The followings are fundamental mathematical results.

Proposition 1 *The system (KS) has the unique classical solution (u, v) for given initial conditions u_0 and v_0 in $\Omega \times (0, T_{max})$ and the solution satisfies that*

$$u(x,t) > 0 \quad and \quad v(x,t) > 0 \qquad for\ (x,t) \in \overline{\Omega} \times (0, T_{max})$$

and that

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx \equiv \lambda.$$

If $T_{max} < \infty$, it holds that

$$\lim_{t \to T_{max}} \max_{x \in \overline{\Omega}} u(x,t) = \lim_{t \to T_{max}} \max_{x \in \overline{\Omega}} v(x,t) = \infty.$$

Here, T_{max} denotes the maximal existence time.

If $\lim_{t \to T} \left(\max_{x \in \overline{\Omega}} u(x,t) \right) = \infty$, we say that the solution blows up at the time T and that T is the blowup time.

If there exist two sequences $\{q_n\} \subset \overline{\Omega}$ and $\{t_n\} \subset (0, T)$ such that $\lim_{n \to \infty}(q_n, t_n) = (q, T)$ and $\lim_{n \to \infty} u(q_n, t_n) = +\infty$, we say that the point q is a blowup point. Moreover, $\mathcal{B}$ denotes the set of blowup points.

Before the explanation of our mathematical results, I shall describe our conjecture of blowup solutions. The conjecture is one of our goal. And our mathematical results mentioned late are evidences that this conjecture is true.

Our Conjecture: *Suppose that the solution blows up in finite time T_{max}.*

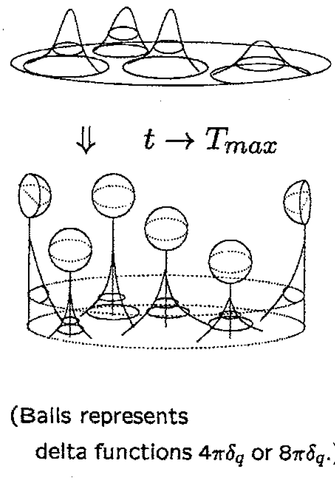

Figure 1: Image of Our Conjecture

Then, the number of blowup points is finite, and the solution satisfies the following.

$$u(\cdot,t) \rightharpoonup \sum_{q\in\mathcal{B}} m_*(q)\delta_q + f \qquad \text{in } \mathcal{M}(\overline{\Omega})$$
$$\text{as } t \to T_{max}.$$

Here, $m_*(q) = \begin{cases} 8\pi & \text{if } q \in \Omega, \\ 4\pi & \text{if } q \in \partial\Omega, \end{cases}$ *,* δ_q *is the delta function whose suppose is the point* q *and* f *is a nonnegative function belonging in* $L^1(\Omega)\bigcap C(\overline{\Omega}\backslash\mathcal{B})$

For our conjecture, I can get the following results.

Theorem 1 (Herrero-Velázquez) *Let* $\Omega = \{x \in \mathbf{R}^2 | |x| < L\}$ *and* $L \in (0,\infty)$. *Then, there exists a radial solution to (KS) satisfying*

$$u(\cdot,t) \rightharpoonup 8\pi\delta_0 + f \qquad \text{in } \mathcal{M}(\overline{\Omega}),$$
$$\text{as } t \to T_{max}(<\infty),$$

where f *is a nonnegative and radial function belonging in* $L^1(\Omega)\bigcap C(\overline{\Omega}\backslash\{0\})$.

Theorem 1 says only the existence of a blowup solution having the delta function singularity. Then, we do not know whether all blowup solutions have such a delta function singularities or not.

Theorem 2 (Nagai-Senba-Suzuki) *Suppose that the solution to (KS) blows up in the finite time and that the blowup points are finite. Then, it holds that*

$$u(\cdot,t) \rightharpoonup \sum_{q\in\mathcal{B}} m(q)\delta_q + f \qquad \text{in } \mathcal{M}(\overline{\Omega}) \qquad \text{as } t \to T_{max},$$

where $m(q) \geqq m_*(q)$ *and* f *is a nonnegative function belonging in* $L^1(\Omega)\bigcap C(\overline{\Omega}\backslash\mathcal{B})$.

Theorem 2 say that all blowup solutions have delta function singularities. However, if our conjecture is true, the assumption of finiteness of blowup points is not necessary, and the constants $m(q)$ must be equal to 8π or 4π.

In order to consider the finiteness of blowup points and the decision of the quantity $m(q)$, we consider the following system.

$$(N)\begin{cases} u_t = \nabla\cdot(\nabla u - u\nabla v) \text{ in } \Omega\times[0,\infty), \\ 0 = \Delta v - v + u \text{ in } \Omega\times[0,T_{max}), \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 \text{ on } \partial\Omega\times[0,T_{max}), \\ u(\cdot,0) = u_0 \text{ in } \Omega. \end{cases}$$

This system is introduced as a simplified system of Keller-Segel system by Professor Nagai. Then, we refer this system as Nagai system. The difference between Nagai system and Keller-Segel system is the second equation. Since the second equation of Nagai system is an elliptic equation, the initial condition v_0 is not necessary.

The analysis of solutions to Nagai system is more easy than one of Keller-Segel system. And we believe that the structure of solutions to Nagai system is similar as one of Keller-Segel system. Then, we investigate the blowup solutions to Nagai system.

Theorem 3 (Suzuki) *If the solution to (N) blows up in the finite time, then the solution satisfies*

$$u(\cdot,t) \rightharpoonup \sum_{q\in\mathcal{B}} m_*(q)\delta_q + f \quad \text{in } \mathcal{M}(\overline{\Omega}) \quad \text{as } t \to T_{max}$$

where f is a nonnegative function belonging in $L^1(\Omega)\bigcap C(\overline{\Omega}\backslash\mathcal{B})$.

We consider the behavior of blowup solutions until now. Next, we shall consider the condition of blowup of solutions for Nagai system.

Theorem 4 (Nagai) *Suppose the following (i) or (ii).*
(i) $\int_\Omega u_0(x)dx > 8\pi$ *and* $\int_\Omega u_0(x)|x-q|^2dx \ll 1$ *for some* $q \in \Omega$.
(ii) $\int_\Omega u_0(x)dx > 4\pi$ *and* $\int_\Omega u_0(x)|x-q|^2dx \ll 1$ *for some* $q \in \partial\Omega$, *and the boundary is line in the neighbourhood of the point* q.
Then, the solution to (N) blows up in the finite time.

By Theorem 3, we obtain that the blowup solutions found by Theorem 4 have delta function singularities.

The conclusion about blowup condition is as follows.

If total mass is less than 4π, the solution can not blowup. In the radial case, if total mass is less than 8π, the solution can not blowup.

If more than 4π mass concentrate near a point on the boundary, the solution blows up. If more than 8π mass concentrate near a point in the domain, the solution blows up.

If the solution blows up in finite time, the delta function appears at each blowup point.

Next, we describe the relation between the structure of solutions and the assumption of the intensity of chemotaxis. In order to describe the relation, we consider the following general case.

$$(N)_\varphi \begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla\varphi(v)) \quad \text{in } \Omega \times [0, T_{max}), \\ 0 = \Delta v - v + u \text{ in } \Omega \times [0, T_{max}), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0, T_{max}), \\ u(\cdot, 0) = u_0 \text{ in } \Omega. \end{cases}$$

In the case where $\varphi(v) = v$, the system is Nagai system.

The first equation represents the change of density of cells. The term ∇u represents the flow due to diffusion of cells, and the term $u\nabla\varphi(v)$ represents the flow due to chemotaxis. Then, we assume that the flow due to chemotaxis is expressed by the term $u\nabla\varphi(v)$, by using a function φ.

We refer to the function φ as the sensitivity function. Since the chemical substance is the attractant, then the differential of φ must be positive.

The typical examples of sensitivity functions are $\log v$, v^p $(p > 0)$, $\frac{v}{1+v}$ and so on.

Then, in order to investigate the relation between the intensity of chemotaxis and the structure of blowup solutions, we investigate the relation between the sensitivity function φ and the structure of blowup solutions.

Here, we treat only the radial case.

Theorem 5 (Nagai-Senba) *Let $\Omega = \{x \in \mathbf{R}^2 \mid |x| < L\}$ $(0 < L < \infty)$ and u_0 be positive and radial in $\overline{\Omega}$.*
(i) Suppose that $\varphi(v) = \log v$ or v^p $(0 < p < 1)$. Then, the radial solution to $(N)_\varphi$ exists globally in time and satisfies

$$\sup\{u(x,t) \mid (x,t) \in \overline{\Omega} \times [0,\infty)\} < \infty.$$

(ii) Suppose that $\varphi(v) = v^p$ $(1 < p)$. If u_0 satisfies

$$\int_\Omega u_0(x)dx > 0 \quad \textit{and} \quad \int_\Omega |x|^2 u_0(x)dx \ll 1,$$

the radial solution blows up in finite time.

Then, the relation between the sensitivity function and blowup of solutions is as follows.

In the case where $\varphi(v) = v$, it holds that the differential of $\varphi(v)$ is equal to 1. Then, if the total mass is less than 4π, the blowup can not occur. If the total mass is more than 4π, the blowup can occur.

In the case where $\varphi(v) = \log v$ or v^p $(0 < p < 1)$, it holds that $\lim_{v\to\infty} \varphi'(v) = 0$. Then, the intensity of chemotaxis is more weak than one of $\varphi(v) = v$, when v is large. In this case, Theorem 5 says that blowup can not occur.

In the case where $\varphi(v) = v^p$ $(p > 1)$, it holds that $\lim_{v\to\infty} \varphi'(v) = \infty$. Then, the intensity of chemotaxis is more strong than one of $\varphi(v) = v$, when v is large. In this case, Theorem 5 says that blowup can occur, even if the total mass is small.

That is to say, the structure of solutions change, when the system changes.